AMSI 2013: MEASURE THEORY
Handout 7
Approximations of Sets and Functions

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INTRODUCTION

We now consider μ a Borel measure on a topological space X. We are interested in whether measurable functions on X can be approximated by continuous functions. Since measurable functions can be naturally expressed as sums of characteristic functions, this leads naturally to the question of approximating arbitrary subsets of X with open and closed sets. Some approximation results hold for arbitrary Borel measures, but we shall see that the strongest results hold for what are called *Radon measures*. Our main concern is the case where X is a metric space, but we will also consider the topological setting. We shall also give some special attention to the case where $\mu = \mathscr{L}^m$ is Lebesgue measure on some $X \subseteq \mathbb{R}^m$.

APPROXIMATION OF SETS

We begin with an interesting and non-obvious manner of capturing the collection $\mathcal{B}$ of Borel subsets of X.

LEMMA 32: Suppose X is a topological space, and suppose $\mathcal{F} \subseteq \wp(X)$ is a collection of subsets of X such that:

(i) $\mathcal{F}$ contains all closed and all open subsets of X;

(ii) $\mathcal{F}$ is closed under countable intersections;

(iii) $\mathcal{F}$ is closed under countable unions.

Then $\mathcal{F} \supseteq \mathcal{B}$. That is, $\mathcal{F}$ contains all Borel subsets of X.

REMARK: If X is a metric space then any closed set C can be written as a countable intersection of open sets:

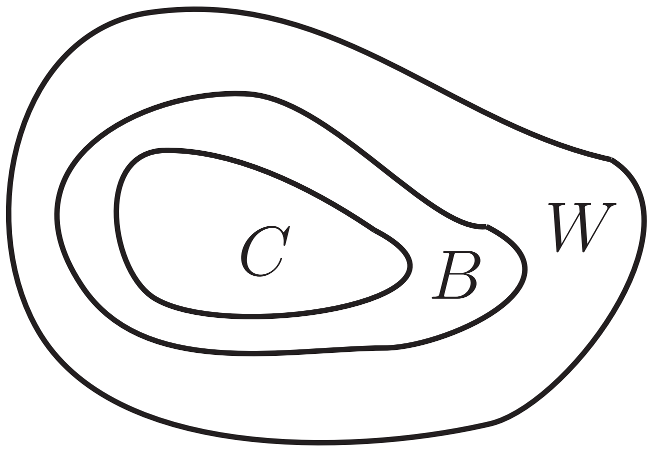
$$C = \bigcap_{j=1}^{\infty} \left\{ x : \operatorname{dist}(x, C) < \tfrac{1}{j} \right\} .$$

Similarly, or by De Morgan, every open subset of X can be written as the countable union of closed sets. Consequently, in applying Lemma 32 in a metric space, one need only show that $\mathcal{F}$ contains *either* the closed sets *or* the open sets.

THEOREM 33: Suppose μ is a Borel measure on a metric space X and that $\mu(X) < \infty$. Then for any Borel set $B \subseteq X$, and for any $\epsilon > 0$, we can find a closed set C and an open set W in X, with $C \subseteq B \subseteq W$ and with

$$(\clubsuit) \qquad \mu(B \sim C) < \epsilon ,$$

$$(\heartsuit) \qquad \mu(W \sim B) < \epsilon .$$



REMARKS:

(a) The Theorem is not generally true for Borel measures on general topological spaces. For example, let $X = \{a, b\}$, and consider the topology $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}\}$ on X. So the closed sets are exactly $\emptyset, \{b\}$ and $\{a, b\}$, and so all subsets of X are Borel; and, counting measure μ^0 is Borel on $(X, \mathcal{T})$, since all sets are μ^0-measurable. But the open set $\{a\}$ cannot be approximated from the inside by a closed set, and the closed set $\{b\}$ cannot be approximated from the outside by an open set.

(b) The Theorem is also not generally true if $\mu(X) = \infty$, even if μ is σ-finite. For example, let $X = \mathbb{R}$ and let $\mu = \sum \mu_j$, where μ_j is the delta measure at $\frac{1}{j}$ for $j \in \mathbb{N}$. Then $(0, \infty)$ cannot be approximated from the inside by closed sets, and the singleton set $\{0\}$ cannot be approximated from the outside by open sets. However, as the next remarks indicate, the Theorem will hold more generally, under suitable finiteness hypotheses, noting in particular the relevance of Lemma 9.

