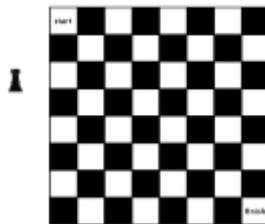


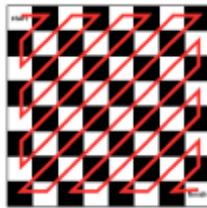
MATHS CHALLENGE SOLUTIONS – PART 3

Burkard Polster and Marty Ross

41. Place one of the major chess pieces (not a pawn) in the top left square of the board. Your mission is to move the piece to the bottom right square of the board, visiting each square of the board exactly once. For which pieces (bishop, knight, rook, queen, king) can you do this? Can you show that it is impossible for the other pieces? Of course, only legal chess moves are permitted. Except for the knight, if a piece moves to a distant square it is considered to have visited all the squares that it has passed through along the way.



Answer: The trip is possible for the king and queen. The diagram shows one of many possible tours for these pieces.



The bishop is obviously impossible because it can only move on squares of a fixed colour. The Rook and knight are also impossible because they change square colour on every move: starting on a white square means that the 64th and last square they visit must be black.

42. In *A Beautiful Mind*, John Nash (Russell Crowe) is quizzing his students. Two bicycles start out 1 mile apart, and are coming together, each traveling at a speed of 10 miles per hour. There is a fly on bicycle A, and he flies towards bicycle B at 20 miles per hour. The fly then travels back to A, then forwards again to B, and so on, until the two bikes collide and the poor fly is squashed. How far does the fly travel in total?

Answer: This is easy when you look at it the right way. The gap between the two bicycles is narrowing at $10 \text{ km/h} + 10 \text{ km/h} = 20 \text{ km/h}$, and the fly is buzzing away at the same speed. This means that the fly will travel exactly 1 mile, the initial distance between the two bicycles.

The “wrong” way to look at this problem is to separately add all the fly’s back-and-forths, resulting in an infinite geometric sum. There is a very famous anecdote concerning this problem and the mathematician John VonNeumann: see <http://www.cut-the-knot.org/arithmetric/999999.shtml>.

43. Federer and Nadal are playing the fifth and final set at Wimbledon. It is 6 games all and there are no tiebreakers: the winner is the first to get two games in front. Federer has a $2/3$ chance of winning a game when he is serving and Nadal has a $3/4$ chance of winning each of his service games. What are the chances that Federer will win this final set?

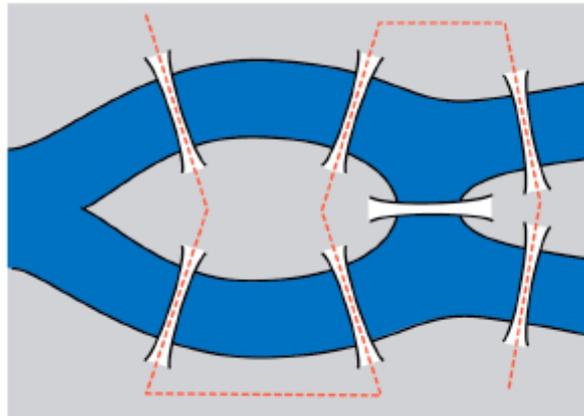
Answer: Call P the probability that Federer wins the set. One way he can do that is by winning the first two games (one of which he serves), giving a probability $2/3 \times 1/4 = 1/6$. Otherwise, Federer has to hope that it gets to 7 games all; this happens if both players win their serve, or both players lose their serve, a probability of $2/3 \times 3/4 + 1/3 \times 1/4 = 7/12$. Once it is 7 games all, Federer is back to the same probability P of winning. So,

$$P = 1/6 + 7/12 P.$$

Solving for P , the probability of Federer winning is $P = 2/5$.

Similar to the previous problem, a more obvious but much longer way to take this problem is as an infinite geometric sum.

44. Here is one of the most famous puzzles of all time. The town of Mathsberg has seven bridges. Can you walk through the town so that you cross each bridge exactly once?



