Function

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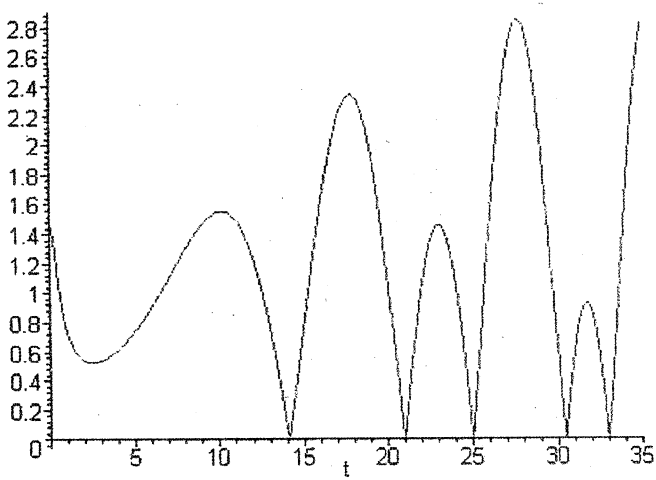

School of Mathematical Sciences – Monash University

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Function is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. It was founded in 1977 by Prof G B Preston, and is addressed principally to students in the upper years of secondary schools, but also more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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* $17 for *bona fide* secondary or tertiary students.

EDITORIAL

This issue of *Function* marks a sad day in the history of our magazine, for it is to be our last. From its inception in 1977 until now, we have published 141 issues under 5 chief editors and have had 31 editors in total. Over 4500 pages have appeared, but now it is time to call a halt to our activities.

Spiralling costs and a dwindling subscriber base have been the reasons for this decision, which was not taken lightly. Moreover, new costing protocols have made it increasingly difficult for Monash to sustain our endeavours, although they have always been most supportive.

After 20 years of *Function*, a special issue was published and in an article in that I listed some of what I thought of as *Function's* very greatest achievements up to that time. To these I would certainly now add Klaus Treitz' article "Apfelmännchen at Königstein" (October 2001) and Jim Cleary's "The Power Triangle and Sums of Powers" (June 2002). But perhaps I am using an unfair criterion in highlighting articles of this type, where the emphasis is on coming up with something truly original. Our main role has been to explain, clearly and accessibly, Mathematics which, while it goes beyond the secondary classroom, can be accessed by students in the final years of High School. As I said in the Special Issue, the mission was to publish "quality exposition of genuine Mathematics".

We have always been especially eager to put into print contributions from members of our target audience and, from the two student feature articles in our very first issue to Anson Huang's solution to Problem 28.1.1 in our August 2004 issue, we have been most pleased to showcase such material.

Our thanks to all those who, in one way or another, have supported *Function* over the years. To those who have subscribed, to those who have written articles for us or sent us other material, and especially to the loyal band of problem-solvers, we express our heartfelt gratitude. And to all those who have been my colleagues on the Editorial Board, my own deep appreciation.

Our cover story is chosen to reflect the occasion of this issue.

Michael A B Deakin
Chief Editor

THE FRONT COVER: The Last Function

If you look up the index in the standard reference work, *Handbook of Mathematical Functions* by Milton Abramowitz and Irene Stegun, you will find the last entry as shown below.

The Riemann Zeta function is a function with especial importance in Number Theory. Something of the basis for this was explained in the History Column for June 2001.

The starting point for work on the Zeta Function (ς-function in what follows) is the definition

$$\varsigma(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

where the series is an infinite one.

However, this simple definition alone does not suffice. The variable s is taken to be complex, but if the real part of s is 1 or less, the series fails to converge. This is to say that, as we add more and more terms to the sum, it does not approach a well-defined number which we could take to be the sum of infinitely many terms.

But this is not the end of the story. To get a feel for what mathematicians do to get around this problem, consider a simpler and more familiar case.

Take the geometric series

$$1 + z + z^2 + z^3 + z^4 + \ldots,$$

where z is complex. That is to say, $z = x + iy$. If $|z| = \sqrt{x^2 + y^2} < 1$, then this series converges and the sum is $\frac{1}{1-z}$. But now notice a somewhat subtle point: wherever the series *does* converge, we can assign to it the sum $\frac{1}{1-z}$. But this latter function is well-defined for all z (apart from the single value $z = 1$). The function $\frac{1}{1-z}$ is said to be an *extension* of that defined by the series. Where both expressions make sense, they agree completely, but the domain of one is larger than that of the other.

Something of the same thing happens with the ς-function. This may be extended in a variety of ways, apparently different, but actually equivalent, but all rather too technical to detail fully here. However, the upshot is much the same. The extension is well-defined everywhere except at $s = 1$. What is possibly the simplest extension goes:

$$\varsigma(s) = \frac{1}{s-1} + \gamma_0 - \gamma_1(s-1) + \frac{\gamma_2}{2!}(s-1)^2 - \frac{\gamma_3}{3!}(s-1)^3 + \dots,$$

but the formulas for the coefficients γ_0, γ_1, γ_2, ... are rather formidable.

[For readers seeking more technical information, H M Edwards' *Riemann's Zeta Function* (New York: Academic Press, 1969) or the website

http://mathworld.wolfram.com/RiemannZetaFunction.html

will do very well.]

The importance of the ς-function in Number Theory stems from a result first propounded (for real s) by Euler, and discussed in the June 2001 History Column. It says that

$$\varsigma(s) = \left(1 - \frac{1}{2}\right)^s \left(1 - \frac{1}{3}\right)^s \left(1 - \frac{1}{5}\right)^s \left(1 - \frac{1}{7}\right)^s \dots,$$

where the denominators on the right are the primes.

It was Bernhard Riemann (1826-1866) who first extended the definition of $\varsigma(s)$ to the case of complex s. His 1859 paper on the subject still forms the basis of all that follows, although there have of course been developments since.

But the early versions of the theory were sufficient to tell is that $\varsigma(s)$ has a number of special values and properties. In particular,

$$\varsigma(-2)=\varsigma(-4)=\varsigma(-6)=\varsigma(-8)= \dots =0,$$

$$\varsigma(0)=\varsigma(1)=-\frac{1}{2},\quad \varsigma(-3)=\frac{1}{120},\quad \varsigma(2)=\frac{\pi^2}{6},\quad \varsigma(4)=\frac{\pi^4}{90},\quad \dots .$$

The first line of this table is *relatively* easily proved, and it gives an infinity of zeroes of the function $\varsigma(s)$. These are now referred to as the *trivial zeroes*. Riemann also looked for other zeroes and was able to prove that these others must all be complex. Indeed he was able to prove more: if $\varsigma(s)=0$, then the real part σ of s had to lie in the region $0\le\sigma\le1$. This region is now called the *critical strip*. A further simplification may be achieved by noting that if $\sigma+it$ is a zero, then so is $\sigma-it$. So only positive values of t need be considered.