(c) If $\mu(X) = \infty$ but $\mu(B) < \infty$, we can obtain closed $C \subseteq B$ satisfying (♣) by applying Theorem 33 to the measure $\mu \llcorner B$. The result continues to hold if B is *locally finite*: if $\mu(\widehat{B}) < \infty$ for every bounded $\widehat{B} \subseteq B$. To see this, fix $a \in X$, and for $j \in \mathbb{N}$ let $B_j = B \cap \{x \in X : j-1 \leqslant d(x,a) < j\}$. B_j is bounded and Borel, and so by assumption we can find closed $C_j \subseteq B_j$ with $\mu(B_j \sim C_j) < \frac{\epsilon}{2^j}$. It is then easy to see that $C = \cup C_j$ is closed and satisfies (♣).

(d) Suppose $B \subset \displaystyle\bigcup_{j=1}^{\infty} V_j$ with each V_j open and $\mu(V_j) < \infty$. Then, applying (♡) to each $\mu \llcorner V_j$, we can obtain open $W_j \supseteq B \cap V_j$ with $\mu\left((W_j \cap V_j) \sim (B \cap V_j)\right) < \frac{\epsilon}{2^j}$. Since each V_j is open, so is $W = \cup(W_j \cap V_j) \supseteq B$, and clearly W satisfies (♡).

(e) It follows from (c) and (d) that (♣) and (♡) hold for Lebesgue measure for any Borel set $B \subseteq \mathbb{R}^m$.

PROOF OF THEOREM 33: The hard work is showing the existence of closed sets satisfying (♣). For a given B, we then obtain an open set W satisfying (♡) by choosing a closed $C \subseteq {\sim}B$ with $\mu(({\sim}B){\sim}C) < \epsilon$; since $({\sim}B){\sim}C = {\sim}C \cap {\sim}B$, we can then set $W = {\sim}C$.

To show the existence of the desired closed sets C, consider

$$\mathcal{F} = \{A \subseteq X : \text{For any } \epsilon > 0 \text{ there is a closed } C \subseteq A \text{ with } \mu(A \sim C) < \epsilon\}\,.$$

We shall apply Lemma 32 to show $\mathcal{F} \supseteq \mathcal{B}$. To this end, first note that obviously all closed sets are in $\mathcal{F}$; and, since X is a metric space, the Remark following Lemma 31 shows that we needn't worry about open sets.

We now have to show $\mathcal{F}$ is closed under countable intersections and countable unions. So, suppose $\{A_j\}_{j=1}^{\infty} \subseteq \mathcal{F}$, fix $\epsilon > 0$, and for each j let $C_j \subseteq A_j$ be a closed set with

$$\mu(A_j \sim C_j) < \frac{\epsilon}{2^j}\,.$$

Note that

$$\left(\bigcap_{j=1}^{\infty} A_j\right) \sim \left(\bigcap_{j=1}^{\infty} C_j\right) \subseteq \bigcup_{j=1}^{\infty} (A_j \sim C_j)\,.$$

(If x is in the set on the LHS, then x is in all of the A_j and $x \notin C_j$ for at least one j: then $x \in A_j \sim C_j$ for that particular j, and thus x is in the RHS union). So, by monotonicity and subadditivity,

$$\mu\left(\left(\bigcap_{j=1}^{\infty} A_j\right) \sim \left(\bigcap_{j=1}^{\infty} C_j\right)\right) \leqslant \sum_{j=1}^{\infty} \mu\left(A_j \sim C_j\right) < \epsilon\,.$$

Since $\bigcap_{j=1}^{\infty} C_j$ is closed, this shows that $\mathcal{F}$ is closed under countable intersections.

To show that $\mathcal{F}$ is closed under countable unions, we calculate similarly:

$$\left(\bigcup_{j=1}^{\infty} A_j\right) \sim \left(\bigcup_{j=1}^{\infty} C_j\right) \subseteq \bigcup_{j=1}^{\infty} (A_j \sim C_j)$$

$$\implies \quad \mu\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \sim \left(\bigcup_{j=1}^{\infty} C_j\right)\right) \leqslant \sum_{j=1}^{\infty} \mu\left(A_j \sim C_j\right) < \epsilon\,.$$

Now, $\bigcup_{j=1}^{\infty} C_j$ may not be closed, but we can approximate the infinite union from below by the finite union $\bigcup_{j=1}^{n} C_j$, which will be closed. Note that any set in $\mathcal{F}$ can be written as a countable union of closed sets together with a null set, and thus will be measurable by the Borelness of μ.[1] Thus, since $\mu(X) < \infty$, we can apply Theorem 8(b) to conclude

$$\lim_{n\to\infty} \mu\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \sim \left(\bigcup_{j=1}^{n} C_j\right)\right) = \mu\left(\left(\bigcup_{j=1}^{\infty} A_j\right) \sim \left(\bigcup_{j=1}^{\infty} C_j\right)\right) < \epsilon\,.$$

So, $\bigcup_{j=1}^{\infty} A_j$ can be approximated from the inside by closed sets, establishing that $\mathcal{F}$ is closed under countable unions, and completing the proof.