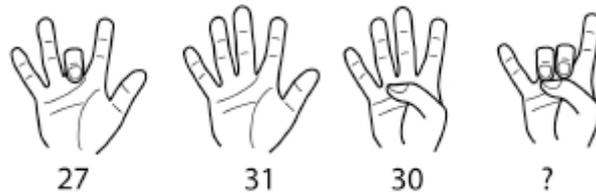
Answer: The trip is impossible. This is the famous Bridges of Königsberg Problem, solved by Leonhard Euler in 1736. Euler did not only solve this particular bridge problem: he determined exactly when such trips are possible or impossible. See <http://www.qedcat.com/mathsnacks/index.html> and http://en.wikipedia.org/wiki/Seven_Bridges_of_Königsberg.

45. You decide to walk up the 10 steps to your apartment differently each day, by mixing 1-step and 2-step jumps. How many different ways can you do this?

Answer: These answer comes from the famous Fibonacci numbers: the number of ways for 10 steps is the 11th Fibonacci number, which is 89.

On your last move up you can either move one step or two steps. This means that the number of ways to walk up 10 stairs equals the number of ways to walk up the first 8 steps (and then do a 2-step), plus the number of ways to walk up the first 9 steps (and then do a 1-step). So, if we let W_n stands for the number of ways to walk up n stairs, then $W_{10} = W_9 + W_8$. And in general $W_{n+2} = W_{n+1} + W_n$. This is the defining formula for the Fibonacci numbers: it's only a question of where we start. Now, there is only one way to walk up one stair and two ways to walk up two stairs. This means that $W_1 = 1$ and $W_2 = 2$. Therefore, $W_3 = W_2 + W_1 = 3$, $W_4 = W_3 + W_2 = 5$, $W_5 = W_4 + W_3 = 8$, and so on. Eventually, we find that $W_{10} = W_9 + W_8 = 89$, giving that there are 89 ways to walk up 10 stairs. (89 is the 11th Fibonacci, because the Fibonacci numbers begin with 1 and 1 rather than 1 and 2).

46.



Answer: The key is Binary numbers. Every hand position corresponds to a five digit binary number: a stretched finger is a 1 and a bent finger is a 0. So

11011 (binary) \Rightarrow 27 (decimal number), 11111 \Rightarrow 31, 11110 \Rightarrow 30. And the answer to our puzzle is 10010 \Rightarrow 18.

47. A class of 30 children are asked if they did their maths homework last night. Each student flips two coins. If both coins come up heads then they lie, and otherwise they tell the truth. 20 students say they did their homework. How many students (roughly) actually did their homework?

Answer: Let D (iligent) and S (lack) be the number of students who did and didn't do their homework. The chances of two heads is $1/4$. So, approximately $1/4$ of the slackers and $3/4$ of the diligent students answered "Yes" to doing the homework. This means

$$\frac{1}{4} S + \frac{3}{4} D = 20.$$

But we also know

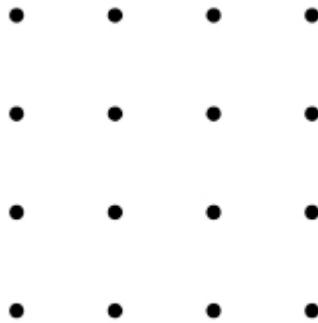
$$S + D = 30.$$

Solving these equations, we find that approximately $D = 25$ students did their homework.

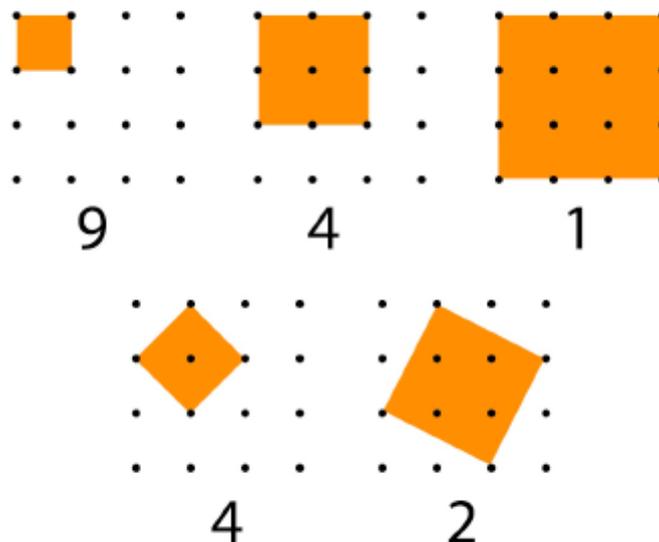
48. A student asks his professor: "What are the ages of your three children?" The professor answers: "If you multiply their ages you get 36, and if you add them you get my house number." "I know your house number, but that's not enough information!" says the student. To that the professor answered: "True. The oldest lives upstairs." What are the ages of the three children?

