Turning the tables: feasting from a mathsnack

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The old wobbly four-legged restaurant table trick – proved!

In *MathSnacks* (*Vinculum* vol. 42 no. 2), we made the following claim, the **Intuitive Table "Theorem"**:

By rotating a square table over uneven ground, you can ensure that all four legs touch the ground.

This elicited an email response from a reader, James Kershaw, which began:

I question the completeness of the "table turning" proof… The problem of fitting coplanar points to an arbitrary surface, from memory, can only be generally solved for three points... i.e., a three pointed stool can always be made stable and the fourth leg on a chair or table requires an extra degree of freedom to match.

James has raised some interesting issues, which we'll address as part of a general discussion of continuity theorems. We'll attempt to keep the discussion uncluttered and intuitive; a couple of details will be relegated to endnotes, and please contact the authors if you have any further thoughts or questions.

Leaps of faith: risible or rigorisible?

Underlying any idea of a *theorem* is some notion of *rigor*, an infallibility in the argument, and some claim to *universality*: a theorem is, by definition, *always* true. Our goal in *Mathsnacks* is to present not proofs but beautiful ideas: we don't pretend that what we have written is *rigorous*, but we hope that it is *rigorisable*. Having said that, the Table (Theorem) is not so easy to nail down. Here, we shall state and outline a proof of a Precise Table Theorem.

Leaps of functions: the limits of plausibility

Underlying the problem we'll consider is the notion of a *continuous function*. Continuity is a natural and extremely important concept in mathematical analysis. In school mathematics it is (too briefly) presented as the intuitive notion of a continuous function being one whose graph can be drawn without your pen leaving the paper. More rigorously, this idea is captured by the

Intermediate Value Theorem (IVT):

If $f:[a,b]\to R$ *is a continuous function, and if* $f(a)<0<f(b)$, *then there is a real number* c *in* (a,b) *with* $f(c) = 0$.

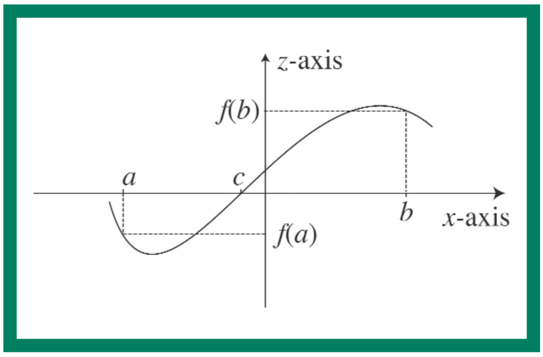

Figure 1: The Intermediate Value Theorem in action.

This effectively characterises continuous functions of one variable, and is the key ingredient in the *To Be or Not to Be* arguments in several of our *Mathsnacks*. The idea is to come up with an appropriate function, and to argue that the function is, or is reasonably assumed, continuous.[1]

Continuous functions of two or more variables are not so easily handled. There is no characterisation as simple as IVT, and the corresponding theorems tend to be harder to understand and much harder to prove.

Turning the tables

Now consider the table-turning problem, noting that it is the four endpoints of the legs that we are actually trying to make touch the ground. So, we consider the square *ABCD* formed by these endpoints, where we let the diagonals $\overline{AC}$ and $\overline{BD}$ have length $2r$. We then want to rotate the square so that the four corners touch the ground simultaneously. The surface of the table, which we temporarily ignore, will be a parallel square.

Next, we have to consider exactly what we mean by a *rotation* of the square, the issue being to

Cliffs (discontinuities) in the floor may prevent a stable table!

preclude a simultaneous *translation* of the table. To do this, imagine the z-axis, as shown in Figure 2, is a spike, and that there is a hole in the centre O of the square. We then place the square on this spike: so, at every stage, O is on the z-axis. We then think of the square rotating around O, but O is free to move up and down the z-axis, and the square can tilt as it rotates.