To obtain approximations of non-Borel sets, we hypothesise a further property of Borel measures.

Definition: μ is a *Borel regular measure* on a topological space X if μ is Borel and if, for every $A \subseteq X$, there is a Borel $B \supseteq A$ with $\mu(B) = \mu(A)$.

REMARK: If A is measurable and $\mu(A) < \infty$ then it follows that $\mu(B \sim A) = 0$. However, if either hypothesis fails to hold, then it is possible that no such B exists.

[1] Alternatively, since we're only interested in showing $\mathcal{F} \supseteq \mathcal{B}$, we can define $\mathcal{F}$ to specifically include only the Borel sets which can be approximated. Then, since μ is Borel, sets in $\mathcal{F}$ will automatically be measurable.

PROPOSITION 34: $\mathscr{L}^m$ is Borel regular.

PROOF: Consider $A \subseteq \mathbb{R}^m$, and for each $n \in \mathbb{N}$ let $\{P_{jn}\}_{j=1}^{\infty}$ be a covering of A by open m-boxes for which

$$\sum_{j=1}^{\infty} \mathscr{L}^m(P_{jn}) = \sum_{j=1}^{\infty} v(P_{jn}) \leqslant \mathscr{L}^m(A) + \frac{1}{n}\,.$$

Let $B_n = \bigcup_{j=1}^{\infty} P_{jn} \supseteq A$. Then B_n is open, and thus Borel, and

$$\mathscr{L}^m(B_n) \leqslant \sum_{j=1}^{\infty} \mathscr{L}^m(P_{jn}) \leqslant \mathscr{L}^m(A) + \frac{1}{n}\,.$$

Taking $B = \bigcap_{n=1}^{\infty} B_n$, it is clear that $B \supseteq A$ is Borel with $\mathscr{L}^m(B) = \mathscr{L}^m(A)$, as desired.

THEOREM 35: Suppose μ is a Borel regular measure on a topological space X.

(a) Theorem 8(a) holds for an arbitrary increasing sequence $\{A_j\}_{j=1}^{\infty}$ of subsets of X:

$$A_j \nearrow A \quad \Longrightarrow \quad \mu(A_j) \nearrow \mu(A)\,.$$

(b) [45] If $A \subseteq X$ is measurable with $\mu(A) < \infty$ then there are Borel sets B and D with $D \subseteq A \subseteq B$ and

(★) $$\begin{cases} \mu(B \sim A) = 0\,, \\ \mu(A \sim D) = 0\,. \end{cases}$$

(c) If $A \subseteq X$ and if *either*

(i) A is Borel,

(ii) [46] *or* A is measurable and $\mu(A) < \infty$,

then $\mu \llcorner A$ is Borel regular.

(d) If X is a metric space then Theorem 33, and the subsequent remarks, hold for any measurable $A \subseteq X$. In particular, suppose $X = \bigcup_{j=1}^{\infty} V_j$ with each V_j open and with

$\mu(V_j) < \infty$. Then for any measurable $A \subseteq X$ with $\mu(A) < \infty$, and for any $\epsilon > 0$, there is a closed $C \subseteq A$ and an open $W \supseteq A$ for which

$$(\clubsuit) \qquad \mu(A \sim C) < \epsilon\,,$$

$$(\heartsuit) \qquad \mu(W \sim A) < \epsilon\,.$$

PROOF OF THEOREM 35:

To prove (a), for each j we choose a Borel $B_j \supseteq A_j$ with $\mu(B_j) = \mu(A_j)$. We would like to apply Theorem 8(a) to the sequence $\{B_j\}_{j=1}^{\infty}$ but this sequence may not be increasing. Instead, we note that

$$k \geqslant j \quad \Longrightarrow \quad B_k \supseteq A_k \supseteq A_j\,.$$

Thus $B_j \supseteq D_j \supseteq A_j$, where $D_j = \bigcap_{k=j}^{\infty} B_k$; in particular $\mu(D_j) = \mu(A_j)$ for each j. Each D_j is Borel, and $\{D_j\}_{j=1}^{\infty}$ is increasing. So, we can apply Theorem 8(a) to conclude

$$\lim_{j\to\infty} \mu(A_j) = \lim_{j\to\infty} \mu(D_j) = \mu\left(\bigcup_{j=1}^{\infty} D_j\right) \geqslant \mu\left(\bigcup_{j=1}^{\infty} A_j\right)\,.$$