Now comes the difficult part. Riemann had evidence to suggest that all the zeroes in the critical strip (the "non-trivial zeroes") were in fact of the form $\frac{1}{2}\pm it$, and he put forward the conjecture that all these zeroes were indeed of this form. This is the proposition known as the Riemann Hypothesis. It is now regarded as the most important outstanding unsolved problem in the whole of Mathematics. (Especially now that Fermat's Last Theorem has at last been laid to rest.)

In 1914, Hardy (whose story was the subject of our History Column in June 1995) proved that there are infinitely many zeroes to be found in the critical strip, and it has also been established, as a result of further theory and by means of absolutely heroic calculations, that the first 250, 000, 000, 000 of these lie on the critical line. In other words, if there is a counterexample to the Riemann Hypothesis, then it refers to a zero further from the point $s = 0$ than all of these. Other, more esoteric, results have also been established.

The famous Hilbert problems, posed at the beginning of the twentieth century, listed the Riemann Hypothesis as part of a more general eighth (of a total of 23). At the start of our own century, the Clay Challenge problems included it as the fourth of seven. For a correct proof, the Clay Mathematics Institute (affiliated with Harvard University) will pay a reward of \$US1, 000, 000. [However, although it is most unlikely ever to be found, a *disproof* would earn nothing!]

The cover picture shows only a minute part of this story. If we were to try to graph $\varsigma(s)$, we would run into trouble, as there are not enough dimensions on the page to put both a real and an imaginary axis for each of the values of s and of $\varsigma(s)$. We have put $\varsigma(s) = u + iv$, and $s = \frac{1}{2} + it$. We used t as the horizontal axis. The vertical axis is $\sqrt{u^2 + v^2}$, which has the value 0 if and only if both u and v are zero. Thus the graph on our cover displays the first five zeroes on the critical line. It was produced using MAPLE, which contains subroutines dealing with the Riemann Zeta Function.

There are also other puzzles connected with this function. Although the Riemann Hypothesis is the most studied and the most important, it is by no means the only one. Readers will note that the table at the top of p 132 is silent on the matter of odd integral values of s, when $s > 1$. The number $\varsigma(3)$ was shown in 1978 to be irrational. The proof was the result of work by a mathematician called Apéry and this number is now called *Apéry's Constant* in his honour. There are still other unsolved questions concerning it and even its computation is far from simple. When we come to $\varsigma(5)$, $\varsigma(7)$, and indeed all $\varsigma(2n+1)$ for which $n > 1$, even the question of rationality is still open.

This article would not be complete if it omitted reference to the persistent and continuing efforts of Louis de Branges (the surname is pronounced "duh BRONZH") to prove the Riemann Hypothesis. Several times he has announced that he had a proof, and a number of alleged proofs have been put, in whole or in part, on the web. In every case, there has either been insufficient detail or else a flaw has been discovered in the reasoning. Readers may check the latest claims at websites linked to that given on p 131, which are updated regularly.

Indeed, by now any other mathematician with such a record of unsubstantiated claim would have lost almost all credibility. But de Branges is not so easily dismissed. In 1984, he proved the long-outstanding "Bieberbach Conjecture", a technical statement about complex functions of complex variables. (See *Function*, February 1985, pp 7, 32.) So de Branges has some runs on the board; nonetheless, one senses in the tone of recent postings on the internet a certain impatience with his latest claims!

THREE CIRCLES AND A POINT

Ken Evans, Dromana, Vic

Circles have many interesting properties. Familiar examples are:

(1) Angles in a segment of a circle have the same magnitude – Figure 1(a),

(2) The magnitude of an exterior angle of a cyclic quadrilateral is equal to the magnitude of the opposite interior angle – Figure 1(b).

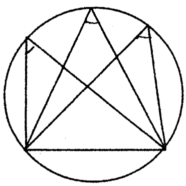


Figure 1(a)

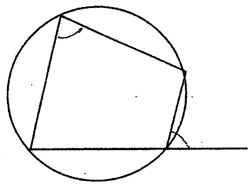


Figure 1(b)

Another less familiar property is needed to help prove the main theorem of this article by Euclidean methods. Such a preliminary theorem is sometimes called a *lemma*.

Lemma 1: If the lines containing two chords, $\overline{AB}$ and $\overline{CD}$ of a circle, intersect at K, then $KA.KB = KC.KD$.

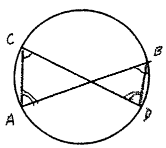


Figure 2(a)

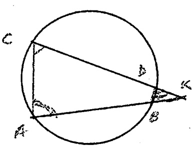


Figure 2(b)

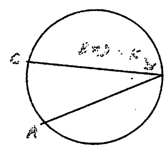


Figure 2(c)

Proof for the case, Figure 2(a), where K is inside the circle.

First draw $\overline{AC}$ and $\overline{BD}$.
Then in the triangles AKC and DKB,

$\text{mag}\angle ACK = \text{mag}\angle DBK$ (angles in a segment)

$\text{mag}\angle CAK = \text{mag}\angle BDK$ (angles in a segment)

Therefore the triangles AKC and DKB are similar.

$$\therefore \frac{KA}{KD} = \frac{KC}{KB}$$

$$\therefore KA.KB = KC.KD.$$

The proof for the case, Figure 2(b), where K is outside the circle is the same as the above except for the reasons why the angle magnitudes are equal. If K belongs to the circle, Figure 2(c), the lemma is trivially true because in that case $KB = KD = 0$, and hence both sides of the equation $KA.KB = KC.KD$ are zero.

Additional Investigations

1. In Figure 2(b), rotate the line $\overleftrightarrow{ABK}$ about K. Four positions are shown in Figure 3, the fourth being the limiting case where the chord-line becomes a tangent. What does the intersecting chords lemma suggest in this limiting case? Your conjecture (if true) needs a separate proof which is not given here.

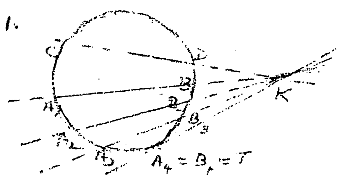


Figure 3

2. After studying the properties of a single circle, a mathematician might look at properties involving two circles, and then three or more circles. For two circles, you might wish to investigate symmetry properties of the common chord (if the two circles intersect) and of common tangents.

3. The property now to be investigated concerns three circles each intersecting the other two. If you have access to Cabri Geometry (or a similar program), it is suggested that you use it for the following. Draw three circles whose radii have different lengths, and such that each circle intersects the other two. Then draw the lines containing the common chords of the three pairs of circles. Make a conjecture about the common chord-lines. Now vary the size and position of the circles and see if your conjecture still appears to be true. Consider also limiting cases where one or two or three pairs of the circles intersect at one point only, i.e. where one or more of the common chords becomes a tangent.

A possible conjecture for Investigation 3 above is:

If three circles are such that each intersects the other two, then the three common chord-lines are either concurrent or parallel.

