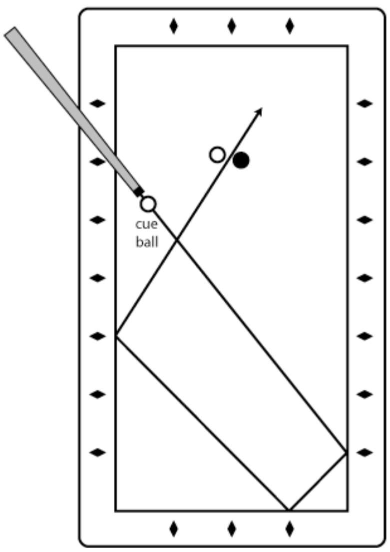
Notes on the Diamond System for playing Three-Cushion Billiards (Excerpt from *Nitpicking in Mathmagic Land* by Burkard Polster and Marty Ross)

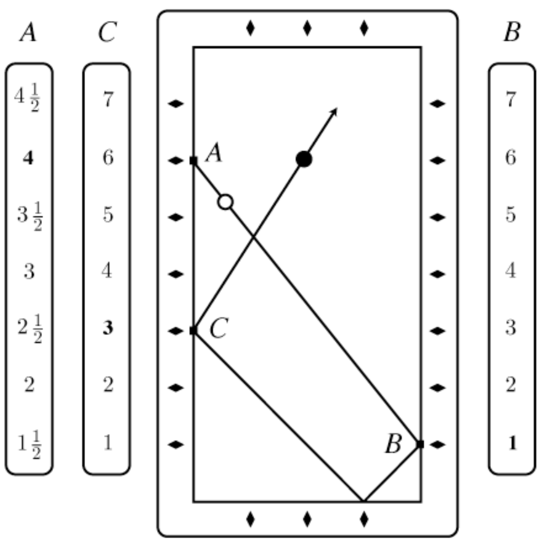
SPIRIT: (Three-cushion) billiards, a mathematical game played on a field of two perfect squares, using three perfect spheres, and a lot of diamonds. In other words billiards….



The diamonds divide the long side of the table into eight equal intervals, and the short side into four.

The game is played with three balls. The player's aim is to make the cue ball hit both the other balls. In addition, the cue ball must contact at least three cushions (or rails) before it hits the last of the three balls.

We'll explain how an expert player uses the diamonds to plan a typical shot, such as the one shown in the diagram. To simplify the discussion, let's use just two balls. In all that follows, the white ball will be the cue ball and the black ball will be the one that the player wants to hit.

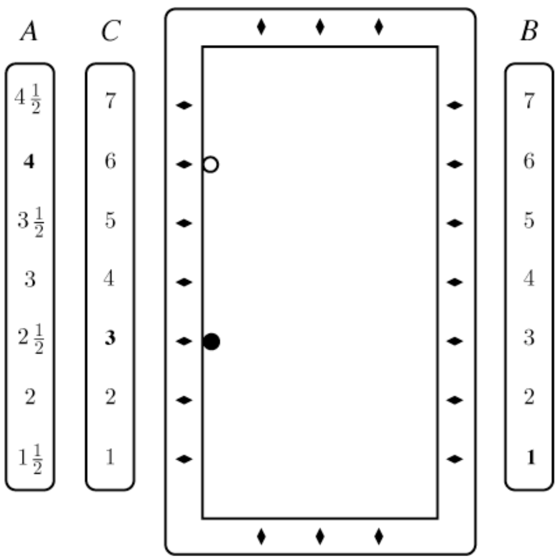


The example on the left is one of the shots discussed in the movie. Here *A* marks the imaginary point on the left rail, where the cue ball is coming from and *B* and *C* mark the points at which the cue ball bounces off the two long rails.

The diamonds on the left side of the table are labelled with numbers in two different ways, as shown in the two columns on the left of the diagram 1 ½, 2, 2 ½, …, 4 ½, and 1, 2, 3, …, 7. The diamonds on the right side of the table are labelled 1, 2, 3, …, 7, as shown on the right.