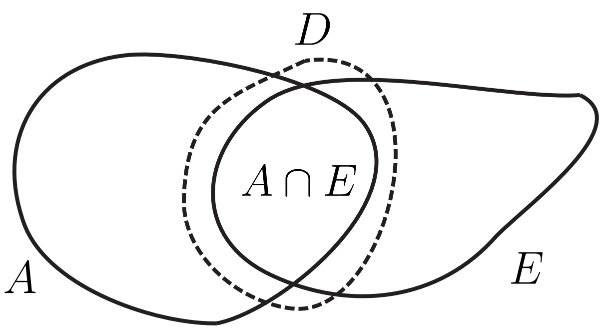
The inequality in the other direction is trivial, and thus (a) is proved.

To prove (c)(i), we assume A is Borel. Then $\mu \lfloor A$ is Borel, by Lemma 9. To show $\mu \lfloor A$ is regular, we consider $E \subseteq X$ and we show that there is a Borel $B \supseteq E$ with

$$\mu \lfloor A(B) = \mu \lfloor A(E)\,.$$

Since μ is regular, there is a Borel $D \supseteq E \cap A$ with $\mu(D) = \mu(E \cap A)$. We then let

$$B = D \cup \sim A\,.$$



Clearly $B \supseteq E$, and since A is Borel, so is B. And,

$$\mu \llcorner A(B) = \mu(A \cap B) = \mu(A \cap D) \leqslant \mu(D) = \mu(E \cap A) = \mu \llcorner A(E)\,.$$

The reverse inequality is trivial, and thus (c)(i) is proved.

Finally, (d) follows immediately from (b) together with Theorem 33.

REMARKS:

- A (not necessarily Borel) measure μ is called *regular* if, for every $A \subseteq X$, there is a measurable $B \supseteq A$ with $\mu(B) = \mu(A)$. So, of course any Borel regular measure is regular. Examining the proof, it is easy to see that (a) above holds for any regular measure.
- The condition in (d) obviously applies to Lebesgue measure, and see Remark (e) after Theorem 33. Thus (♣) and (♡) will hold for any $\mathscr{L}^m$-measurable $A \subseteq \mathbb{R}^m$, whether or not $\mathscr{L}^m(A)$ is finite.

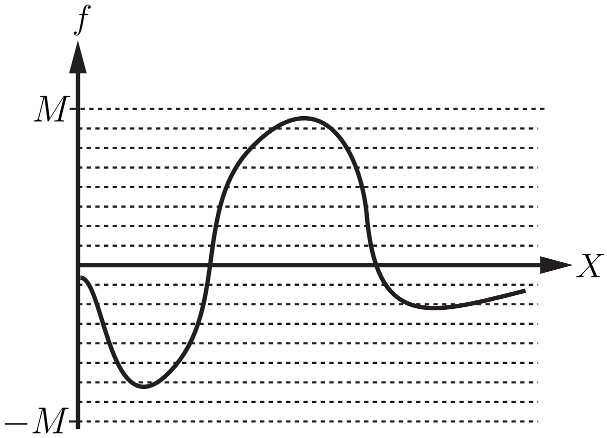
APPROXIMATION OF FUNCTIONS

We now want to consider the approximation of measurable functions by continuous functions. We begin with the simple and general result that we can approximate functions in L^p by finite-valued functions.

LEMMA 36: Suppose μ is a (not necessarily Borel) measure on a set X, and suppose $1 \leqslant p \leqslant \infty$. Then for any $f \in L^p$ and for any $\epsilon > 0$ there is a real-valued $\phi \in L^p$ with finite range for which $\|f - \phi\|_p < \epsilon$.

PROOF: First consider $p = \infty$, and set $M = \|f\|_\infty = \operatorname{ess\,sup}|f|$. Choose $N \in \mathbb{N}$ with $\frac{M}{N} < \epsilon$, and set

$$\begin{cases} D_j = \left\{x : \frac{jM}{N} \leqslant f(x) < \frac{(j+1)M}{N}\right\}\,, \\ \phi = \displaystyle\sum_{j=-N}^{N} \frac{jM}{N} \chi_{D_j}\,. \end{cases}$$



Since f is measurable, each D_j is measurable, and thus ϕ is as well. And, since $|f| \leqslant M$ a.e., it easily follows that $\|f - \phi\|_\infty \leqslant \frac{M}{N} < \epsilon$, as desired.