Answer: This is another puzzle from the movie *Fermat's Room*. The product of the three children's ages being 36 means there are the following possibilities: (1, 1, 36), (1, 2, 18), (1, 3, 12), (1, 4, 9), (1, 6, 6), (2, 2, 9), (2, 3, 6), and (3, 3, 4). Since the student knows the house number, and since this is not enough information, two of the triples must sum to the house number. Checking the triples, the only possibility is that the house number is 13, with the ages being (1, 6, 6) or (2, 2, 9). The final clue is that there is only one oldest child, and so the ages must be (2, 2, 9).

49. How many squares can you make using any four points from the grid below as corners? How many equilateral triangles?



Answer: There are 20 squares:



There are no equilateral triangles (pretty obvious once you give it a try).

50. Five students go to the teacher to get back their tests. But the teacher is so confused that each of the students receives someone else's test. How many ways are there to do this?

Answer: One way to have each of the students collect the wrong homework is to imagine them all in a circle with their correct homework, and then have each of them pass their homework to the left. There are $4 \times 3 \times 2 \times 1 = 24$ ways of arranging 5 students in a circle (the position of the first student in the circle doesn't matter), and so this gives 24 ways of distributing the homework so that none is with the correct owner.

Is there another way? Yes, but any such way will similarly involve circles. Imagine Jeff has Sally's homework, and Sally has Bill's, and Bill has June's, and so on. Eventually, we must cycle back to Jeff, and we have a circle of students.

The only question is how many circles do we have? The only choices are one circle of 5 students, or a circle of 3 students together with a "circle" of 2 students. (A circle of 4 students together with a "circle" of 1 student is impossible: a circle of 1 means that student in fact has their own homework).

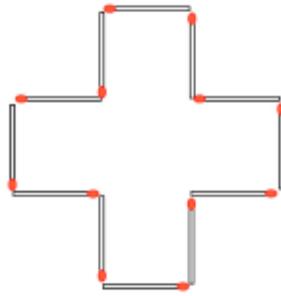
There are 10 ways to split the students into a group of 3 and a group of 2, and then there are 2 ways of organizing the group of 3 students into their circle. This gives $10 \times 2 = 20$ more ways to distribute the homework, giving a total of 44 ways.

Distributions such as these, where every piece of homework is in the wrong place, are called *derangements*. See <http://en.wikipedia.org/wiki/Derangement>. If we have N students then the number of possible derangements is

$$N! \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^N \frac{1}{N!} \right)$$

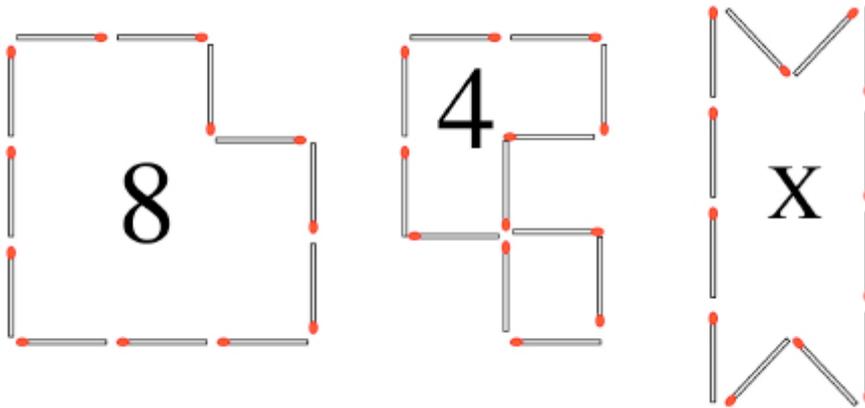
Here $N!$ stands for " N factorial": $N! = 1 \times 2 \times 3 \times \dots \times N$. Also, the total number of ways of distributing the homework is $N!$, This can be used to show that if you have a large number of students the probability that a random distribution turns out to be a derangement is about $1/e$, where e is Euler's number (see Problem 54).