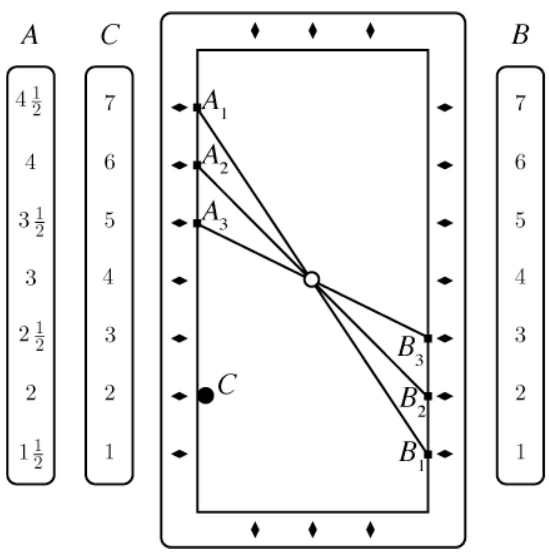
We associate numbers with the points *A, B*, and *C*, using the different labellings, as indicated at the top of the columns. Then, the Magic Rule at the core of the *diamond system* for planing shots is $A - B = C$. In this example, $A = 4$, $B = 1$, and $C = 3$. And, indeed, $4 - 1 = 3$.

It is important to realize that the diamond system is just a rule of thumb, that in reality A – B is only approximately equal to C, and that the accuracy of this approximation depends in a complicated manner on the relative positions of the balls and the manner in which the cueball is struck. However, in many typical situations the rule A – B = C is accurate enough for planing shots on a real pool table. Just as in the movie, for now we will make things easy for ourselves by pretending that A – B = C exactly, and ignore some of the subtleties in its real application.

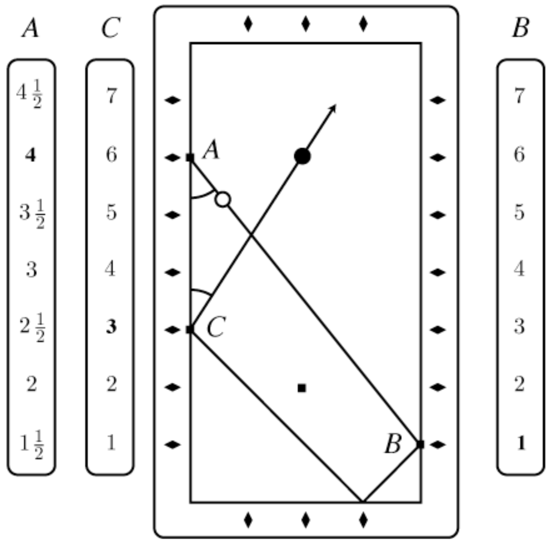


We now describe how a player uses the diamond system to plan a shot.

In the simplest case, both the white ball and the black ball are on the left rail. This means that the white ball coincides with the point A and the black ball with C. In the example on the left, we see $A = 4$ and $C = 3$. Therefore, $B = A - C = 1$. This means that if we aim the white ball at the first diamond on the bottom right, we can be sure to hit the black ball (and the path of the white ball will be exactly the same as in the previous diagram).



Next, the white ball could be somewhere in the middle of the table and the black ball on the rail. To find the value of A and B, the player swivels an imaginary line through the white ball and keeps subtracting the right end point from the left until the resulting subtraction gives the value C. In our example, C = 2 and the player's guesses are
$A_1 - B_1 = 4\ ½ - 1 = 3\ ½$,
$A_2 - B_2 = 4 - 2 = 2$, and
$A_3 - B_3 = 3\ ½ - 3$.
Therefore, the second subtraction is the one that works, and the Magic Rule says we should aim at the second diamond at the bottom right to make our shot.



Finally, in the case that the black ball is in the middle of the table, things are tricky and there is no foolproof way to determine the correct values for *A*, *B*, and *C* that relies solely on the diamond system. However, an experienced player (such as the one in the movie) with a good intuitive feel for how balls bounce off the rails, will have a good idea of how the angle at *C* depends on the angle at *A*. *Here it is important to realize that it is usually not the case that the angle at A is equal to the angle at C, as a non-expert might suspect; in practice, the relationship is more complicated than that. However, since the angle relationship does not depend upon the position of the white ball, it is easy to develop a good feeling for this relationship*. The player can use the swivel technique together with his knowledge of the angle relationship to determine the correct values as follows: Every swivel position corresponds to an *A* and a *B* and an angle at *A*, the player calculates *C* via the Magic Rule, and guesses the angle at *C* based on the angle at *A*. Once the position of *C* and the angle at *C* are in line with the black ball, he knows that the values are correct and can make his shot.