Two proofs of this conjecture are to be given: the first uses Euclidean methods, in particular Lemma 1, and the second uses Cartesian (co-ordinate) methods.

(a) First Proof

Let C_1, C_2, C_3 be the intersecting circles (Figure 4).

(i) If no two of the common chord-lines intersect, then the three common chord-lines are parallel.

Additional Investigation

4. Find a condition regarding the centres of the three circles which is sufficient to make the common chord-lines parallel.

(ii) If it is not true that the common chord-lines are parallel, then two (at least) intersect. Suppose these two are $\overleftrightarrow{AB}$, the common chord-line of C_1 and C_3, and $\overleftrightarrow{CD}$, the common chord-line of C_2 and C_3.

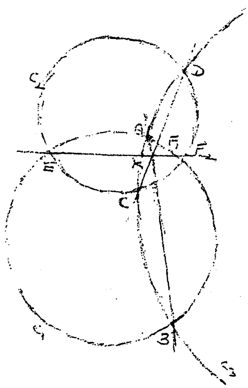


Figure 4

Let K be the point of intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. It remains to prove that $\overleftrightarrow{EK}$ is the common chord-line of C_1 and C_2. So as not to assume what is to be proved, let $\overleftrightarrow{EK}$ intersect C_1 at F_1 and C_2 at F_2. (Figure 4 has been slightly distorted to show this.)

Now $KA.KB = KE.KF_1$ (intersecting chord-lines of C_1, Lemma 1)
Also $KC.KD = KE.KF_2$ (intersecting chord-lines of C_2)
But $KA.KB = KC.KD$ (intersecting chord-lines of C_3)
$\therefore KE.KF_1 = KE.KF_2$
$\therefore KF_1 = KF_2$
$\therefore F_1 = F_2 = F$ (say).

Because $F = F_1 \in C_1$ and $F = F_2 \in C_2$, $F \in C_1 \cap C_2$.
Hence $\overleftrightarrow{EKF}$ is the common chord-line of C_1 and C_2.
Hence $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{EF}$ intersect at K, i.e. are concurrent.
From (i) and (ii), the common chord-lines are concurrent or parallel.

For the co-ordinate proof some preliminary notions are introduced. Firstly, a point is identified with an ordered pair (x, y) of real numbers, and a curve is identified with a relation $\{(x,y): f(x,y)=0\}$ where $f(x,y)$ is an appropriately chosen expression in x, y. This means that a point (X,Y) belongs to a curve $\{(x,y): f(x,y)=0\}$ if its co-ordinates satisfy the defining equation $f(x,y) = 0$; and conversely, if a point (X,Y) belongs to a curve $\{(x,y): f(x,y)=0\}$, then $f(X,Y)=0$.

E.g. Because $2^2+(-1)^2-5=0$, $(2,-1)\in\{(x,y): x^2+y^2-5=0\}$.
Because $2\times 2+(-1)\times(-1)-5=0$, $(2,-1)\in\{(x,y): 2x-y-5=0\}$.
If $(X,Y)\in\{(x,y): x^2+y^2-5=0\}$ then $X^2+Y^2-5=0$, and conversely if $X^2+Y^2-5=0$, then $(X,Y)\in\{(x,y): x^2+y^2-5=0\}$

Secondly, two further lemmas, the first of which applies to curves in general and the second to circles, are now proved.

Lemma 2

If $(X,Y)\in\{(x,y): f_1(x,y)=0\}=C_1$, and also
If $(X,Y)\in\{(x,y): f_2(x,y)=0\}=C_2$, i.e. if (X,Y) is a point of intersection of C_1 and C_2, then
$(X,Y)\in\{(x,y): \alpha f_1(x,y)+\beta f_2(x,y)=0\}=C$ for real numbers α, β not both zero.
(If α, β are both zero, C is not a curve.)

Proof Because $(X,Y)\in C_1$, $f_1(x,y)=0$.
Because $(X,Y)\in C_2$, $f_2(x,y)=0$.
$\therefore \alpha f_1(x,y)+\beta f_2(x,y)=\alpha\times 0+\beta\times 0=0.$
$\therefore (X,Y)\in\{(x,y): \alpha f_1(x,y)+\beta f_2(x,y)=0\}=C.$
i.e. if the two curves C_1, C_2 intersect at a point, then curve C passes through that point.
The above argument applies to each point Y of the intersection of C_1 and C_2. Hence C passes through all points of intersection of C_1 and C_2.

Lemma 3 If C_1 and C_2 are intersecting circles with equations
$f_1(x,y)=x^2+y^2+a_1x+b_1y+c_1=0$
$f_2(x,y)=x^2+y^2+a_2x+b_2y+c_2=0$,
then $(a_1-a_2)x+(b_1-b_2)y+(c_1-c_2)=0$
is the equation of the common chord line.

Proof Consider the equation

$$\alpha f_1(x,y)+\beta f_2(x,y)=0,\ (\alpha,\ \beta\ \text{not both zero}). \qquad (1)$$

After simplifying the left side, (1) becomes

$$(\alpha+\beta)x^2+(\alpha+\beta)y^2+(\alpha a_1+\beta a_2)x+(\alpha b_1+\beta b_2)y+(\alpha c_1+\beta c_2)=0. \qquad (2)$$

By Lemma 2, (2) defines a *curve* through both points of intersection of C_1 and C_2. Furthermore (2) has the required form (in general) to be the equation of a circle. So (2) can be thought of as defining a set of circles (one circle for each choice of α, β) through the points of intersection of C_1 and C_2. There is however a special case which stands out, viz. if $\beta = -\alpha$. In this case (1) becomes

$$\alpha f_1(x,y) = -\alpha f_2(x,y) = 0$$
$$\Leftrightarrow f_1(x,y) = -f_2(x,y) = 0 \qquad (\alpha \neq 0 \text{ so } \beta \neq 0)$$

$$\Leftrightarrow (a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2) = 0. \tag{3}$$

(Multiplying both sides by -1 shows that (3) is equivalent to

$$(a_2 - a_1)x + (b_2 - b_1)y + (c_2 - c_1) = 0.)$$

Because (3) is a first-degree equation, it defines a line, and, from the above, the line passes through both points of intersection of C_1 and C_2. Furthermore there is only one line through two distinct points, so (3) defines the common chord-line of C_1 and C_2.

Note that for Equation (3) we cannot have both $a_1 - a_2 = 0$ and $b_1 - b_2 = 0$. Show that if both $a_1 = a_2$ and $b_1 = b_2$, then C_1 and C_2 are concentric and so do not intersect.

(b) Second Proof of the Theorem of Three Intersecting Circles

Let

$$C_1 = \{(x,y) : x^2 + y^2 + a_1x + b_1y + c_1 = 0\}$$
$$C_2 = \{(x,y) : x^2 + y^2 + a_2x + b_2y + c_2 = 0\}$$
$$C_3 = \{(x,y) : x^2 + y^2 + a_3x + b_3y + c_3 = 0\}$$

be the intersecting circles.