51.



The matches make a cross of area 5. Is it possible to rearrange the matches into a shape of area exactly 8? What about an area of 4? What about an area of π ?

Answer: All the requested areas are possible. In particular, consider the last diagram. By squeezing together the two vertical rows of matches, it is possible to produce any area X between 0 and 8.



52. A watch has an hour hand and a minute hand. What is the first time after midnight when the two hands will be pointing in exactly opposite directions? A second watch has an hour hand, a minute hand and a second hand. Is there a time of day when the three hands are symmetrically placed, with 120 degrees between each pair of hands?

Answer: In M minutes, the hour hand moves M notches and the minute hand moves $M/12$ notches. We want the minute hand to be 30 notches ahead. So, we solve

$$M - M/12 = 30.$$

This gives $M = 360/11$, corresponding to $32 \frac{8}{11}$ minutes past midnight.

It is impossible for the three hands to be equally spaced. The easiest way to see this is to identify all the different times at which the hour and minute hands are 20 minutes apart, and then check that at those times the second hand is not in the right position.

54. Suppose we invest \$1 at 100% annual interest. Then at the end of the year we will have \$2. However, if instead we receive 50% interest after each six months, then the compounding means that at the end of the year we will have \$2.25. What if we receive 25% interest after each three months? As the time increments get smaller and smaller, how does the final amount change? Does the amount grow as large as we want, or do the amounts level off?

Answer: The amounts level off to 2.71828 ..., known as Euler's number e . In fact, the deep and very difficult problem is to show that the amounts level off: Euler's number is actually *defined* by our interest rate puzzle, to be whatever these amounts level off to.

One approach to seeing that the amounts level off is as follows. If we divide the year into N periods then the total amount we have at the end of the year is

$$\left(1 + \frac{1}{N}\right)^N .$$

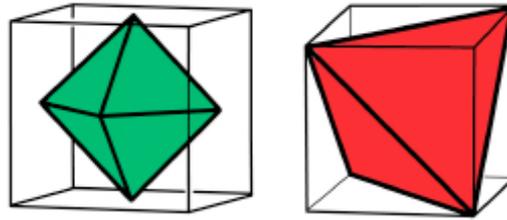
Now suppose we double the number of periods to $2N$. Then the amount after 1 year would be.

$$\left(1 + \frac{1}{2N}\right)^{2N} .$$

The second number is larger (more interest compounded on the interest), but not much larger. And if we double the number of periods again, the newly added amount is smaller still.

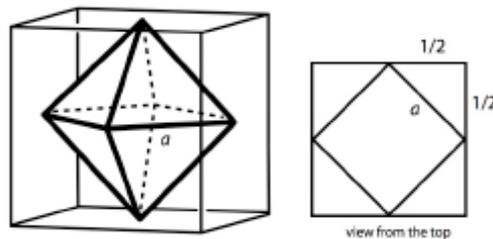
We can now imagine summing all these increases as we keep doubling the number of periods. With some clever fiddling, one can show that the sum of all these increases does indeed level off (by a comparison to a geometric sum).

55. The cubes have side length 1. What are the volumes of the two bodies inscribed in the cubes? The six corners of the octahedron are at the midpoints of the faces of the cube. The four corners of the tetrahedron are at the corners of the cube.



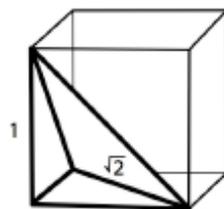
Answer: The green shape is called an octahedron and the red one a tetrahedron. These are two of the famous five regular polyhedra (the cube is another one).

The octahedron inscribed in the cube is made up of two identical square pyramids. The volume of a square pyramid is $\frac{1}{3} a^2 h$, where h is its height, and a is the length of a side of its square base. So, to calculate the volume of the inscribed octahedron, we need to figure out what a and h of our square pyramids are. Clearly, $h = 1/2$. If you look at the cube from the top, you see that, by Pythagoras's Theorem $a = 1/\sqrt{2}$.