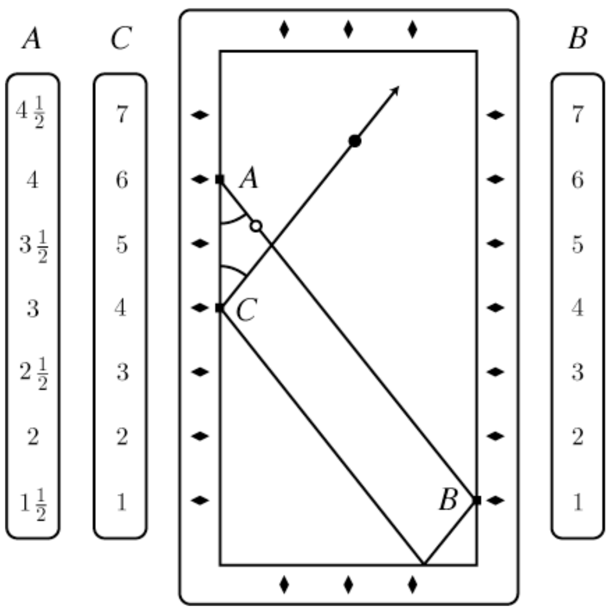
The way the spirit explains the diamond system will confuse anybody unfamiliar with billiards. Both of the examples that get discussed are of the complicated type. However, listening to the explanation, we get the impression that it is easy to determine the points *C* and *A* without using the Magic Rule, and that the player only needs to calculate *B* with the Magic Rule to guarantee a successful shot. Of course, this does not make any sense, because if the white ball is in the middle of the table, then its position together with that of *A* determines the shot, and so *B* and *C* would already be determined, without using the rule at all! Here is the relevant part of the Spirit's explanation, which refers to the situation shown in the previous diagram.

Spirit: … he uses the diamond markings on the rail as a mathematical guide. First he figures the natural angle (= C) for hitting the object balls (= the black ball). And then he finds that his cue ball (= the white ball) must bounce of the no. 3 diamond. Next he gets ready for the shot, and he needs a number for his cue position (= A). This calls for a different set of numbers (= different labelling). You see, the cue position (= A) is 4. Now, a simple subtraction 4 – 3 = 1. So, if he shoots for the first diamond (= B), he should make it.

What is even funnier is that when Donald is trying to use the diamond system, he is in effect attempting to use the swivel technique and gets scolded by the spirit for doing so.

There are a few other issues that get glossed over. First, and we already mentioned this, the Magic Rule is really just a rule of thumb and not as mathematically infallible as is suggested by the spirit. Without a lot of experience at the table to adjust the suggestions of the diamond system, a player cannot hope to be successful. Second, for the diamond system to work, a player is supposed to provide the cue ball with some sidespin so that the rebound angles are predictably larger than the approach angles (this is the case in all our diagrams so far). Third, the diamond system, as described in the movie and by us, is only part of the official diamond system which also covers the case in which *A* has to be chosen somewhere along the top rail. Fourth, according to many modern books on billiards, the diamond system works best if you aim directly at the diamonds and not at the rail positions adjacent to the diamonds, as described in the movie and by us. Fifth, taking into account that balls have nonzero radius would change all our diagrams slightly; however, this last point is more of an issue with our set-up than that in the movie, as there the paths of balls are drawn very thick anyway.

Still, to most laymen it would appear strange that we should have to resort to a rule of thumb at all in a "mathematical" game such as billiards? Shouldn't we be able to come up with some *precise* rule? The problem with formulating such a rule is that when a ball bounces off a rail, then, contrary to most laymen's intuition, the rebound angle is not the same as the approach angle.