Now suppose $p < \infty$. We'll assume that $f \geqslant 0$: the result for general f then follows easily by writing $f = f^+ - f^-$, approximating f^+ and f^-, and applying the triangle inequality.

By Lemma 20, we can write

$$f = \sum_{j=1}^{\infty} a_j \chi_{A_j} \qquad a_j > 0\,.$$

Now let

$$\phi_n = \sum_{j=1}^{n} a_j \chi_{A_j}\,.$$

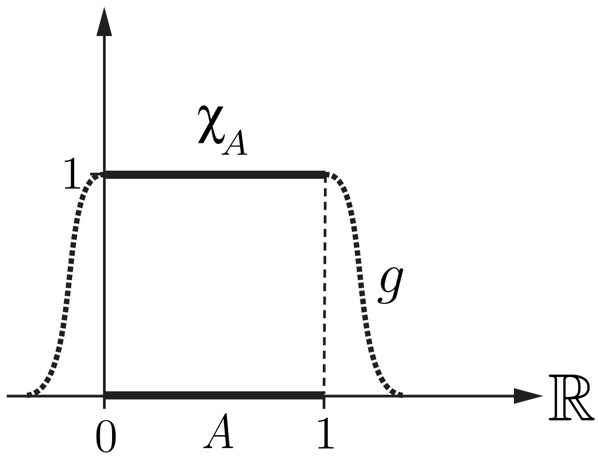
Then $0 \leqslant \phi_n \leqslant f$ and $\phi_n \to f$ a.e. By assumption $f \in L^p$, and so we can apply the Dominated Convergence Theorem (Theorem 22) to give

$$\lim_{n\to\infty} \|f - \phi_n\|_p^p = \lim_{n\to\infty} \int |f - \phi_n|^p = 0\,.$$

Thus, for n large we have $\|f - \phi_n\|_p < \epsilon$, as desired.

So, quite generally, a function $f \in L^p$ can be approximated by a finite sum $\phi = \sum a_j \chi_{A_j}$ of characteristic functions. We now ask: if μ is a Borel measure on a topological space, can f be approximated by continuous functions? Of course, it is enough to approximate ϕ as given by Lemma 36. So, by the triangle inequality on L^p, it is enough to consider the approximation of a characteristic function χ_A of a measurable $A \subseteq X$ with $\mu(A) < \infty$.

We first consider L^∞, where in general there is no hope of approximating by continuous functions. For example, consider Lebesgue measure $\mathscr{L}$ on $\mathbb{R}$ and let $A = [0, 1]$. Since any continuous $g : \mathbb{R} \to \mathbb{R}^*$ cannot instantaneously leap from 1 to 0, it is clear that we have $\|\chi_A - g\|_\infty \geqslant \frac{1}{2}$.



This is quite generally the case, since for X a compact topological space, the space $C(X)$ of continuous functions on X is complete with respect to the *sup* norm. Thus, as long as $\mu(V) > 0$ for every non-empty open $V \subseteq X$, $C(X)$ will be a closed subspace of L^∞. This implies that *no* discontinuous functions in L^∞ can be approximated by continuous functions.[2]

Now what about approximating by continuous functions in L^p for $p < \infty$? Here, we certainly have more hope. For example, it is clear that $\chi_{[0,1]}$ can be approximated in $L^p(\mathscr{L})$ by a continuous g, by setting $g = 1$ on $[0, 1]$, and then ensuring g decays to 0 sufficiently quickly. This is the key idea, and we shall show it works in broad generality, but one has to be careful. For example if A is not a closed set, then setting a continuous g to be 1 on A still means that g will be 1 on all of the closure $\overline{A}$ of A, which could result in an awful approximation: if $A = \mathbb{Q}$, the continuous extension g of $\chi_\mathbb{Q}$ cannot take into account the measure-triviality of $\chi_\mathbb{Q}$. This problem is not always solvable:

47 There is a Borel regular measure μ on $\mathbb{R}$ (usual metric) and a μ-measurable $A \subseteq \mathbb{R}$ with $\mu(A) < \infty$, such that χ_A cannot be approximated by continuous functions in L^p for any $p < \infty$.

