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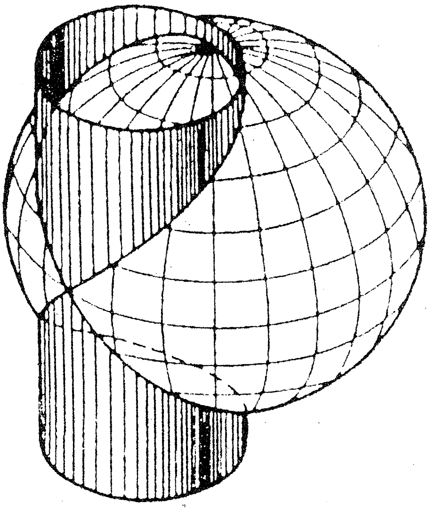

School of Mathematical Sciences – Monash University

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Function is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. It was founded in 1977 by Prof G B Preston, and is addressed principally to students in the upper years of secondary schools, but also more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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* $17 for *bona fide* secondary or tertiary students.

THE FRONT COVER

The front cover for this issue comes from *The Australian Mathematics Teacher* where it appeared in their issue for March 2002. It formed part of an article by Paul Scott and Peter Brinkworth and it appears with the kind permission of these authors and of the editors of *The Australian Mathematics Teacher.*

It depicts a curve called "Viviani's Curve" and the manner of its construction. A sphere of radius a intersects a cylinder of diameter a, so placed that a diameter of the cylinder coincides with a radius of the sphere. The resulting intersection between the two surfaces takes the form of a twisted figure-8, and this is what is known as Viviani's Curve.

Viviani, after whom it is named, was Vincenzo Viviani (1622-1702). He was a member of a circle of companions and pupils of Galileo and his best-known disciple, Torricelli. Indeed he succeeded Torricelli as lecturer in the Academy of Design in Florence. He remained in Florence under the patronage of the Grand Duke of Tuscany, who appointed him as his mathematician, and stayed in this post despite offers from the kings of France and Poland to leave it and enter their service instead.

He is perhaps best remembered as the experimenter who first produced an accurate measure of the velocity of sound. As this might perhaps suggest, his interests tended to concentrate on engineering and architecture. The curve that bears his name may be said to follow from this latter study, as he took an interest in the construction of domes.

If we take the radius of the sphere, a, as 2, which we may do without any real loss of generality, then we may assign to the sphere the equation

$$x^2 + y^2 + z^2 = 4.$$

The cylinder may then be assigned the equation

$$(x-1)^2 + y^2 = 1$$

in the same co-ordinate system, and these two equations together serve to define the curve.

Alternatively we may give a set of three equations. Any point on the curve will also be a point on the sphere. Suppose its latitude on the sphere is φ. Then the co-ordinates x, y, z of any point on the curve may be given in terms of φ by the equations

$$x = 1 + \cos 2\varphi$$
$$y = \pm \sin 2\varphi$$
$$z = \sin \varphi.$$

A (very brief) account of Viviani's life may be found at

http://www-groups.dcs.st-and.ac.uk/history/Mathematicians/Viviani.html

and accounts of the curve and some generalisations of it at

http://mathworld.wolfram.com/VivianisCurve.htm

and

http://mathworld.wolfram.com/Cylinder-SphereIntersection.htm

A HINGED DISSECTION OF A TRIANGLE TO A SQUARE

M J Englefield, Monash University

Can you cut up a triangle into pieces that can be rearranged to form a square? A hinged dissection also requires that the rearrangement can be achieved by rotating the pieces about chosen points (the hinges) of the triangle. The recent book "Hinged Dissections" (CUP, 2002) by G.N.

Fredericksen discusses such problems. It begins with the figure below illustrating the conversion of an equilateral triangle to a square.

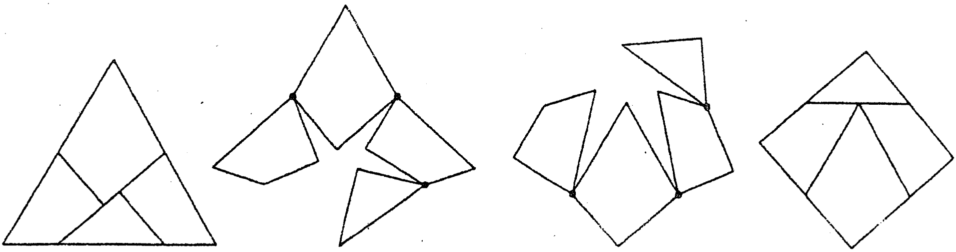
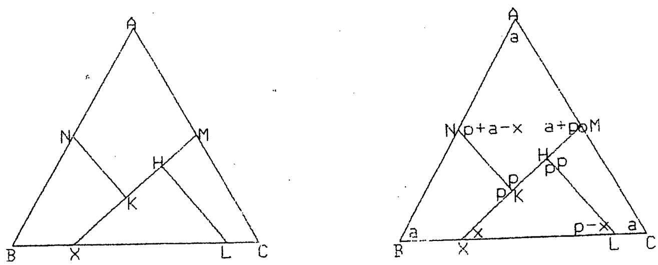