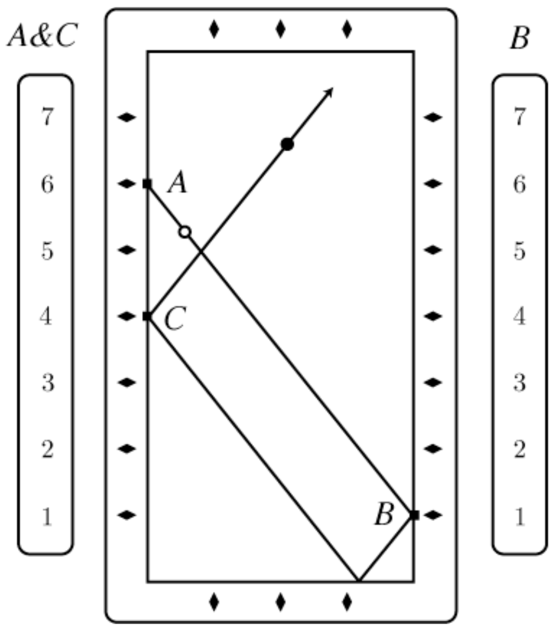


This is also the reason why it is very hard to pin down exactly why and to what extent the Magic Rule works in practice, and we will not even attempt to do so. On the other hand, it is very insightful, and a lot of fun, to figure out what the diamond system would look like in an ideal game of mathematical billiards in which rebound and approach angles are always the same and balls are points (never mind that we cannot play this kind of game on a real billiard table). Using the same labellings as before, we get an *exact* Magic Rule

$$A - B = C/2 + 1.$$

For example, on the left, we have $A = 4$, $B = 1$, $C = 4$, and, indeed, $4 - 1 = 4/2 + 1 = 2$.

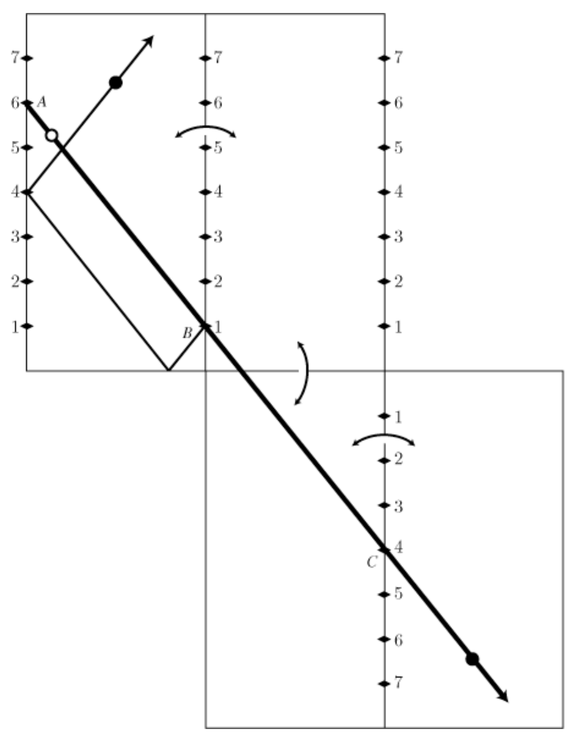
We can use exactly the same strategies as the ones outlined above to plan shots in mathematical billiards. Furthermore, even the situation in which both balls are off the rails becomes easy to control because the angles at *C* and *A* are always equal (as you can easily check for yourself).

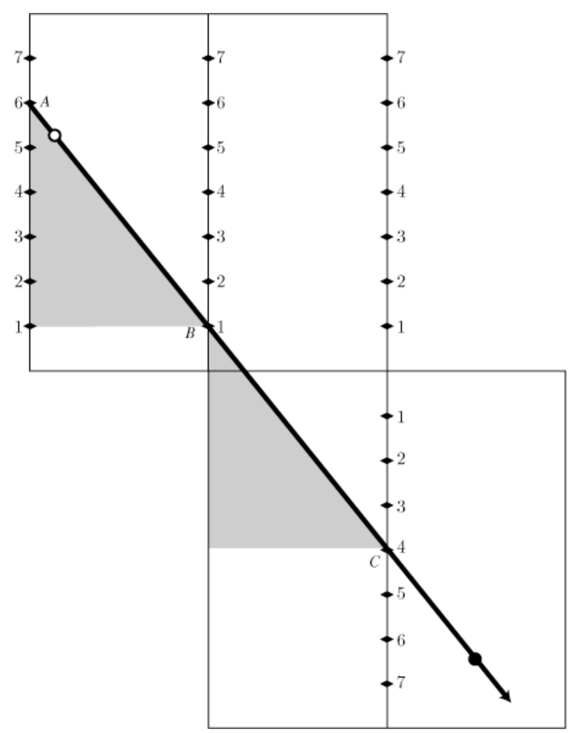


As one further simplification, we can get rid of the labelling of the diamonds on the far left, and stick with the "natural" labellings of the diamonds for A, B, and C. If we do this, then the *precise* Magic Rule looks takes the form

$$A - 2B = C.$$

It is very easy to see why this last rule should be true. Just unfold the path of the white ball into a straight line, as shown in the following diagram on the left. Then it is clear that the two grey right-angled triangles highlighted in the diagram on the right are congruent. In particular, this means that the lengths of the two vertical sides of these triangles are equal. Hence $A - B = C + B$, or $A - 2B = C$. Q.E.D.