From Lemma 3, the common chord-lines of C_1 and C_2, C_2 and C_3, C_3 and C_1 respectively are:

$$\{(x,y):(a_1-a_2)x+(b_1-b_2)y+(c_1-c_2)=0\}=L_1$$
$$\{(x,y):(a_2-a_3)x+(b_2-b_3)y+(c_2-c_3)=0\}=L_2$$
$$\{(x,y):(a_3-a_1)x+(b_3-b_1)y+(c_3-c_1)=0\}=L_3.$$

If no two of the chord-lines intersect, the lines are parallel. If this is not the case, at least two of the lines, say L_1 and L_2, must intersect. Let (X,Y) be the point of intersection of L_1 and L_2.

Because $(X,Y)\in L_1$, $(a_1-a_2)X+(b_1-b_2)Y+(c_1-c_2)=0$ (4)
Because $(X,Y)\in L_2$, $(a_{21}-a_3)X+(b_{21}-b_3)Y+(c_2-c_3)=0$ (5)
From (4) and (5), by addition,

$$(a_1-a_3)X+(b_1-b_3)Y+(c_1-c_3)=0$$
$$\Leftrightarrow (a_3-a_1)X+(b_{31}-b_1)Y+(c_3-c_1)=0 \qquad (6)$$

From (6), $(X,Y)\in\{(x,y):(a_3-a_1)x+(b_3-b_1)y+(c_3-c_1)=0\}=L_3$.

Hence L_1, L_2, L_3 are concurrent.

Further Investigations

5 Make a conjecture about L_1, L_2, L_3 if (i) one pair, (ii) two pairs, (iii) all three pairs, of circles have single points of intersection, i.e. touch.

6 Show that whether C_1 intersects C_2 or not, L_1 is perpendicular to the line containing the centres of C_1 and C_2.

7 Find the centre and radius-length of C_1. Use these results and the Theorem of Pythagoras to find an expression for the length of a tangent from an exterior point (X, Y) to the point of contact of circle C_1. Show that, if the tangents to C_1 and C_2 (whether C_1 and C_2 intersect or not), from an exterior point (X, Y) are equal in length, then $(X,Y)\in L_1$. By reversing your argument, prove the converse: if a point (X, Y) exterior to C_1 and C_2 belongs to L_1, the lengths of the tangents from (X, Y) to C_1 and C_2 are equal.

OLYMPIAD NEWS

Hans Lausch, Monash University

The 2004 International Mathematical Olympiad

This year's International Mathematical Olympiad (IMO) took place in Athens. On 12 and 13 July, 486 secondary-school students from 85 countries sat the contest, which consisted of two sets of three problems each. China came first, with a total of 220 out of possible 252 points, USA (212 points) finished second and Russia (205) third. Australia (125 points) took the 27th position, just behind the host country Greece (126) points). Australian students were successful in winning a total of 1 gold medal, 1 silver medal, 2 bronze medals and 1 Honourable Mention, as follows:

Ivan GUO, Sydney Boys High School, NSW, year 12	- GOLD
Laurence FIELD, Sydney Grammar School, NSW, year 12	- SILVER
Alex HUA, Scotch College ,VIC, year 12	- BRONZE
Daniel NADASI, Cranbrook School, NSW, year 12,	- BRONZE
Graham WHITE, James Ruse Agricultural High School, NSW, year 10	- HM

Congratulations to all!

Here are this year's IMO problems:

Day 1

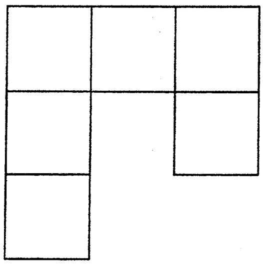
1. ABC is acute angled triangle with $AB \neq AC$. The circle with diameter BC intersects the lines AB and AC respectively at M and N. O is the midpoint of BC. The bisectors of $\angle BAC$ and $\angle MON$ intersect at R.

Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the line BC.

2. Find all polynomials P with real coefficients such that, for all reals a, b, c such that $ab + bc + ca = 0$, we have the relation

$$P(a-b)+P(b-c)+P(c-a)=2P(a+b+c).$$

3. Define a "hook" to be a figure made up of six unit squares as shown in the figure below, or any of the figures obtained by rotations and reflections to this figure.



Determine all $m \times n$ rectangles that can be covered without gaps and without overlaps with hooks such that no point of a hook covers area outside the rectangle.

Day 2

4. Let $n \geq 3$ be an integer. Let $t_1, t_2, \ldots, t_n$ be positive real numbers such that $n^2+1>(t_1+t_2+\ldots+t_n)\left(\frac{1}{t_1}+\frac{1}{t_2}+\ldots+\frac{1}{t_n}\right)$.

Show that, for all distinct i, j, k, t_i, t_j, t_k are the side-lengths of a triangle.

5. In a convex quadrilateral $ABCD$, the diagonal BD bisects neither $\angle ABC$ nor $\angle CDA$. A point P lies inside $ABCD$ and satisfies $\angle PBC = \angle DBA$ and $\angle PDC = \angle BDA$. Prove that $ABCD$ are concyclic if and only if $AP=CP$.

6. A positive integer is alternating if every two consecutive digits in its decimal representation are of different parity. Find all positive integers n such that n has a multiple which is alternating.

The 2004 Senior Contest
of the Australian Mathematical Olympiad Committee (AMOC)

The AMOC Senior Contest is the first hurdle for mathematically talented Australian students who wish to qualify for membership of the team that represents Australia in the following year's International Mathematical Olympiad. This year that four-hour competition took place on 10 August.

These are the questions:

1. Consider 8 points in a plane consisting of the 4 vertices of a square and the 4 mid-points of its edges. Each point is randomly coloured red, green, or blue with equal probability. Show that there is a more than a 50% chance of obtaining a triangle whose vertices are 3 of these points coloured red.

2. Let $a_1, a_2, \ldots, a_{2004}$ be any non-negative real numbers such that $a_1 \geq a_2 \geq \ldots \geq a_{2004}$ and $a_1 + a_2 + \ldots + a_{2004} = 1$.
Prove that $a_1^2 + 3a_2^2 + 5a_3^2 + 7a_4^2 + \ldots + 4007a_{2004}^2 \leq 1$..

3. Let $f(n)$ be the integer closest to $\sqrt{n}$. Determine

$$\frac{1}{f(1)} + \frac{1}{f(2)} + \ldots + \frac{1}{f(10000)}$$

4. Let x, y, z be positive integers that satisfy $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{z^2}$.

Prove that 20 divides xy.

5. Let AB be the diameter of semicircle S, and let C and D be points on S other than A or B, with B closer to C than to D. Further, let AC and BD intersect in E and let AD (extended) and BC (extended) intersect in F. We let G and H be the midpoints of AE and BE, respectively, and O the circumcentre of triangle ABE. Suppose that DG is parallel to CH.

