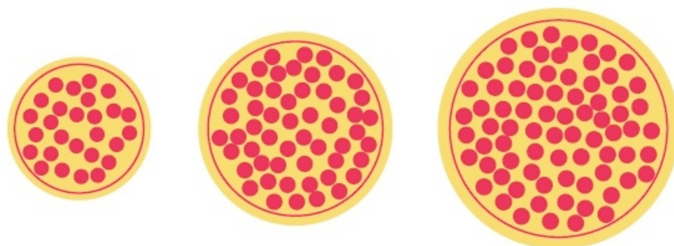
SUMMER MATHS QUIZ SOLUTIONS – PART 2

MEDIUM 1

You have three pizzas, with diameters 15cm, 20cm and 25cm. You want to share the pizzas equally among your four customers. How do you do it? What if you want the pizzas sliced into the minimum number of pieces?



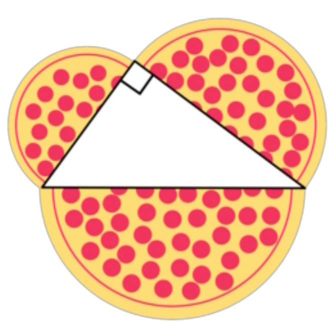
Solution: We can obviously cut each pizza into quarters, making 12 pieces in total, and then everybody gets a quarter of each pizza. However, with a little thought we can see that six pieces will do. It's trickier, but in fact five pieces is enough.

The diameters are in the ratio 3:4:5, and $3^2 + 4^2 = 5^2$. That means the large pizza has exactly the area of the two smaller ones combined:

$$\pi\,(15/2)^2 + \pi\,(20/2)^2 = \pi\,(25/2)^2.$$

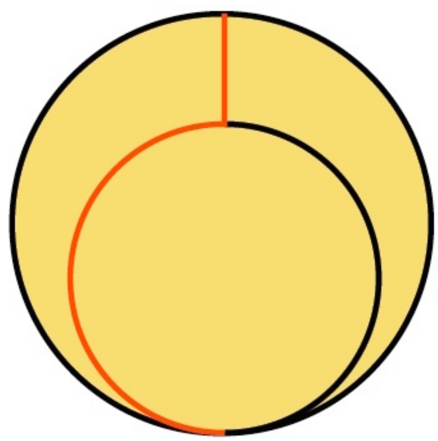
So, cutting each pizza in half, two customers get a large half, and the other two customers each get a small half and a medium half.

Though you don't need it to obtain the solution, hiding underneath all of this is Pythagoras's Theorem:

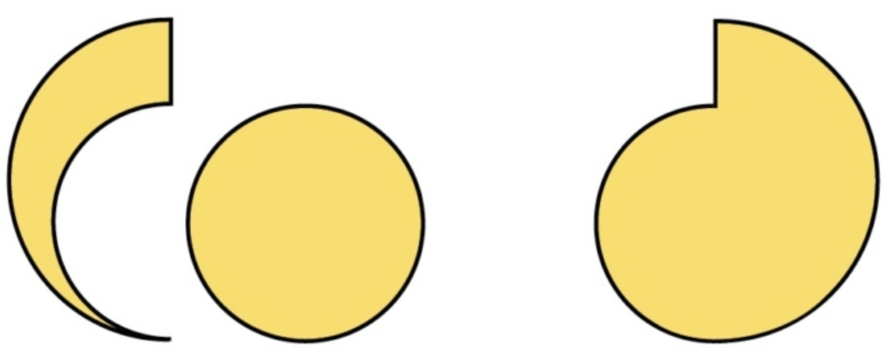


This also suggests how we can get away with cutting the pizzas into a total of five pieces.

Cut the large pizza into two halves, as above. Next, place the small pizza on top of the medium pizza: it doesn't matter exactly how, but easiest is to have the edges of the pizzas touching at a point, leaving a crescent of the medium pizza showing. Now, use the small pizza to cut the medium pizza, halfway around and then to the edge, as indicated by the red lines.



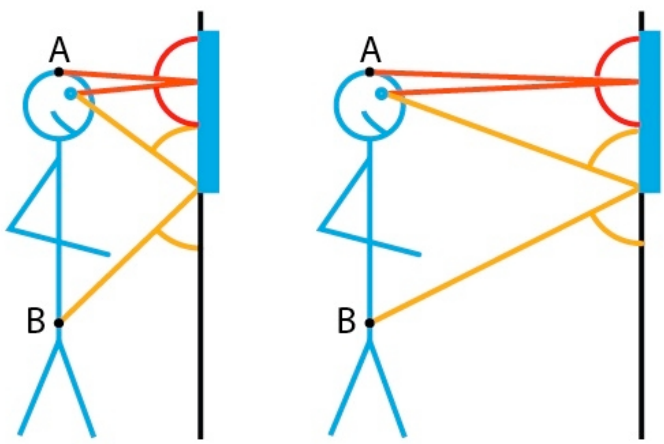
The medium pizza is now in two pieces (making a total of five pieces overall). The smaller piece together with the small pizza has exactly the area of the larger piece, which is exactly the area of a half of the big pizza.