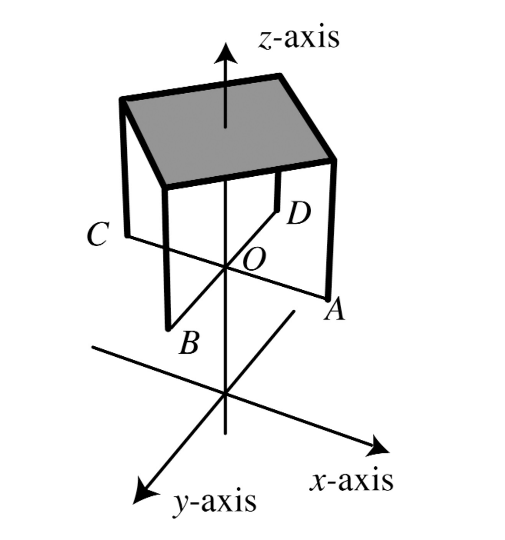

Figure 2: The table set-up

We next consider the *ground*, which we take to be defined by a function $g{:}R^2 \rightarrow R$. Our task is to determine conditions on g that guarantee the four corners can simultaneously touch the ground. As James pointed out, it is *not* true that we can balance the table on any ground function.

For example (a variation of James's suggestion), consider $g(\vartheta)$, a rotationally symmetric function of the angle ϑ about the z-axis, with

$$g(\vartheta) = \begin{cases} 2 & \text{if } 0 \le \vartheta < \dfrac{\pi}{2} \text{ or } \pi \le \vartheta < \dfrac{3\pi}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

So, as shown in Figure 3, the ground consists of four quadrants, two at height 2 and two at height 1. Note that g is a discontinuous function. It is easy to prove that a table cannot be balanced on such a cliff-like region.

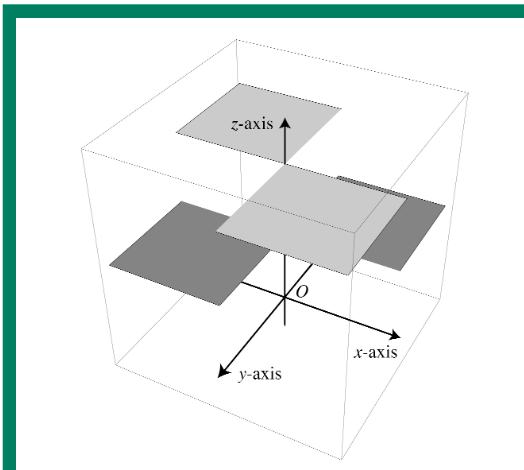

Figure 3: A discontinuous ground, upon which the table cannot be balanced.

What if we assume g is continuous?

We don't know! We know of no continuous counter-example, but there are difficulties with filling in the details of the argument indicated in our *Mathsnacks.* We'll go through the proof, indicating the difficulties, and showing how a further assumption, a *gradient hypothesis*, guarantees the successful balancing of the table.

The ground g is a function of two variables, and the square is free to elevate, tilt and rotate. Thus, on its face, we have a multi-dimensional problem in continuity, which is likely to be difficult. Our approach treats the problem as a succession of IVT arguments, taking one "dimension" at a time. This puts into effect James's intuition that to successfully place the four corners, we need four degrees of freedom, four separate motions of the table.

The first corner A: easy!

Consider the first corner A. Initially, as shown in Figure 2, we take the square to be horizontal, high above the ground, with A hovering directly above the positive x-axis. We then simply