Prove that DG is parallel also to FO.

"The contemplation of the various steps by which mankind has come into possession of the vast stock of mathematical knowledge can hardly fail to interest the mathematician. He takes pride in the fact that his science, more than any other, is an *exact* science, and that hardly anything ever done in mathematics has proved to be useless. The chemist smiles at the childish efforts of alchemists, but the mathematician finds the geometry of the Greeks and the arithmetic of the Hindus as useful and admirable as any research of to-day. He is pleased to notice that though, in the course of its development, mathematics has had periods of slow growth, yet in the main it has been pre-eminently a *progressive* science."

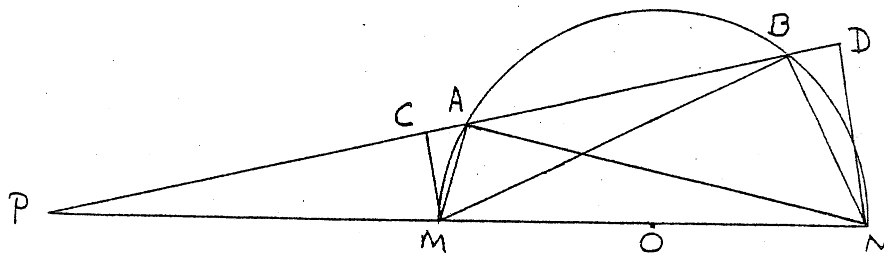
Florian Cajori
A History of Mathematics

SOLUTIONS TO PROBLEMS

We here give the solutions to the problems set in our last three issues. We regret that it has not been possible to allow the usual time-lapse in all cases.

SOLUTION TO PROBLEM 28.2.1

This problem, submitted jointly by Willie Yong (Singapore), Jim Boyd (USA) and Richard Palmaccio (USA) asked for a proof that in the diagrammed figure $\dfrac{AM \times BM}{AN \times BN} = \dfrac{PM}{PN}$ where MN is the diameter of a semi-circle $MABN$ and P is the intersection of MN and AB (extended in both cases).



We received solutions from Šefket Arslangić (Bosnia), John Barton, David Shaw and the proposers. They were all essentially the same and we here print a composite.

The diagram already displays construction lines MC and ND, both perpendicular to AB. In the triangles AMB, ANB, the angles AMB, ANB are equal as they stand on the same chord AB. Then the area of the triangle AMB is ½ $AM.BM \sin \angle AMB$ = ½ $AB.MC$. Similarly, the area of the triangle AND is ½ $AN.BN \sin \angle ANB$ = ½ $AB.ND$. Hence

$$\frac{AM.BM}{AN.BN} = \frac{MC}{ND}$$

But the triangles *PMC*, *PND* are similar, and so this ratio equals $\dfrac{PM}{PN}$ as required.

SOLUTION TO PROBLEM 28.2.2

This problem has an interesting history. The journal *School Science and Mathematics* posed the problem of simplifying the expression

$$\left\{\sqrt[3]{3+\sqrt{5}}+\sqrt[3]{3-\sqrt{5}}+\sqrt[3]{2}\right\}^{-1}$$

They received three different answers:

$$A=\frac{1}{2}\left\{\sqrt[3]{2}\left[\sqrt[3]{1+\sqrt{5}}+\sqrt[3]{1-\sqrt{5}}\right]\right\}$$
$$B=\frac{1}{2}\left\{\sqrt[3]{14+6\sqrt{5}}+\sqrt[3]{14-6\sqrt{5}}-\sqrt[3]{6+2\sqrt{5}}-\sqrt[3]{6-2\sqrt{5}}\right\}$$
$$C=\frac{1}{10}\left\{\sqrt[6]{3+\sqrt{5}}-\sqrt[6]{3-\sqrt{5}}\right\}\left\{\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}\right\}\sqrt{5}.$$

all of which are correct, and so equal to one another. They wondered if a more direct proof was possible.

We received solutions from Šefket Arslangić (Bosnia), John Barton, Julius Guest and David Shaw. Shaw's came closest to providing a direct proof, although it may not strike readers as being simpler than the proof via the original problem. But here it is.

Begin by considering A.

Note that $\left(1+\sqrt{5}\right)^2=6+2\sqrt{5}$, from which it follows that

$$1+\sqrt{5}=\sqrt{6+2\sqrt{5}} \text{ and that } \sqrt[3]{1+\sqrt{5}} = \sqrt[6]{6+2\sqrt{5}}.$$

Similarly, $\sqrt[3]{1-\sqrt{5}} = -\sqrt[6]{6-2\sqrt{5}}$.

Therefore

$$\sqrt[3]{1+\sqrt{5}} + \sqrt[3]{1-\sqrt{5}} = \left(\sqrt[6]{3+\sqrt{5}} - \sqrt[6]{3-\sqrt{5}}\right)\sqrt[6]{2}$$

and so

$$A = \frac{1}{\sqrt{2}}\left(\sqrt[6]{3+\sqrt{5}} - \sqrt[6]{3-\sqrt{5}}\right).$$

Next consider C.

$$\left(\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}}\right)^2 = 6 + 2\sqrt{4} = 10,$$

so $\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}} = \sqrt{10}$. Likewise $\sqrt{3+\sqrt{5}} - \sqrt{3-\sqrt{5}} = \sqrt{2}$.

Therefore

$$C = \frac{1}{10}\left(\sqrt[6]{3+\sqrt{5}} - \sqrt[6]{3-\sqrt{5}}\right)\sqrt{5}.\sqrt{10}$$
$$= \frac{1}{\sqrt{2}}\left(\sqrt[6]{3+\sqrt{5}} - \sqrt[6]{3-\sqrt{5}}\right) = A.$$

Now $\sqrt{3+\sqrt{5}} + \sqrt{3-\sqrt{5}} = \sqrt{5}\left(\sqrt{3+\sqrt{5}} - \sqrt{3-\sqrt{5}}\right)$, so C may be rewritten:

$$C = \frac{1}{10}\left\{\sqrt[6]{3+\sqrt{5}} - \sqrt[6]{3-\sqrt{5}}\right\}\left\{\sqrt{3+\sqrt{5}} - \sqrt{3-\sqrt{5}}\right\}\sqrt{5}.\sqrt{5}$$
$$= \frac{1}{2}\left\{\sqrt[6]{3+\sqrt{5}} - \sqrt[6]{3-\sqrt{5}}\right\}\left\{\sqrt{3+\sqrt{5}} - \sqrt{3-\sqrt{5}}\right\}$$

Now, by expanding, we obtain B because, for example,

$$\sqrt[6]{3+\sqrt{5}}.\sqrt{3+\sqrt{5}} = \left(3+\sqrt{5}\right)^{1/6}.\left(3+\sqrt{5}\right)^{1/2} = \left(3+\sqrt{5}\right)^{2/3} = \sqrt[3]{14+2\sqrt{5}}$$

and

$$\sqrt[6]{3+\sqrt{5}}.\sqrt{3-\sqrt{5}} = \left(3+\sqrt{5}\right)^{1/6}.\left(3-\sqrt{5}\right)^{1/6}\left(3-\sqrt{5}\right)^{1/3}$$
$$= 4^{1/6}\left(3-\sqrt{5}\right)^{1/3} = 2^{1/3}\left(3+\sqrt{5}\right)^{1/3} = \sqrt[3]{6+2\sqrt{5}}$$

Thus $A = B = C$.