[2] If the measure μ trivialises the topology, then discontinuous functions can be approximated in an analogously trivial sense. For example, let $X = \mathbb{R}$ and consider the measure $\mu = \sum_{j \in \mathbb{Z}} \mu_j$, where μ_j is the Dirac measure at j. Then μ ignores everything between the integers, which means we can approximate any $f \in L^\infty$ by a continuous $g \in L^\infty$, simply by setting $g(j) = f(j)$ and then joining the dots by straight lines. But this is not a true approximation, because $g \equiv f$ in L^∞. That is, f and g are simply different representatives of the same equivalence class $\widehat{f} = \widehat{g} \in L^\infty$.

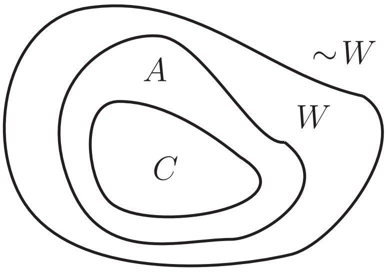
To obtain the desired continuous approximations, the first step is to try to replace A by a closed set $C \subseteq A$ with

$$\mu(A \sim C) < \epsilon\,.$$

Then $\|\chi_C - \chi_A\|_p < \epsilon^{\frac{1}{p}}$, and it is thus sufficient to approximate χ_C. So, we want to set $g = 1$ on C and then have g quickly descend to 0. The second step is to ensure that the region where g dies out has small measure. This will be possible as long as:

- We can find an open $W \supseteq C$ with $\mu(W \sim C) < \epsilon$;
- Given such C and W, we can find a continuous $g\colon X \to [0,1]$ such that

$$g(x) = \begin{cases} 1 & x \in C\,, \\ 0 & x \in \sim W\,. \end{cases}$$



Any such continuous g will give the desired approximation to χ_A, since

$$\|g - \chi_A\|_p \leqslant \|g - \chi_C\|_p + \|\chi_C - \chi_A\|_p \leqslant \|\chi_W - \chi_C\|_p + \|\chi_C - \chi_A\|_p < 2\epsilon^{\frac{1}{p}}\,.$$

For X a metric space, Theorems 33 and 35(d) are exactly designed to give us the needed approximating sets C and W. What about g? It is not obvious, once we have defined g on C and W, how we can extend g to a continuous function on all of X. However, in a metric space, this turns out to be easy: we can simply define

(✠) $$g(x) = \frac{\operatorname{dist}(x, \sim W)}{\operatorname{dist}(x, \sim W) + \operatorname{dist}(x, C)}\,.$$

(If $W = X$ this formula makes no sense, but in that case we can just set $g \equiv 1$ everywhere). It is easy to check that g has the desired properties. Thus, putting all of the above discussion together, we have

THEOREM 37: Suppose μ is a Borel regular measure on a metric space X, and suppose that $X = \bigcup_{j=1}^{\infty} V_j$ with each V_j open and $\mu(V_j) < \infty$. Then $C(X)$ is dense in L^p for $1 \leqslant p < \infty$.

Theorem 37 in particular applies to Lebesgue measure on Euclidean space (see the discussion below). For general metric spaces, it is not quite clear what are the natural hypotheses to make. The following (not immediately obvious) Lemma motivates the subsequent definition:

48 **LEMMA 38**: Suppose X is a locally compact and separable metric space. Then we can write $X = \bigcup_{j=1}^{\infty} V_j$ with V_j open and $\overline{V}_j$ compact.

Definition: Suppose μ is a Borel regular measure on a locally compact and separable metric space. We say μ is a *Radon measure* if $\mu(K) < \infty$ for every compact $K \subseteq X$.

Lebesgue measure is the obvious example of a Radon measure. And, the purpose of Lemma 38 is that it implies Theorem 37 applies for all Radon measures. However, in the Radon setting we can actually prove more.

If X is a separable and locally compact metric space then, with the notation of Lemma 38, any closed $C \subseteq X$ is the countable union of the compact sets $C \cap \overline{V}_j$. Then, if μ is Radon, we can apply Theorem 8(b) to obtain a refinement of Theorem 35(d): If $A \subseteq X$ is measurable with $\mu(A) < \infty$ then there is a *compact* $K \subseteq A$ with $\mu(A \sim K) < \epsilon$. As before, we can then find open $W \supseteq K$ with $\mu(W \sim K) < \epsilon$. But the compactness of K, ensuring that finitely many V_j are sufficient to cover W, means that we can also demand that $\overline{W}$ be compact. Then the continuous function g defined by ($\maltese$) will be 0 outside of $\overline{W}$, and so will have compact support.[3] We thus have proved:

THEOREM 39: Suppose that X is a separable and locally compact metric space, and suppose that μ is a Radon measure on X. Then, for $1 \leqslant p < \infty$, the space $C_0(X)$ of compactly supported continuous functions on X is dense in L^p.