The Intermediate Value Theorem (IVT) smooths the way …

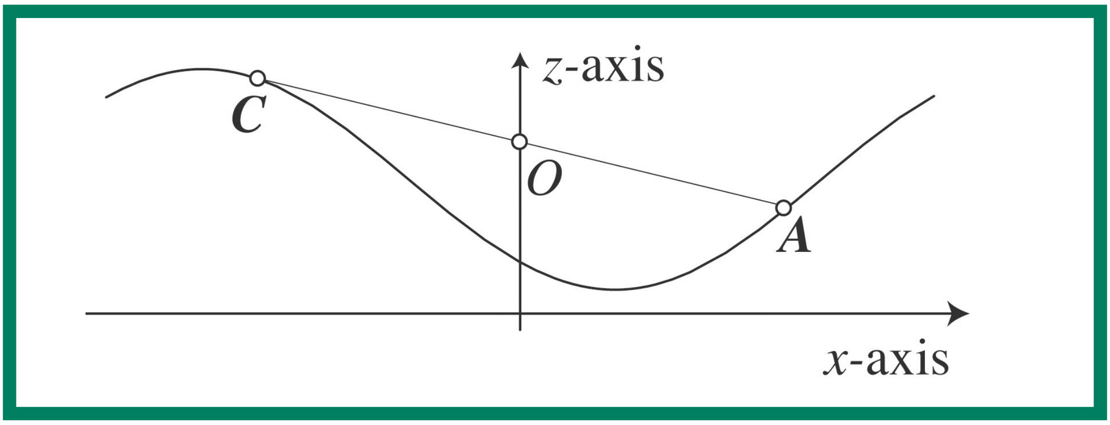

Figure 4: Both A *and* C *touch the ground.*

translate the square vertically downward until A touches the ground. Done!

The second corner C: easy, but …

With A always lying above the x-axis, we now try to tilt $\overline{AC}$ so that C also touches the ground, as shown in Figure 4. (In the process, we slide A along the ground, closer to the z-axis, with O sliding up or down the z-axis). Can we do this? Yes! Since the ground is continuous, this process is continuous, and the claim follows from IVT.[2]

However, there is a problem: if the ground is too steep, there may be more than one tilt angle for which C touches the ground. Usually, we wouldn't care, with the more solutions the merrier. But here, when we consider the last two corners, B and D, we need to have kept careful track of the manner in which A and C have been made to touch the ground: we want there to be a *unique* way to make C touch. By its nature, IVT alone (and thus continuity alone) can never provide us with this uniqueness.

In order to establish this uniqueness, *we shall assume that the ground function* g *satisfies a* ***gradient hypothesis***: there is a positive real constant k such that, for any two points P and Q in the plane,

$$\frac{|g(P)-g(Q)|}{\|P-Q\|} \leq k\,.$$

The gradient hypothesis is exactly what it sounds like: in any direction, between any two points, the gradient of the ground is at most k. Note that the gradient hypothesis guarantees that g is continuous, but a function may be continuous and still fail the gradient hypothesis.[3] It is not obvious, but if the ground satisfies the gradient condition with $k = 1$, then there is a unique way to slide A so that C touches the ground.[4]

The third and fourth corners: equal hovering

So, A and C are fixed and touching the ground. We *could* now tilt the square around the diagonal $\overline{AC}$ so that B also touches the ground. However, it turns out to be better to tilt so that B and D are the same height above the ground. By another IVT argument, we can show that this is possible; see Figure 5.

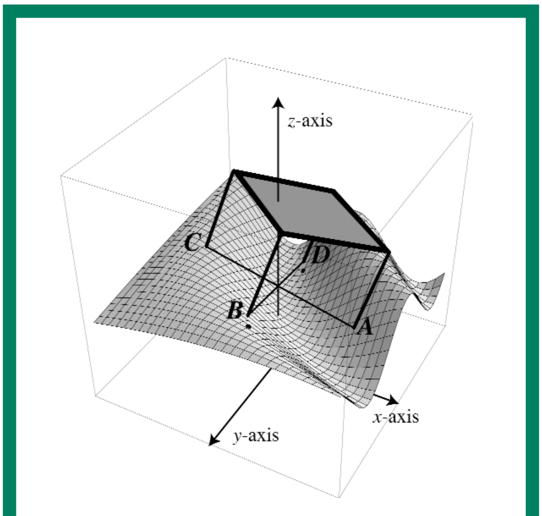

Figure 5: Both A *and* C *touch the ground, while* B *and* D *are hovering the same distance above the ground.*

However, as for the argument above for C, we want there to be a *unique* tilt for which B and D are hovering at the same height. It takes some calculation, but it turns out that this can be guaranteed, if we assume a gradient hypothesis with

$$k = \tfrac{1}{\sqrt{2}}\,.$$

… and rectangular tables? or five-legged tables? …

The third and fourth corners: touchdown!

We now have two corners, A and C, touching the ground, with the other corners, B and D, hovering the same distance above (or below) the ground. We want B and D to actually touch the ground. To arrange this, we apply one last degree of freedom: the rotation of the table around the z-axis! We can arrange for B and D to touch with one last application of IVT.