Playing Billiards With Jugs

by Burkard Polster

In the movie *Die Hard: With a Vengeance* John (Bruce Willis) and Zeus (Samuel Jackson) are having a little problem with jugs.

58:00

SIMON (via a mobile phone): I trust you see the message. It (the bomb) has a proximity circuit. So please, don't run.
JOHN: Yeah, I got it. We're not gonna run. How do we turn this thing off?
SIMON: On the fountain, there should be 2 jugs, do you see them? A 5 gallon and a 3 gallon. Fill one of the jugs with exactly 4 gallons of water and place it on the scale and the timer will stop. You must be precise, one ounce more or less will result in detonation. If you're still alive in 5 minutes, we'll speak.
JOHN: Wait, wait a second. I don't get it. Do you get it?
ZEUS: No.
JOHN: Get the jugs. Obviously, we can't fill the 3-gallon jug with 4 gallons of water.
ZEUS: Obviously.
JOHN: All right. I know, here we go. We fill the 3-gallon jug exactly to the top, right?
ZEUS: Huhuhu.
JOHN: Okay, now. We pour that 3 gallons into the 5-gallon jug. Given us exactly 3 gallons in the 5-gallon jug, right?
ZEUS: Right, then what?
JOHN: All right. We take the 3-gallon jug and fill it at the 3rd of the way...
ZEUS: No, he said: Be precise. Exactly 4 gallons.
JOHN: Shit. Every cop within 50 miles is running his ass off and I'm out here playing kids games in the park.

……………

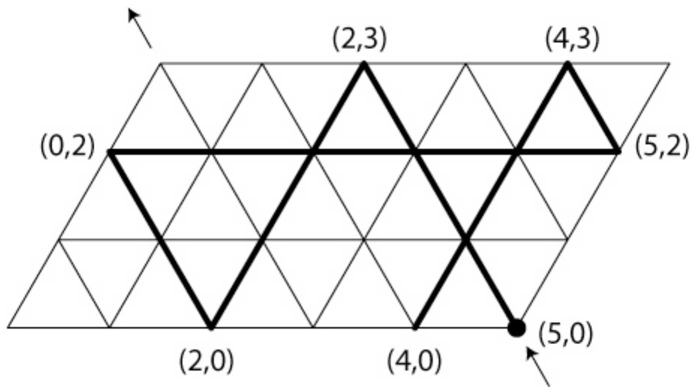
1:00:02

JOHN: Look, we can't take this off, it will detonate. Just wait, wait a second. I got it. I got it. Exactly 2 gallons in here, right?
ZEUS: Right.
JOHN: Leaving exactly 1 gallon of empty space, right?
ZEUS: Yeah.
JOHN: A full 5 gallon here, right?
ZEUS: Right.
JOHN: You pull 1 gallon out of 5 gallon in there, we have exactly 4 gallons in here.
ZEUS: Yes.
JOHN: Come on. Don't spill any. Good, good, good. Exactly 4 gallons.
ZEUS: You did it McClane.

There is a lovely way of tackling this kind of problem. It is based on playing billiards on a very special kind of table.

Billiards Solution of the Die Hard Problem

We use the special parallelogram-shaped billiards table shown in the following diagram.



The dimensions of the table are 5 by 3 units, the angle in the lower left corner is 60 degrees. We coordinatize the table in the natural way, such that the lower left corner is the origin and the upper right corner is the point with coordinates (5,3). We now shoot a ball located at the lower right corner as indicated. As the ball travels across the table, we note down the coordinates of the points on the upper and lower rail in the order that the ball bounces off them.

(5,0) (2,3) (2,0) (0,2) (5,2) (4,3) (4,0) etc.