Figure 1

The angles and lengths required for this example were not given, but a construction is given on p.24 of the book *Mathematical Models* by Cundy and Rollett (2nd edition, OUP, 1961). However, even there "the formal proof is left to the reader".

This note presents a determination of the quantities, which reveals a generalisation from square to rectangle. In Figure 2(a) the points of the triangle shown in Figure 1 are labelled. We assume that *N*, *M* are the midpoints of *AB* and *AC*, that *X* and *L* are arbitrary points on *BC*, and that *K* and *H* are obtained by requiring that *NK* and *LH* are perpendicular to *XM*. Once *X* is chosen, angle *MXC* (= *x*, say) determines all other angles as in Figure 2(b), where $p = 90°$, $a = 60°$. When *L* is also chosen, all other lengths can be expressed in terms of *x* and *XL*.



Figures 2(a), 2(b)

Now consider the stages in Figure 1 corresponding to the three successive rotations, about hinges at *M*, *N* and *L*. These hinge points are marked by small circles in Figures 2(b), 3 and 4 respectively.

(i) Cut along *XM*, rotate *XMC* anticlockwise through 180° about *M*. In Figure 3, the label 2 denotes the new positions of points that have moved. Because $AM = MC$, *C*2 and *A* coincide ; lengths between two moved points are unchanged, e.g. $|C2\_X2| (= AX2) = CX$. The two right angles at *H* have moved to $H2$. The four right angles in Figure 2 have to end up as the angles of the square.

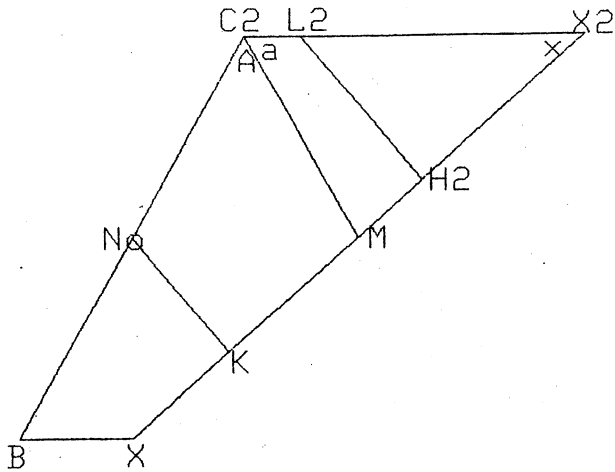


Figure 3

(ii) In Figure 3, cut along *NK*, rotate *NKXB* clockwise through 180° about *N*. In Figure 4, the label 3 denotes the new positions of *B*, *X* and *K*. Because $AN = BN$, *B*3 and *A* coincide. One of the two right angles that were at *K* has moved to $K3$. The three 60^o angles at *A* show that $X3$ lies on the straight line A_X2.

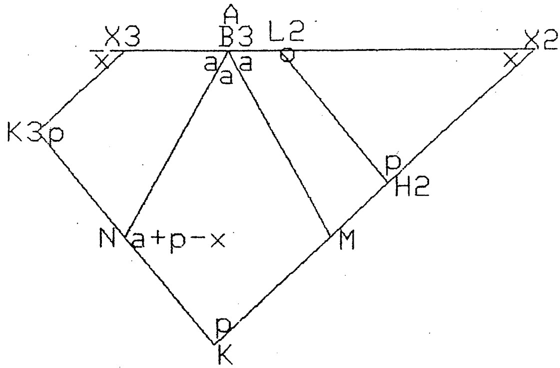


Figure 4

(iii) Finally, cut along $L2\_H2$, rotate $L2\_X2\_H2$ anticlockwise through 180° about $L2$, to get Figure 5. The new positions of $H2$ and $X2$ are labelled $H4$ and $X4$. Figure 5 is evidently not a square, so the procedure suggested by Figure 1 has failed. However Figure 5 does have four right angles, and would be a rectangle if $X3$ coincided with $X4$.