We shall close with some brief remarks on the special case of Lebesgue measure on Euclidean space, and on the more general case of Radon measures on topological spaces. Again, we consider $L^p(X)$ for $1 \leqslant p \leqslant \infty$.

LEBESGUE MEASURE

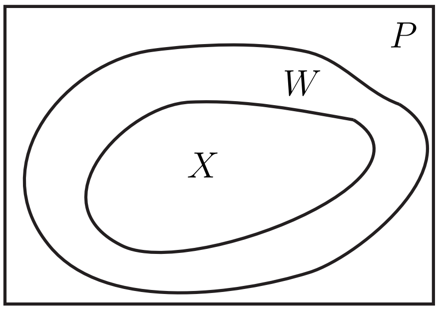
Suppose $X \subseteq \mathbb{R}^m$ is $\mathscr{L}^m$-measurable. Measurable functions on X can be thought as measurable functions on $\mathbb{R}^m$ by extending them to be 0 off of X. So, by our previous remarks, $C(\mathbb{R}^m)$ is dense in $L^p(X)$, and thus $C(X)$ is dense in $L^p(X)$. Theorem 39 also

[3]The support of a continuous function g is defined to be the closure of $\{x : g(x) = 0\}$.

promises us that $C_0(\mathbb{R}^m)$ is dense in $L^p(X)$, but is it the case that $C_0(X)$ is dense in $L^p(X)$? That is, can we take the approximating continuous functions to have support within X? By Theorem 39, this will be true if X is locally compact; so for example, it is true if X is either closed or open.[4]

Even if X is not locally compact, we can be more precise. Given $\epsilon > 0$, we can always find an open $W \supseteq X$ with $\mathscr{L}^m(W \sim X) < \epsilon$. Then $C_0(W)$ will be dense in $L^p(X)$.

Suppose now that X is bounded. Then we can place X and $W \supseteq X$ in an open and bounded m-box P: extending any $g \in C_0(W)$ to be 0 outside W, we can then regard g a function on P. But then the Stone-Weierstrass Approximation Theorem (Handout 0) says that any such g can be approximated uniformly, and thus in L^p, on P by polynomials.[5] The polynomials themselves can be approximated by polynomials with rational coefficients: since there are only countably many such polynomials, this proves that $L^p(X)$ is separable.



Suppose now that X is not necessarily bounded. A given $g \in C_0(W)$ on an open $W \supseteq X$ can still be uniformly approximated by polynomials with rational coefficients within any box P containing the support of g. We can then cut off any such polynomial so that it has support only slightly larger than P: with these cut-off polynomials, we can then approximate anything in $L^p(X)$. Thus, whether or not X is bounded, $L^p(X)$ is separable, and a countable dense subset can be chosen from $C_0(\mathbb{R}^n)$.[6]

RADON MEASURES ON GENERAL TOPOLOGICAL SPACES

On a general topological space X, we effectively define Radon measures to be those for which Theorem 39 holds. We consider X locally compact and Hausdorff. Then a measure μ on X is defined to be *Radon* if μ is Borel, and if:

(a) $\mu(K) < \infty$ for every compact $K \subseteq X$;

(b) $\mu(V) = \sup\{\mu(K) : K \subseteq V, K \text{ compact}\}$ for every open $V \subseteq X$;

(c) $\mu(A) = \inf\{\mu(W) : W \supseteq A, W \text{ open}\}$ for every $A \subseteq X$.

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[4]Of course, if X is compact then $C_0(X) = C(X)$.

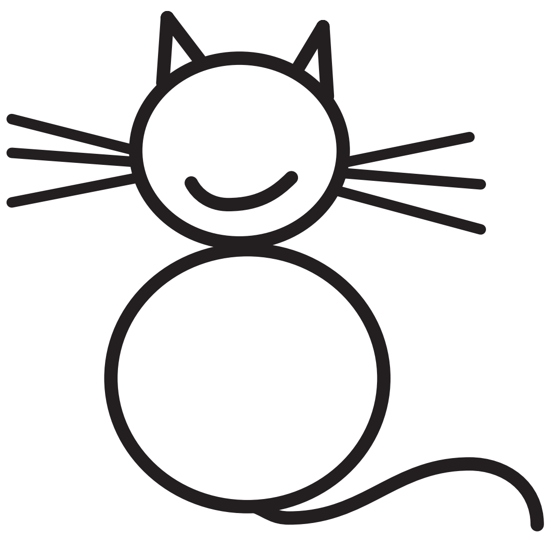
[5]The Stone-Weierstrass Approximation Theorem is very general: see, for example, Ch 9 of *Real Analysis* by H. Royden (Prentice Hall, 3rd ed., 1988). So, in fact the discussion here applies to many settings beyond Euclidean space.