MEDIUM 2

A familiar frustration is looking into a bathroom mirror: you can see only part of your body. In an attempt to solve the problem, you step backwards a pace. Does this result in you seeing more or less of yourself?

Solution: Pretty much, it doesn't matter where you stand. In the diagram below, the two red angles are always equal, as are the two orange angles. This suggests that, whatever distance away, you see everything between points A and B.

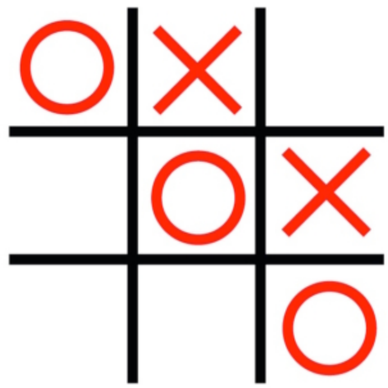


There are two qualifications included in our "pretty much". First of all, if you're *really* close to the mirror then your field of vision won't extend to the full length of the mirror, and you'll see less of yourself.

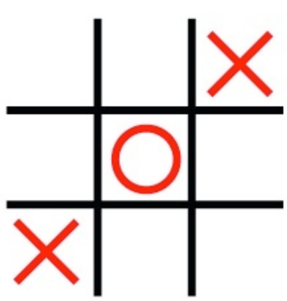
Secondly, if you are further away then you can see further around the curve at the top of your head. You never quite get to see the very top of your head (the "North Pole"), but you see a tiny bit more as you walk further away.

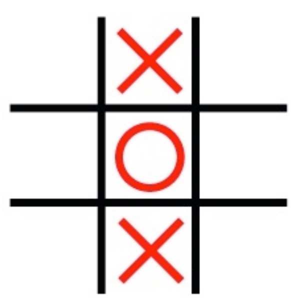
MEDIUM 3

Here's a new version of Noughts and Crosses. The board is the same as usual. However, on each turn the player can choose to draw either a nought or a cross. The winner is the person to complete three in a row of either noughts or crosses. The usual version of Noughts and Crosses should always end in a draw (as long as no one goofs). What about for this new version?



Solution: Martin Gardner discusses this puzzle in his *6th Book of Mathematical Diversions*. The first player can always win by playing in the centre square. (Any other move allows the second player to force a draw). If she plays O in the centre, then the second player must play X to avoid losing on the very next move. Whether the X is in a corner or on a side, the first player plays an X opposite, as indicated in the diagrams. In the left diagram the second player must give away a win on the next move. In the right diagram, the second player's only option is to play X on a third side, the first player responds with X on the fourth side, and then the second player is stuck.





MEDIUM 4

A chocolate shop is selling chocolate frogs, but they forget to include the 10% GST in the price on the packages. They are also offering a 20% discount on bulk purchases of strawberry chocolate frogs. You stagger up to the cashier with a sackful of strawberry chocolate frogs, and you are given a choice: you can have the GST added first and then calculate the bulk discount, or the other way around. Which do you choose?

Solution: It doesn't matter. To add in the 10% GST you multiply the price by 1.1, and to incorporate the 20% discount you multiply by 0.8. So, overall, you multiply by either 1.1 x 0.8 or by 0.8 x 1.1. In either case, the frogs are 88% of the marked price.

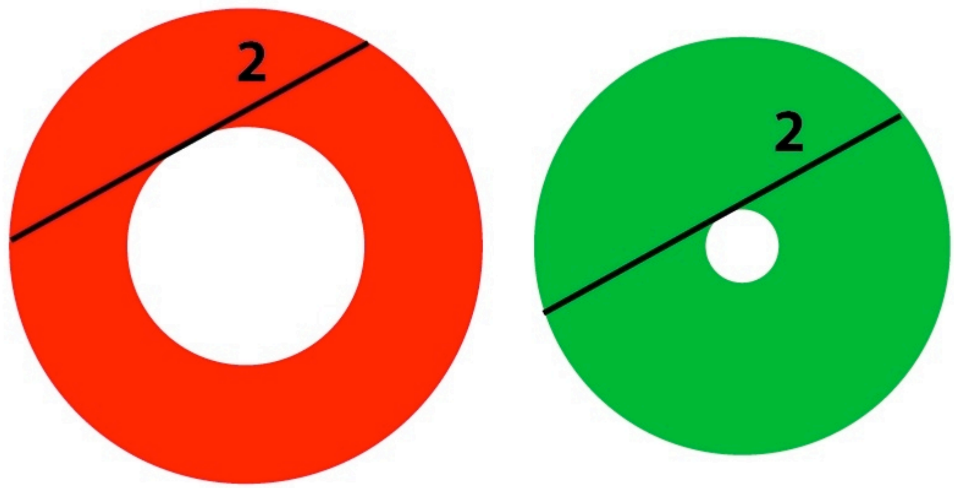
MEDIUM 5

Write down the largest number possible using each of the digits 1, 2 and 3 exactly once, and no other mathematical symbols.