This sequence of coordinates corresponds to a solution of our problem as follows:

(5,0) = fill the 5-gallon jug
(2,3) = fill the 3-gallon jug from the 5-gallon jug, leaving two gallons in the 5-gallon jug.
(2,0) = empty the 3-gallon jug
(0,2) = empty the two gallons in the 5-gallon jug into the 3-gallon jug.
(5,2) = fill the 5-gallon jug
etc.

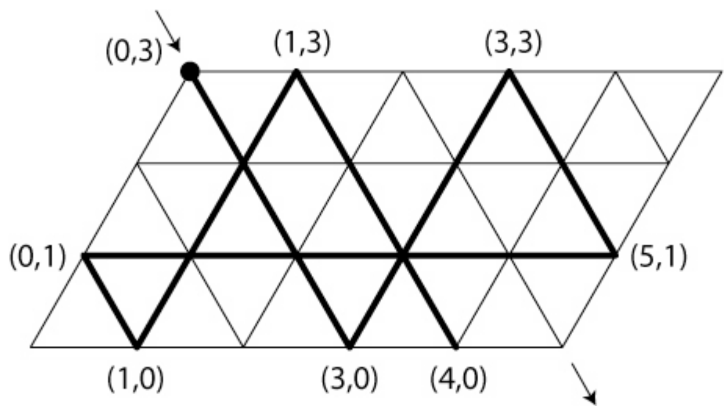
If we keep following the ball, we notice that it traverses every single one of the lines drawn on the table once each before it hits the upper left corner. In particular, this means that the path of the ball includes the following points with integer coordinates on the boundary of the table. In particular, it contains the points

(1,0), (2,0), (3,0), (4,0), (5,0) and
(0,3), (1,3), (2,3), (3,3), (4,3).

We conclude that even if Simon had asked John and Zeus to produce exactly 1, 2, 3, 4, 5, 6 (= 3+3), 7 (= 4+3) gallons, the puzzle would have had a solution (in the cases 6 and 7 they would have to put both jugs on the scales). Of course, 0 and 8 are also possible.

The Other Solution

Hidden in the diagram is a second solution to our problem: just put the ball in the upper left corner, as in the following diagram, and shoot as indicated by the arrow. What we are doing here is traversing the path that we considered before in reverse.



The new path corresponds to the solution

(0,3) (3,0) (3,3) (5,1) (0,1) (1,0) (1,3) (4,0)

Half-Turn Symmetry

The complete path from the bottom right to the top left gives rise to the following sequence:

(5,0) (2,3) (2,0) (0,2) (5,2) (4,3) (4,0) (1,3) (1,0) (0,1) (5,1) (3,3) (3,0) (0,3)

As we have already pointed out, the second complete path is just this first one traversed in reverse, that is, it corresponds to the following sequence, which is just the reverse of the above sequence:

(0,3) (3,0) (3,3) (5,1) (0,1) (1,0) (1,3) (4,0) (4,3) (5,2) (0,2) (2,0) (2,3) (5,0)

However, because of the half-turn symmetry of the table that turns the first path into the second path, you also get the second sequence by subtracting every element of the first sequence from (5,3): (5,3) – (5,0) = (0,3), (5,3) – (2,3) = (3,0), (5,3) – (2,0) = (3,3), etc.

Algorithm

We can summarize the first graphical solution as an algorithm, the key steps of which are the following:

1. If the 3-gallon jug is full, then empty it.
2. Pour as much as possible from the 5-gallon jug into the 3-gallon jug.
3. If there is water left in the 5-gallon jug, then return to step 1.

4. Fill the 5-gallon jug and return to step 2.

In particular, this means that the water always flows as follows:

fountain >> 5-gallon jug >> 3-gallon jug >> fountain

To get the algorithm corresponding to the second solution, in the above algorithm replace every 3 by a 5 and every 5 by a 3. Also, when you apply this new algorithm you will notice that the direction in which the water flows is reversed:

fountain >> 3-gallon jug >> 5-gallon jug >> fountain

Equation

Let's mark those coordinates in the first and second solutions in which jugs get filled from the fountain or are emptied into the fountain:

>	(5,0)			(0,3)	<
	(2,3)	>		(3,0)	
	(2,0)			(3,3)	<
	(0,2)		<	(5,1)	
>	(5,2)			(0,1)	
	(4,3)	>		(1,0)	
	(4,0)			(1,3)	<
				(4,0)	

On the left, we see that the 5-gallon jug gets filled twice and the 3-gallon jug gets emptied twice before we are left with four gallons. This corresponds to the equation

$$2\cdot 5 - 2\cdot 3 = 4$$

Similarly, on the right we have the 3-gallon jug being filled three times and the 5-gallon jug being emptied once. Hence,

$$3\cdot 3 - 1\cdot 5 = 4.$$

Note that adding $5\cdot 3 - 3\cdot 5 = 0$ to the left side of the first equation gives the second equation.