Mr Shaw remarks that the index notation helps to keep track of the multiplications.

Readers may also like to check the derivations of the three different answers. The details are to be found in the October 2003 issue of *School Science and Mathematics* on pp 309 and 310.

SOLUTION TO PROBLEM 28.2.3

This problem, from Angela Dunn's *Mathematical Bafflers*, asked for the integral solutions of the equation $x^2 = \dfrac{y^2}{y+4}$.

We received solutions from Šefket Arslangić (Bosnia), John Barton, Julius Guest and David Shaw. Here is Barton's.

Substitute $y = \eta - 4$, and rewrite the equation as $x^2 - \eta + 8 = \dfrac{16}{\eta}$.

The left-hand side of this new equation is integral, while the right is integral if $\eta = \pm 1,\ \pm 2,\ \pm 4,\ \pm 8$ or ± 16. Check these ten possibilities against the rewritten equation to find five solutions:

$$(x,\eta) = (3,1),\ (-3,1),\ (0,4),\ (8,16) \text{ or } (-3,16)$$

and so

$$(x,y) = (3,-3),\ (-3,-3),\ (0,0),\ (8,12) \text{ or } (-3,12).$$

SOLUTION TO PROBLEM 28.2.4

This problem came from the same source, and asked for integers A and B such that both solutions of both the equations $x^2 + Ax + B = 0$ and $x^2 + Bx + A = 0$ were also integral.

Solutions were received from Bernard Anderson, John Barton, Julius Guest and David Shaw. Here is Anderson's.

There are three cases: (1) A (or B) is zero, (2) Both A and B are positive, (3) B (or A) is negative.

In Case (1), either $B = 0$ or $B = -n^2$ (not counting imaginary numbers as integers). Or we could have $B = 0$ and $A = -n^2$.

In Case (2), suppose without loss of generality that $A \leq B$, and consider the two subcases (2a) $A < B$ and (2b) $A = B$.

Note that in both these subcases, both the roots of both quadratics must be negative. Call the roots of the first quadratic $-r$ and $-s$, and those of the second quadratic $-t$ and $-u$, where r, s, t, u are all positive. Then $r + s = A = tu$ and $rs = B = t + u$.

Now in Case (2a) we must have $tu < t + u$. But now if both t and u exceed 1, then $A = tu \geq 2\max(t,u) \geq t + u = B$, contrary to hypothesis. Thus either u or t equals 1. Suppose without loss of generality that $u = 1$. Then $A = t = r + s = B - 1 = rs - 1$. I.e. $rs = r + s + 1$. In this equation, neither r nor s can equal 1. Nor can both exceed 2, for if this were the case we would have $rs \geq 3\max(r,s) > r + s + 1$, which again is not possible. Thus one or other, r say, must equal 2. In this case, $s = 3$, and so we have $A = 5$, $B = 6$. (Had we assumed that B was the smaller of the two, then we would have found $A = 6$, $B = 5$.)

Turning now to Case (2b), and using a similar analysis, we find $A = rs = r + s$, from which it follows that $\frac{r}{s} = r - 1$, and so $\frac{r}{s}$ must be integral. This shows that $r = ks$ for some integer k, and $k + 1 = ks$. But then $s = 1 + \frac{1}{k}$ and this must be integral. Hence $k = 1$. So $s = r$, and $r^2 = 2r$, and thus $r = 2$. This gives us another solution: $A = B = 4$.

Turn now to Case 3, and, without loss of generality, set $B < 0$. The roots of the first quadratic are therefore opposite in sign. Call them $-r$ and s, where r and s are positive. Then $A = r - s$ and $B = -r s$. We now consider two subcases (3a) $A < 0$, and (3b) $A > 0$.

In Case (3a), the roots of the second quadratic must be of opposite sign. Call them $-t$ and u. As both A and B are negative in this subcase, we may without loss of generality suppose that $-A \geq -B$. But $-A = ut$ and $-B = u - t$. This implies that $u(t+1) \leq t$, which is impossible. Thus Case (3a) yields no solutions.

Case (3b) now tells us that the second quadratic has roots of the same sign, indeed both positive. Call them t and u. Then $B = -(t+u)$ and $A = tu$. But we found above that $A = r - s$ and $B = -rs$. So $r - s = ut > 0$ and $rs = u + t$. If now $t \geq 2$, then $tu \geq t + u$, and thus $r - s \geq rs$, or $s(r+1) \leq r$, which is impossible. Thus $t = 1$. It follows that $B = -(A+1)$. The first quadratic becomes $x^2 + Ax - (A+1) = 0$, with roots 1 and $-A-1$. The second becomes $x^2 - (A+1)x + A = 0$, with roots 1 and A. Had we assumed that A was negative (rather than B), we would have reached the solution $A = -(1+B)$. Represent the integers A and B in these last expressions by n.

Collecting all our different solutions gives the following list:

$$(A, B) = (0, 0),\ \left(0, -n^2\right),\ \left(-n^2, 0\right),\ (5, 6),\ (6, 5),\ (4, 4),\ (n, -1-n)\ \text{or}\ (-1-n, n).$$

SOLUTION TO PROBLEM 28.3.1

This problem was badly misprinted when first posted. A correction was published in our last issue. What the problem should have asked for was a proof that all the inflection points of the curve $y = \sin\left(\frac{1}{x}\right)$ lie on the curve $y^2 = \dfrac{4x^2}{1+4x^2}$

Three correspondents (Šefket Arslangić, Derek Garson and Joseph Kupka) pointed out the existence of an error. John Barton tactfully did not, but sent us nonetheless a solution of the correctly stated problem. We follow it here.

It is a necessary condition for the existence of an inflection point is that $\frac{d^2y}{dx^2}=0$ at the point in question. Now if $y=\sin\left(\frac{1}{x}\right)$ then

$$\frac{dy}{dx}=-x^{-2}\cos\left(\frac{1}{x}\right)$$

and

$$\frac{d^2y}{dx^2}=2x^{-3}\cos\left(\frac{1}{x}\right)-x^{-4}\sin\left(\frac{1}{x}\right)$$

The zeroes of this expression occur where

$$2x^{-3}\cos\left(\frac{1}{x}\right)-x^{-4}\sin\left(\frac{1}{x}\right)=0, \tag{1}$$

that is to say where $\tan\left(\frac{1}{x}\right)=2x$, an equation with infinitely many solutions, all having magnitude less than 1. Put $\sin\left(\frac{1}{x}\right)=y$, so that $\cos\left(\frac{1}{x}\right)=\pm\left(1-y^2\right)^{1/2}$ and from Equation (1)

$$\pm 2x^{-3}\left(1-y^2\right)^{1/2}-x^{-4}y=0,$$

$$y=\pm 2x\left(1-y^2\right)^{1/2}, \quad y^2=4x^2\left(1-y^2\right), \quad y^2=\frac{4x^2}{1+4x^2}$$

This is the answer (that should have been) sought, but Barton went on to point out that as $\sin\left(\frac{1}{x}\right)$ (for all non-zero x) is an odd function, then so is its second derivative, and thus the locus of the points of inflection may be more exactly stated as

$$y = \frac{2x}{\left(1+4x^2\right)^{1/2}}$$

for $x \neq 0$.