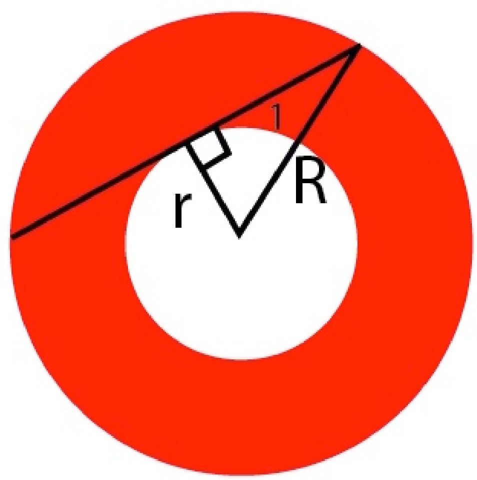
Solution: All you can do is string the digits to make big numbers, and make powers. The only real contenders are 3^{21} and 2^{31}, and 3^{21} = 10,460,353,203 wins.

MEDIUM 6

The indicated lines have length 2. Which circular ring has the greater area?



Solution: The rings have the same area.



Take one of the rings, and write *R* and *r* for the radii of the outer and inner circles. Then the area of the ring is $\pi R^2 - \pi r^2 = \pi (R^2 - r^2)$. However, by Pythagoras's theorem, we have $R^2 - r^2 = 1^2$. It follows that the area of the ring is π. This works for either ring, and so the areas of the two rings are equal. Note that the areas are the same as if there is no hole at all, and we just have a disk of diameter 2.

MEDIUM 7

What is the secret to this ordering of the numbers from 0 to 9?

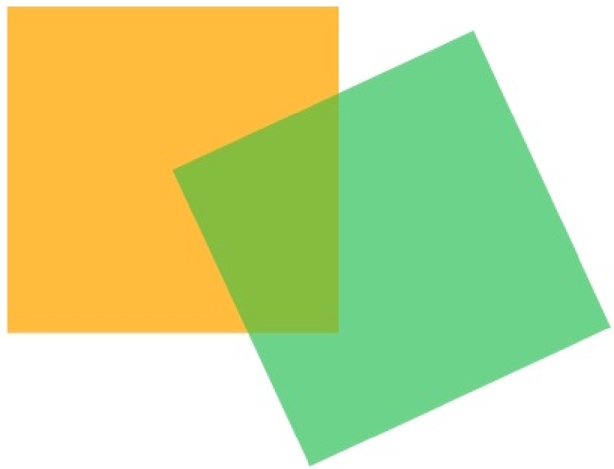
8 5 4 9 1 7 6 3 2 0

Solution: We've written the numbers as words, and then put them in alphabetical order:

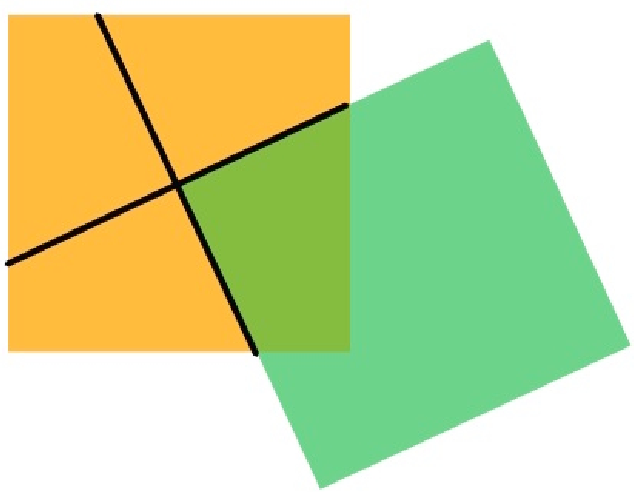
Eight, five, four, nine, one, seven, six, three, two, zero.

MEDIUM 8

The orange square and the green square each have area 1. If a corner of the green square coincides with the centre of the orange square, how large is the area where the squares overlap?