By combining all this information, we find that the volume of the inscribed octahedron is two times the volume of one of the pyramids, that is, $2/3 h a^2 = 1/6$. This means that the volume of the inscribed octahedron is one sixth of the volume of the cube.

To figure out the volume of the inscribed tetrahedron, note that the tetrahedron arises from the cube by cutting off four identical corners. We have highlighted one of these corners in the following diagram.



So, the volume of the tetrahedron equals the volume of the cube minus 4 times the volume of one of these corners. Moving them together, the four corners combine into a square pyramid with $h = 1$ and $a = \sqrt{2}$. Therefore the volume of the tetrahedron is $1 - 1/3 h a^2 = 1/3$. This means that the volume of the inscribed tetrahedron is just one third of the volume of the cube.

56. 666, 36, 1316, 11131116, 31133116, ???

What's the next number?

Answer: This is based upon the Look and Say Sequence, due to the mathematician John Conway. See <http://mathworld.wolfram.com/LookandSaySequence.html>.

DESCRIBE the first number in words "three six(es)" \Rightarrow 36 (which is the second number). Describe this second number 36 to get the third number: "one three, one six" \Rightarrow 1316. Next is "one one, one three, one one, one six" \Rightarrow 11131116, and so on. So, the number we are looking for is 1321232116.

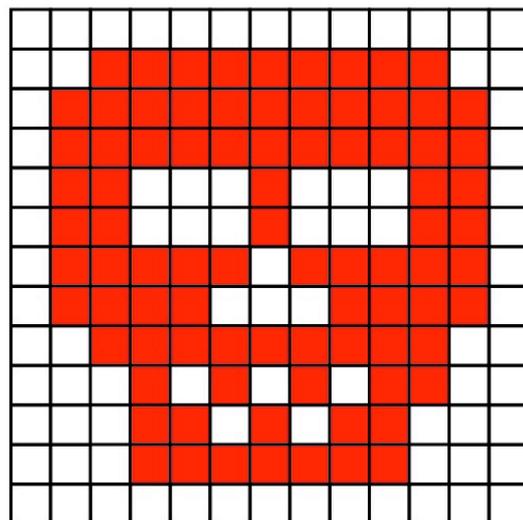
57. Decipher the following message:

00000000000000111111110001111111111001111111110011000100
 01100110001000110011110111110011110001111000111111110000010
 101010000001101011000000111111100000000000000000.

Answer: This is another puzzle from the movie *Fermat's Room*. There are $169 = 13 \times 13$ digits in the sequence, and this is the only non-boring way to write 169 as the product of two positive integers. So, the key is to writing the sequence in 13 rows of 13 digits and interpreting 0's as one colour and 1's as a second colour. The result is a striking picture of a skull, which is the required answer. In *Fermat's Room*, one of the characters uses the front and back of Mahjong pieces.

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0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 1 1 1 1 1 1 1 1 1 1 0 0
0 1 1 1 1 1 1 1 1 1 1 1 0
0 1 1 1 1 1 1 1 1 1 1 1 0
0 1 1 0 0 0 1 0 0 0 1 1 0
0 1 1 0 0 0 1 0 0 0 1 1 0
0 1 1 1 1 1 0 1 1 1 1 1 0
0 1 1 1 1 0 0 0 1 1 1 1 0
0 0 1 1 1 1 1 1 1 1 1 0 0
0 0 0 1 0 1 0 1 0 1 0 0 0
0 0 0 1 1 0 1 0 1 1 0 0 0
0 0 0 1 1 1 1 1 1 1 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0
    
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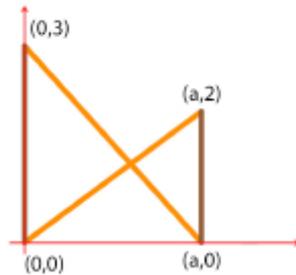


This puzzle is probably inspired by the famous Arecibo Message. This message is a string of 0's and 1's that was beamed to outer space, in the hope that some alien civilization would intercept it and decode it into a 23×73 pixel picture.

58. John places two poles in the ground, one of them 3 meters high and one 2 meters high. He then ties ropes from the top of each pole to the base of the other pole. How high off the ground do the ropes cross? Let's say the two poles are a distance a apart. Then we can interpret the ropes as lines in the xy -plane:



Answer:



The line through the points $(0,0)$ and $(a,2)$ is given by the equation $y = (2/a)x$.