Surviving the General Case

So, what if tomorrow it's your turn, and Simon gives you a p-gallon jug and a q-gallon jug and then asks you to defuse a bomb by filling exactly r gallons in one of the jugs? Of course, here p, q, and r are supposed to be natural numbers as in the in the special case that we have been focusing on up to now ($p = 5$, $q = 3$, $r = 4$). Using a parallelogram-shaped table of dimensions p x q, and imitating what we did, we expect to be able to find solutions in many other cases. Let's figure out what exactly is possible and what is not.

To start with, it is clear that we may as well say our prayers if $p = q$ and r is not equal to the first two numbers.[1] So, let's assume that $p > q$ and also, because we want r gallons to fit into the larger of the two jugs and we want a bit of a challenge, that $p > r$.

It is clear from what we said before that a solution to such a problem, derived via our billiards approach, corresponds to an equation of the form

$$x \cdot p - y \cdot q = r,$$
or
$$x \cdot q - y \cdot p = r,$$

where x and y are *nonnegative integers*.

On the other hand:

1. It is well-known that the equation $X \cdot p + Y \cdot q = r$ has *integer* solutions X and Y if and only if r is a multiple of the greatest common divisor of p and q. This immediately implies that it is prayer time if r is not of this form.
2. It is clear that *if* the equation $X \cdot p + Y \cdot q = r$ has integer solutions X and Y, then one positive and the other non-negative. This means that the equation at least looks like an equation corresponding to a solution of our problem. So, for example, if X is positive, then the equation is of the form $x \cdot p - y \cdot q = r$, where x and y are *nonnegative integers*.
3. Finally, it is also not too hard to see that if $x \cdot p - y \cdot q = r$, where x and y are nonnegative integers, then by applying the algorithm that corresponds to the *first* path until we have filled the p-gallon jug x times and emptied the q-gallon jug y times, we will be left with exactly r gallons. Similarly, if $x \cdot q - y \cdot p = r$, where x and y are nonnegative integers, then by applying the algorithm that corresponds to the *second* path until we have filled the q-gallon jug x times and emptied the p-gallon jug y times, we will be left with exactly r gallons.

Summarizing these observations we come up with the following general result.

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[1] Actually, even in this special case things may not be completely hopeless, but you will have to use ideas that are different from those that we are discussing right now. For example, if both jugs are transparent, of the same shape and hold two gallons each, then you can very precisely measure one gallon by pouring water into the second jug until the water level in both jugs coincides. This means that you can survive the case in which $p = q = 2$ and $r = 1$ using this new trick.

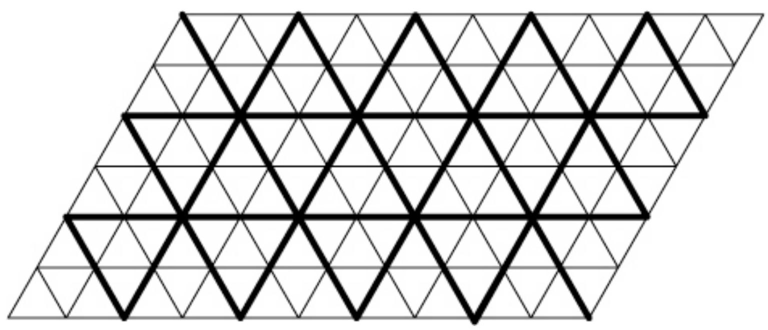
The General Jug Problem

Let p, q, and r be natural numbers with $p>q, r$. If Simon gives you a p-gallon jug and a q-gallon jug and then asks you to fill one of the jugs with r gallons of water, then our billiards approach will save your life iff r is a multiple of the greatest common divisor of p and q.

So, what if you are confronted with a situation in which our billiards method does not apply, that is, where r is not a multiple of the greatest common divisor of p and q? Is there perhaps another way of solving the puzzle. Well, we have not really specified what we are allowed and not allowed to do. Therefore, it is impossible to say what is possible. However, if we assume that we are only allowed to use the operations of filling and emptying the jugs that we have used so far combined in any conceivable way, then it is quite easy to see that we are still doomed.