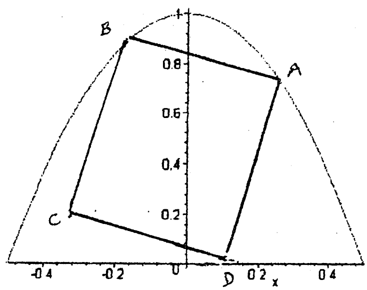
For completeness, it should be recorded that the proposer (who, of course, was not spooked by *Function's* misprints) also sent in a correct answer.

SOLUTION TO PROBLEM 28.3.2

The problem (proposed by Keith Anker) sought the maximal area of a rectangle contained entirely between the x axis and the parabola $y = 1 - 4x^2$.

No complete solution was received, although the proposer noted that the problem is surprisingly difficult. See the diagram below.

It is easy to show that the points A, B, D must lie on the boundaries of the region, as illustrated. Now let



$$A = (x_1, y_1) = \left(x_1,\ 1-4x_1^2\right),$$
$$B = (x_2, y_2) = \left(x_2,\ 1-4x_2^2\right).$$

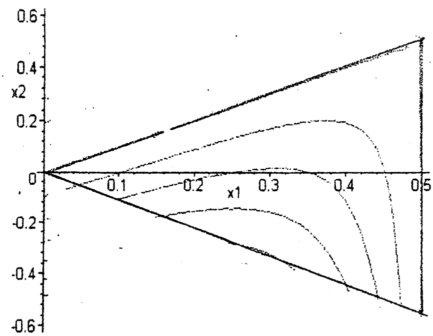
The slope of the line AB is then readily shown to be $4(x_1 + x_2) = \tan\alpha$, say.

Then the length of AB is $(x_1 - x_2)\sec\alpha$. Similarly, the length of AD is $y_1 \sec\alpha$. So the area of the rectangle $ABCD$ is

$$(x_1 - x_2)y_1 \sec^2\alpha = (x_1 - x_2)\left(1+4x_1^2\right)\left\{1 + 4(x_1 + x_2)^2\right\}.$$

This is the function that needs to be maximised. There are constraints on the values of x_1, x_2 in that $x_1 \leq \frac{1}{2}$ and we may assume without loss of generality that $x_1 \geq 0$ and $x_1 \geq x_2 \geq -\frac{1}{2}$. More subtly we may use symmetry considerations to restrict consideration to the region in which $x_1 \geq -x_2$.

A contour map of the area function was produced using MAPLE. Here it is.



The absence of closed contours shows that no maximum occurs inside the region of interest. It follows that the required maximum is to be found on a boundary. The boundaries are $x_1 = \frac{1}{2}$, $x_2 = \pm x_1$. On two of these boundaries, the area works out to be zero. The other is the boundary $x_2 = -x_1$. Substituting this into the expression for the area gives

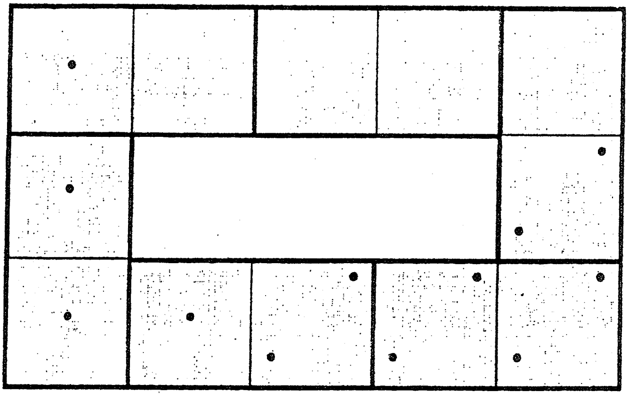
$$\text{Area} = 2x_1\left(1 - 4x_1^2\right)$$

This may be analysed by means of simple calculus. The maximum occurs when $x_1 = \frac{1}{2\sqrt{3}}$, giving an area of $\frac{2}{9}\sqrt{3}$.

It may be remarked that the condition $x_2 = -x_1$ corresponds to the case in which the vertex C lies on the x-axis. Many people would assume that this would give the required maximum but, as the proposer remarked, proving it is quite hard (unless someone can come up with a simpler proof).

SOLUTION TO PROBLEM 28.3.3

This problem was sent by Paul Grossman. Let us define a *Domino set of rank n* as a set of tiles, the rectangular faces of which are separated into two squares, each marked with dots representing numbers from zero to *n*, such that no two tiles contain the same pair of numbers and all combinations of pairs are represented.



The figure shows a set of rank 2 laid out in a closed chain. The contacting squares on adjoining tiles have matching numbers and each tile was placed in the clockwise direction at the end of the previous tile, either in the same direction or at right angles. Now:

1. Prove that a closed chain with the above conditions can be established with a set of rank 6 (the standard domino set) but not with sets of rank 3, 4 or 5.

2. Show what ranks will allow a continuous chain to be formed with matching numbers on adjoining squares and tiles placed at the end of the previous tile.

We received solutions from Derek Garson and the proposer. Here is the proposer's.

The solution to Question 2 will answer Question 1. Two conditions must be fulfilled:

> **(a)** The numbers on the adjoining squares on the last and first tiles must match.
> **(b)** The last tile placed must run into the end or side of the first.

Condition (a)

For this purpose let us consider the tiles to be lines joining numbered points. Tiles with repeated numbers are of no interest; they represent lines that start and finish at the same point. We are thus left with $\frac{n(n+1)}{2}$ lines with all the combinations of pairs of points at their ends. The problem may be envisaged as choosing $n + 1$ points on a circle, constructing a polygon by consecutively joining neighbouring points and then drawing all the diagonals without lifting the pencil.

For every line entering a point another line must be leaving. Each corner of our polygon is joined to n other points; therefore n must be even and the polygon has an odd number of corners. It remains to show that for every such polygon we can find at least one path to get back to the starting point after visiting all points repeatedly. If $n + 1$ is a prime number, we can join successive points until we reach the starting point again, then go round twice in steps of 2 and proceed until we have returned after steps of $n/2$. If $n + 1$ is not prime, we proceed in the same manner until we come to a factor q of $n + 1$, then use steps of $n + 1 - q$ and resume with $q + 1$.