[6]If $X \subseteq \mathbb{R}^m$ is $\mathscr{L}^m$-measurable with $\mathscr{L}^m(X) > 0$, then it is easy to show that $L^\infty(X)$ is *not* separable.

It is immediate that a Radon measure is Borel regular, and the discussion before Theorem 39 shows that this definition agrees with our previous definition of a Radon measure on a separable metric space. It is not obvious that (b) holds for all measurable $A \subseteq X$: with a little work one can show this is true if $\mu(A) < \infty$.[7]

More easily, if A is measurable and if $\mu(A) < \infty$ then A can be approximated from the outside by an open W with $\overline{W}$ compact; then, by (c), W can be approximated from the inside by a compact K. This is exactly the type of set-up we created to prove Theorem 37 and Theorem 39. The remaining question is, can we find a continuous $g : X \to [0, 1]$ with $g = 1$ on K and $g = 0$ on $\sim W$? In the topological setting this is not so obvious: we cannot simply write down a formula such as (✠). However, if X is normal and second countable then X is metrizable by the *Urysohn Metrization Theorem* (Handout 0), and we can use (✠) as above. In fact, if X is locally compact and Hausdorff then such a g still exists, by *Urysohn's Lemma*.[8] We can therefore still conclude

Theorem 40: Suppose μ is a Radon measure on a locally compact and Hausdorff topological space. Then $C_0(X)$ is dense in L^p for $1 \leqslant p < \infty$.



[7]See §2.2.5 of *Geometric Measure Theory* by H. Federer (Springer, 1991).

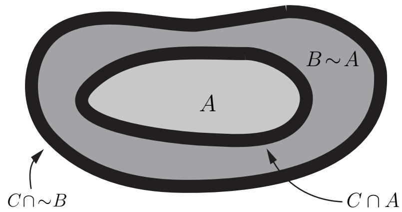
[8]See Handout 0 and, for example, §§8.3,9.5 of *Real Analysis* by H. Royden (3rd ed., Prentice Hall, 1988). Note that it is critical here that one of the closed sets, K or $\sim W$, be compact: that is, a locally compact Hausdorff space need not be normal. One counterexample is a creature called the *Deleted Tychonof Plank*: see *Counterexamples in Topology* by L. Steen and J. Seebach (Dover, 1995).

SOLUTIONS

45 Suppose μ is a Borel regular measure on a topological space X. We want to show that if $A \subseteq X$ is measurable with $\mu(A) < \infty$ then there are Borel sets B and D with $D \subseteq A \subseteq B$ and

$$\begin{cases} \mu(B \sim A) = 0\,, \\ \mu(A \sim D) = 0\,. \end{cases}$$

The existence of $B \supseteq A$ follows from the previous exercise. Then, we can also find a Borel $C \supseteq B \sim A$ with $\mu(C \sim (B \sim A)) = 0$. We now let $D = B \sim C$.



Then D is Borel and

$$\begin{aligned} & \sim C \subseteq \sim (B \sim A) \\ \implies \quad & D = B \sim C \subseteq B \sim (B \sim A) = A\,. \end{aligned}$$

Also, since $A \subseteq B$,

$$A \sim D = A \cap \sim (B \sim C) = A \cap (\sim B \cup C) = C \cap A \subseteq C \sim (B \sim A)\,.$$

Thus $\mu(A \sim D) = 0$.

47 Let $X = \mathbb{R}$ with the usual topology, and define $\mu = \mu_0 \llcorner \mathbb{Q}$ be counting measure restricted to the rationals. From Theorem 35, it is immediate that μ is Borel regular. It is also clear that if f is continuous and $f \in L^p$ for some $1 \leqslant p < \infty$, then $f \equiv 0$: otherwise, we would have $|f| > \delta$ for some $\delta > 0$ and on some open interval, and since the interval contains infinitely many rationals, we would immediately have $\int |f|^p = \infty$. Thus, we cannot approximate *anything* non-trivial in L^p with continuous functions. For example, $\chi_{\{0\}}$ is L^p but cannot be approximated by continuous functions.

Note that μ is a σ-finite measure, since $\mathbb{R} = \sim\mathbb{Q} \cup \bigcup_{q_n \in \mathbb{Q}} \{q_n\}$. However, we cannot write $\mathbb{R}$ as a countable union of *open* sets of finite μ-measure.