Initially, we started with the square horizontal, and with $\overline{OA}$ lying in the xz-plane. We now consider the square having first been rotated counter-clockwise an angle ϑ about the z-axis (so that $\overline{OA}$ projects to an angle ϑ with the positive x-axis). For any such ϑ, we can (uniquely) arrange for the corners A and C to touch the ground, with B and D hovering the same height above (or below) the ground. Furthermore (because of the uniqueness), the positions of the corners are continuous functions of ϑ.

Now, suppose that when $\vartheta = 0$, B and D are above the ground, as in Figure 5 (a similar argument will work if they are below the ground). Consider the angle ϑ^* (approximately $\frac{\pi}{2}$) for which $\overline{OB}$ lies in the xz-plane. *Then, for this angle ϑ^*, B and D are now below the ground!*

Why? Whether above or below the ground, we can translate the square vertically until B and D are touching the ground. But this means that B and D are now exactly in the original ($\vartheta = 0$) positions of A and C; and, symmetrically, A and C have taken the original positions of B and D, and thus are above the ground. This means we must have translated upwards, and so the positions of B and D for ϑ^* must be below the ground.

We now have the easy finish. For $\vartheta = 0$, B and D are above the ground, and for $\vartheta = \vartheta^*$, B and D are below the ground. So by IVT, there is some angle ϑ between 0 and ϑ^* for which B and D, and thus all four corners, are touching the ground. Done!

To summarise, we have sketched a proof of the

Precise Table Theorem:

Suppose the ground is described by a (continuous) function $g: \mathbb{R}^2 \to \mathbb{R}$ which satisfies the gradient hypothesis

$$\frac{|g(P) - g(Q)|}{\|P - Q\|} \leq \frac{1}{\sqrt{2}}\ .$$

Then, given any square, there is a way to place the square so that the centre of the square is on the z-axis, and the four corners of the square touch the ground.

On our last legs: final remarks

We emphasise that the gradient condition is essential to make the *proof* of our Precise Table Theorem work, but we don't know whether the *theorem* is true in other contexts. It is possible that a cleverer, completely different proof will work more for more general ground functions.

Finally, we consider how long the legs must be to ensure that, for the solution we have found, the surface of the table is not cutting through the ground. We have assumed the diagonal of the table surface is of length $2r$, so that every point on the table is at most a distance r from its centre. On the other hand, with our gradient hypothesis, the most the ground can rise over a distance r is $\frac{r}{\sqrt{2}}$. Thus, if the table happens to be horizontal, legs of length $\frac{r}{\sqrt{2}}$ will suffice to ensure that the surface of the table is above the ground. A messy but straight-forward calculation then shows that legs of length $\frac{r}{\sqrt{2}}$ will suffice, at no matter which angle the table is tilted.

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1 For example, consider the Mathematical Monk (*Vinculum*, last issue). However he travels, we can consider his distance up the mountain $d(t)$ on the first day, and $D(t)$ on the second day, as functions of time. Failing miraculous intervention, the monk's journeys will be continuous. Now define a new continuous function $f(t) = d(t) - D(t)$. Note that $f(0) < 0$ and $f(12) > 0$.

2 If A touches the ground at coordinates $(t, 0\ g(t,0))$, then for C to also touch the ground it will have to have coordinates $(-t, 0\ g(-t,0))$. So, we consider the distance-squared:

$$D(t) = \|(t,0,g(t,0)) - (-t,0,g(-t,0))\|^2 = 4t^2 + (g(t,0) - g(-t,0))^2.$$

Since $|\overline{AC}|^2 \geq 4r^2$, we are looking for a t between 0 and r for which $D(t) = 4r^2$. But $D(0) = 0$ and $D(r) \geq 4r^2$, and so the existence of t follows from IVT.

3 For example, the function $f(x) = \sqrt[3]{x}$ is continuous, but f doesn't satisfy the gradient hypothesis, no matter how we choose k.

4 The gradient hypothesis with $k = 1$ guarantees that the function $D(t)$ defined in note 2 is an increasing function of t for $t > 0$. Thus, there is exactly one value of t for which $D(t) = 4r^2$.