Similarly, the line through $(0,3)$ and $(a,0)$ has the equation $y = -(3/a)x + 3$.

Eliminating x from these two equations gives $y = 1 \frac{1}{5}$ (meters), the height at which the two ropes/lines intersect. Note that this height is independent of the distance between the two poles.

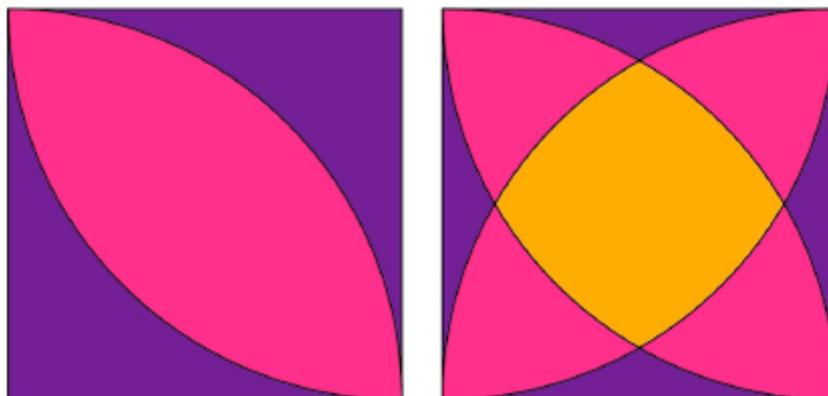
59. Imagine a rubber rope one meter long. A worm starts at one end and travels along the rope at 1 centimeter each second. At the end of each second, the rope is stretched, so that it is one meter longer than before. The worm is carried along with the stretching. Does the worm ever reach the end of the rope?

Answer: Let's consider what fraction of the rope the worm has travelled after each second. After 1 second, he travels one centimetre of the meter rope, amounting to $1/100$ of the trip. When the rope is stretched an extra meter, the worm goes along as well, and so has still travelled $1/100$ of the total length. For the next second the worm travels 1 centimeter, which is $1/200$ of the total length of 2 meters. So, he has now travelled $1/100 + 1/200$ of the total length. After N seconds, the fraction of the total length he has travelled is

$$\frac{1}{100} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \right)$$

As N grows, this sum grows without bound, as large as we want (ask us why). So, eventually it must grow to beyond 1, and the worm has completed his journey. How long does it take? The age of the Universe, many times over.

60. The squares have area 1. What are the areas of the pink lens in the first square, and the bulging yellow square inside the second square?



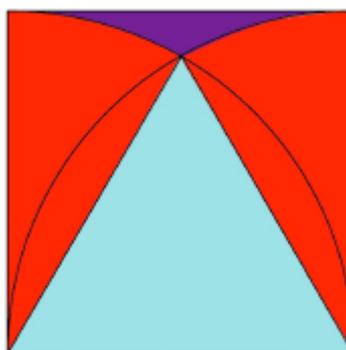
a) Two quarter circles cover the square once, except the pink lens is covered an extra time. So

$$\text{lens} = 2 \times \text{quarter-circle} - \text{square} = 2 \times \pi/4 - 1 = \pi/2 - 1.$$

b) Similarly, the area of the yellow bulging square is equal to

$$\text{bulging square} = 2 \times \text{lens} + 4 \times \text{purple sliver} - \text{unit square}$$

We just calculated the area of the lens, which leaves us with figuring out the area of one of these purple slivers.



Note the blue triangle is equilateral, and the red sectors each have an angle of 30° , and so are each twelfth of a circle. So

$$\text{purple sliver} = \text{square} - \text{equilateral triangle} - 2 \times \text{twelfth-circles} = 1 - \sqrt{3}/2 - \pi/6.$$

Combining with the area of the lens, we finally find that the area of the yellow bulging square is $1 - \sqrt{3} + \pi/3$.

Many regions can be approached with these add-and-subtract ideas. For a beautiful and related calculation of the area of a dodecagon, see *Kürschak's Tile*: <http://agutie.homestead.com/FILES/kurschak1.html>.