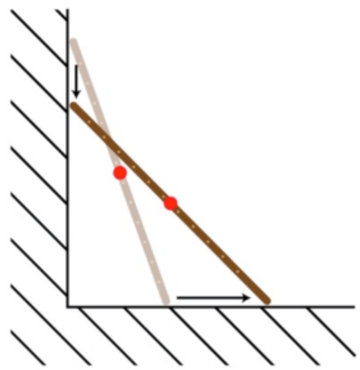


Solution: Using the green square to draw a rectangular cross, the orange square is cut into four equal pieces. The overlap coincides exactly with one of these pieces, and so has area 1/4.

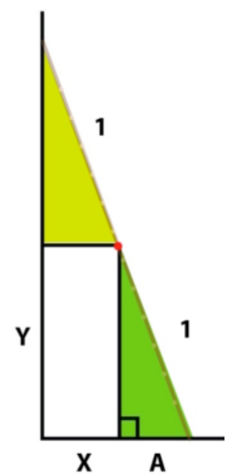


MEDIUM 9

A ladder is leaning against a wall. The base of the ladder starts sliding away from the wall, with the top of the ladder sliding down the wall. As the ladder slides, you watch the red point in the middle of the ladder. What figure does the red point trace? What about other points on the ladder?



Solution: Suppose the red point has coordinates (X, Y), with the base of the ladder a further A units along, and suppose the ladder has length 2. In the diagram below, the two green triangles are equal, and so $A = X$.



And, Pythagoras's Theorem applied to the darker triangle gives $A^2 + Y^2 = 1$. So, $X^2+Y^2 = 1$, and we see that the red point moves along a quarter circle.

The endpoints of the ladder obviously describe straight lines. For other points, the diagram is as above, but the triangles are similar rather than equal. Think of a dot splitting the ladder into a lower piece of length L, and an upper piece of length $2 – L$. Then, by Pythagoras, we have $A^2 + Y^2 = L^2$. Using the similar triangles, we also have $A / L = X / (2 – L)$. Solving for A and plugging into the previous equation, we get the equation for an ellipse.

MEDIUM 10

What, if anything, does this square root monster evaluate to?

$$\cdots\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}}$$

Solution: Your Maths Masters had quite a discussion about this one. The answer might be 2, or 4, or either, or neither. The problem, as always with dots, is to decide what is really meant by those dots.

In brief, the dots should tell us how to sneak up on the completed expression. As a simpler and more familiar example, consider the infinite decimal 0.99999… Here, the dots instruct us to consider the sequence of dot-free decimals 0.9, 0.99, 0.999, 0.9999, and so on. These finite decimals level off, and the number to which they level off is (by definition) the value of the infinite decimal. (The decimal happens to have value 1, but that doesn't matter for this story).

Now, to our root monster. The process is to repeatedly take the square root and add 2. This gives us three natural ways to sneak up on the monster:

(a) $2,\ \sqrt{2},\ 2+\sqrt{2},\ \sqrt{2+\sqrt{2}},\ 2+\sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \cdots$ OR

(b) $2,\ 2+\sqrt{2},\ 2+\sqrt{2+\sqrt{2}},\ \cdots$ OR

(c) $\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \cdots$

Probably (a) is the most natural, since it is spelling out every step. For (b), we stop after each "add 2", and for (c) we stop after each "take the square root".

If (shudder) we plug these expressions into a calculator, we get a good sense of where the sequences are going. Rounding off, we have

$$\text{(a)} \quad \mathbf{2\,,\ 1.412\,,\ 3.412\,,\ 1.848\,,\ 3.848\,,\ 1.962\,,\ \cdots} \qquad \text{OR}$$

$$\text{(b)} \quad \mathbf{2\,,\ 3.412\,,\ 3.848\,,\ \cdots} \qquad \text{OR}$$

$$\text{(c)} \quad \mathbf{1.412\,,\ 1.848\,,\ 1.962\,,\ \cdots}$$

We'll give a bit of justification below, but let's begin by accepting the calculator as guide. Then the (c) sequence seems to approach 2, and so we might give the value of the monster as $M = 2$. Then, the numbers in (b) are 2 more than the numbers in (c), and so we also have the possibility $M = 4$.

Finally, what about (a)? Here, every second number is close to 2, and every second number is close to 4. So, what's THE answer? It depends who you ask. There's an argument that both 2 and 4 are legitimate answers: exactly since we can approach as in either (b) or (c). But there's also an argument that two answers means no answer! Given the numbers are alternatingly close to both 2 and 4, they're not specifically close to anything!

To tidy up a bit, let's argue without a calculator. We'll just consider the (c) sequence, and write M for its end result: we suspect that $M = 2$. The key idea is that the underlying rule of (c) – "add 2 then take the square root" –also makes no difference to M. That is,

$$M = \sqrt{2 + M}$$

(This takes a bit of justification, but we're just being skimpy on the details here, not actually cheating). We can see that $M = 2$ does indeed solve this equation, and a bit more work shows that 2 is the only solution. (Squaring both sides of the equation, we can factorise to completely analyse the possible solutions).