Thus the condition is fulfilled if and only if the rank n is even.

Condition (b)

This condition can be fulfilled only if the number of tiles N is even. To demonstrate this, consider the Cartesian co-ordinates of the centre of each tile's first square. Let the origin be the centre of the first tile's first square and let the length of a tile be two units. As each successive tile is added to the end of the previous one, one co-ordinate remains constant while the other increases or decreases by 2. After N such steps, we must reach (0, 0) again, which means that there must be as many steps resulting in positive changes in x as in negative changes. Thus the total number of tiles oriented in the x-direction is even and so is the number oriented in the y-direction.

An even number is necessary but not sufficient. Two tiles cannot form a chain. However, four can and it is readily seen that if the condition is fulfilled for N, it must be fulfilled for $N + 2$. Any tile may be severed from its neighbour and rotated by 90°; two new tiles may then be added leading back to the original neighbour.

A set of rank n has N_n tiles where

$$N_n = \frac{n(n+1)}{2} + (n+1) = (n+1)\left(1 + \frac{n}{2}\right)$$

The reader may now verify that, to obtain an even N_n when n is even as required by Condition (a), n must not be a multiple of 4.

SOLUTION TO PROBLEM 28.3.4

This problem, posed by Šefket Arslangić (Bosnia), considered three non-negative numbers x, y, z for which $x^2 + y^2 + z^2 + 2xyz = 1$, and sought a proof that

$$x^2 + y^2 + z^2 \geq 3/4.$$

We received proofs from John Barton (2 proofs), Julius Guest, Joseph Kupka, David Shaw (2 proofs) and the proposer, who also sent two. Here is the first of these two, which is the same as Barton's second.

Let $A = x^2 + y^2 + z^2$ and $B = 2xyz$. By the inequality connecting the arithmetic and the geometric means[1]

$$\frac{x^2 + y^2 + z^2}{3} \geq \sqrt[3]{x^2 y^2 z^2}$$

so that $\left(\frac{x^2 + y^2 + z^2}{3}\right)^3 \geq x^2 y^2 z^2$ or $4\left(x^2 + y^2 + z^2\right)^3 \geq 27 \times 4x^2 y^2 z^2$.

This tells us that $4A^3 \geq 27B^2$. But the data tell us that $A + B = 1$, so that $4A^3 \geq 27(1 - A)^3$. This means that $4A^3 - 27A^2 + 54A - 27 \geq 0$. We can factorise the left-hand side of this equation and so obtain

$$4\left(A - \frac{3}{4}\right)(A - 3)^2 \geq 0,$$

so that $A \geq \frac{3}{4}$ as required.

Professor Arslangić notes that equality holds if $x = y = z = \frac{1}{2}$ and indeed it may be shown that this is the only case of equality.

SOLUTION TO PROBLEM 28.4.1

The problem (from *Australian Senior Mathematics Journal*) asked for all solutions of the equation

$$\left(x^2 - 5x + 5\right)^{x^2 - 9x + 20} = 1.$$

Despite the very short time available, our regular correspondent Julius Guest sent us a solution. There are two very obvious ways in which the desired equality may be achieved. The first is to set

$$x^2 - 9x + 20 = 0.$$

[1] See *Function*, *Vol* **8**, *Part* **1**, p 15.

This entails $(x-4)(x-5)=0$, so that $x=4$ or 5. Then $x^2-5x+5=1$ or 5 and, as both 5^0 and 1^0 are equal to 1, we have two solutions. Alternatively, we could set $x^2-5x+5=1$, which implies that

$$x^2-5x+4=0.$$

This entails $(x-4)(x-1)=0$, so that x = 4 or 1. So x = 1 is also a solution.

More subtly, we could set $x^2-5x+5=-1$, which implies that

$$x^2-5x+6=0.$$

This entails $(x-2)(x-3)=0$, so that $x=2$ or 3. Then

$$x^2-9x+20=6 \text{ or } 2$$

and both $(-1)^6$ and $(-1)^2$ are equal to 1 and so we have two more solutions. The question arises as to whether there might not be yet others. This could perhaps be achieved if both

$$\begin{aligned} x^2-5x+5 &= \varpi, \\ \text{and } x^2-9x+20 &= n, \end{aligned}$$

where ϖ is a complex number, and n an integer, such that $\varpi^n=1$. There are in fact no further solutions of this type, although we leave the proof to the reader. So the complete list of solutions is 1, 2, 3, 4, 5.

We may remark in passing that the source offered no solution although it implied that it is easy to miss possibilities. They also claimed that the solution was easier than it looked. It isn't quite clear what they meant by this!

PARTIAL SOLUTION TO PROBLEM 28.4.2

The source (*Mathematical Bafflers*, Ed Angela Dunn) asked for all pairs of rational numbers (x, y) such that $x^y = y^x$.

The following partial solution follows that given in the source.

Set $y = xr$, where r is necessarily rational. Then $x^{xr} = (xr)^x$, so that $x^{x(r-1)} = r^x$ and $x^{r-1} = r$. Therefore either $r = 1$ or else $x = r^{1/(r-1)}$. The first possibility yields the trivial solutions $x = y$, the other is more subtle. To be sure to have x rational when r is rational, set $1/(r-1)$ integral. Call it n say. Then $x = \left(1+\frac{1}{n}\right)^n$ and $y = \left(1+\frac{1}{n}\right)^{n+1}$, where $n \neq 0$ or -1. In fact the negative values of n are unnecessary, as they merely regenerate the same set of solutions as the positive values, but with x and y interchanged. [The restriction to positive n corresponds to making the convention that $y > x$.] Notice that as $n \to \infty$, both x and y tend to the irrational number e.

[*While this gives an infinite set of solutions, it does not however prove that these are the only ones! Eds*]

SOLUTION TO PROBLEM 28.4.3

Vasile Berinde (in *Exploring, Investigating and Discovering Mathematics*) asked for all prime numbers n such that $n + 4$ and $n + 8$ are both also prime.

The following solution is based on that given in the source.

Clearly n must be odd. So put $n = 2k + 1$ $(k \geq 0)$. Then k must have one or another of the forms: $k = 3p$, $k = 3p+1$ or $k = 3p-1$. If $k = 3p$, then $n+8 = 6p+9$, which is not prime. If $k = 3p+1$, then $n = 6p+3$, which is prime only if $p = 0$. This yields the triple 3, 7, 11, as a solution to our problem. If $k = 3p-1$, then $n+4 = 6p-3$, which is prime only if $p = 1$, but in this case $n = -1$, which is not prime. There is thus only one possible case: 3, 7 and 11.

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Note that a Cumulative Index to the first 20 volumes of *Function* was included in our Special Issue of 1996. Further indexes were published in the October issues for 1997, 1998, 1999, 2000, 2001 and 2003. Between them, these indexes cover all 28 years of *Function's* life.

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