The Greatest Common Divisor

Let's consider an example: $p = 10$, $q = 6$. The greatest common divisor of 10 and 6 is 2. This means that it is prayer time if r is an odd number. On the other hand, things are looking good if r it is even. Here is the corresponding billiards table and complete path.



As we have seen, exactly those points $(x,0)$ are contained in the path for which x is a multiple of the greatest common divisor of p and q. This translates into the following lovely graphical method for calculating the greatest common divisor of two numbers p and q:

Finding the Greatest Common Divisor

Draw the parallelogram-shaped billiards table of dimensions p x q and draw in the path of the ball. Then the greatest common divisor of p and q is the smallest positive x such that $(x,0)$ is contained in the path.

The Least Common Multiple

There is also a graphical billiards-based method to calculate the *least common multiple* of two natural numbers p and q. Of course, since

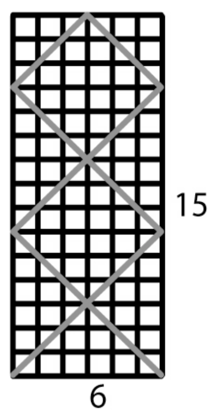
$$\text{least common multiple of } p \text{ and } q = p \cdot q \;/\; \text{greatest common divisor of } p \text{ and } q,$$

or

$$\text{lcm}(p,q)=p\cdot q/\text{gcd}(p,q),$$

any method for calculating the greatest common divisors, such as the billiards-based one in the previous section, can also be considered as a method for calculating the least common multiple. Here is a different billiards-based method:

1. Start by making up a rectangular billiards table that is p units wide and q units long.
2. Put a ball in the lower left corner.
3. Shoot the ball such that the angle between its initial trajectory and a horizontal is 45 degrees; see the following diagram (here $p = 6$ and $q = 15$).



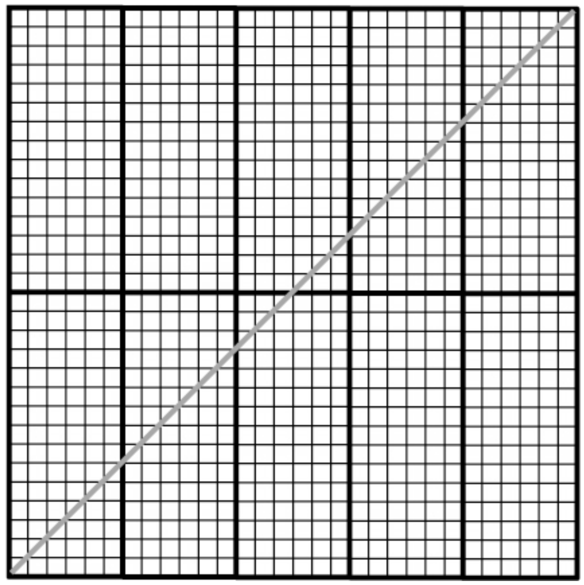
Finding the Least Common Multiple

The least common multiple of p and q is p times the number of times the ball moves between the two vertical cushions, until it hits a corner for the first time.

In our example, the ball moves five times back and forth between the vertical cushions. Hence, the least common multiple of 6 and 15 is $5 \cdot 6 = 30$. Alternatively, the least common multiple is also q times the number of times the ball moves between the horizontal cushions. In our example, the ball moves two times between the horizontal cushions, and, indeed, $2 \cdot 15 = 30$.

Sketch of a proof that this billiard ball trick works: If we tile the upper right quarter of the *xy*-plane with copies of our billiard table as in the following diagram, then we can "unfold'" the path that the billiard ball takes onto the diagonal of the *xy*-plane.

Let's move along this diagonal, starting at the lower left corner. Since we are moving on the diagonal, the *x*- and *y*-coordinates of our position are always equal. This means that every time we hit a corner of one of the tiles we know that



x-coordinate
= p times the number of tiles to the left of us
= q times the number of tiles below us
= y-coordinate

This implies that this number is a common multiple of p and q. Since we get all common multiples in this way, the least common multiple corresponds to the first time that we come across a corner of a tile on our journey along the diagonal. In turn, this insight translates back into our billiards-ball trick in a straightforward manner. Q.E.D.