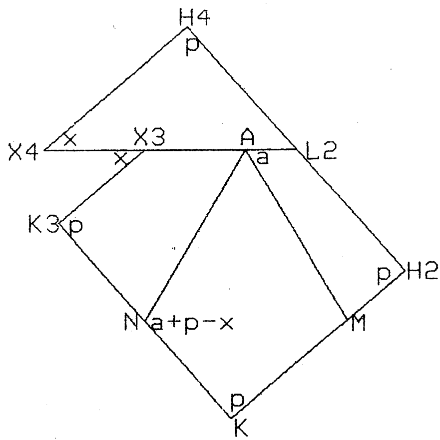


Figure 5

Figure 5 can be altered by re-choosing the point L and the angle x in Figure 2. The condition $X4 = X3$ can be traced back through the figures:

	$\lvert L2\_X4\rvert = \lvert L2\_X3\rvert$ in Figure 5
is	$\lvert L2\_X2\rvert = AX3 + AL2 = \lvert B3\_X3\rvert + AL2$ in Figure 4,
which is	$\lvert L2\_X2\rvert = BX + \lvert C2\_L2\rvert$ in Figure 3,
giving	$LX = BX + CL$ in Figure 2(b).

Thus $LX = BC - LX$, $BC = 2(LX)$, which is the final form of the condition for a rectangle. Then Figure 6 is obtained from Figure 2.

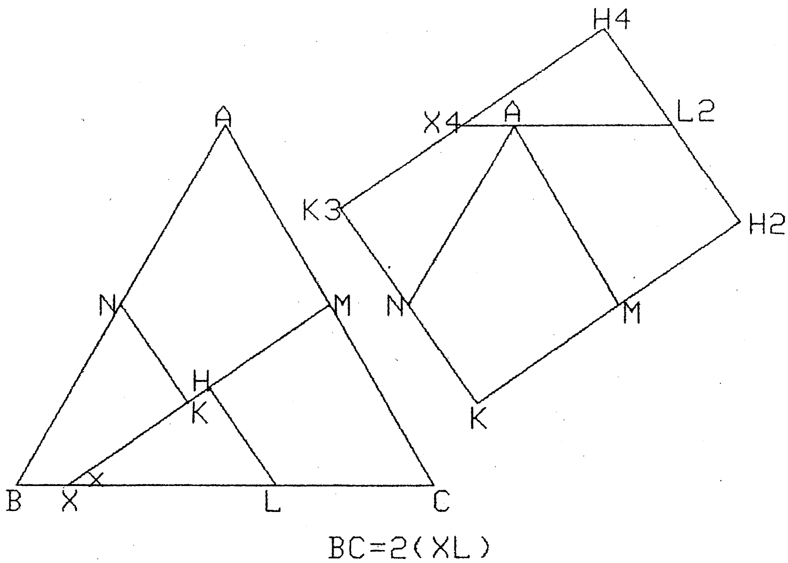


Figure 6

The hinged dissection in Figure 6 requires only that $XL = \frac{1}{2}BC$, so X can be any point from B to the midpoint of the base of the triangle. When (Figure 7) X and B coincide, so do K and H, and MB is perpendicular to AC. All acute angles are either 30° or 60°. Choosing units so the triangle has side 4, then $KL = NK = 1$, $BK = KM = \sqrt{3}$. The hinges, in order, are M, N, L. The sides of the rectangle are 2 and $2\sqrt{3}$ ($|K4\_K2|$ and $|K\_K2|$).

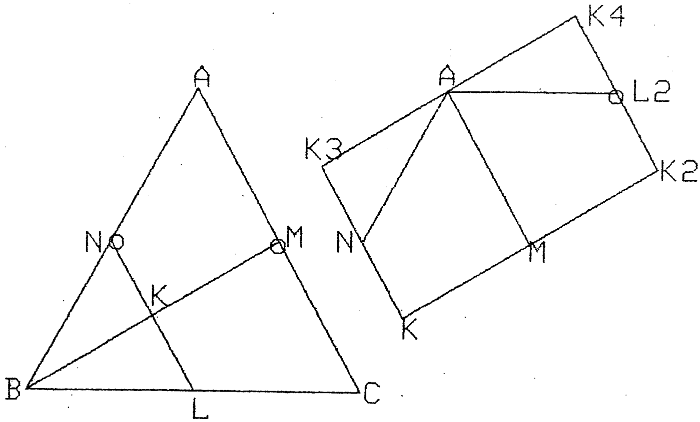


Figure 7

In all cases the rectangle and the triangle have the same area which, from the triangle, is $\frac{1}{2}$(base)(height) $= \frac{1}{2}(BC)\frac{1}{2}(BC)\sqrt{3}$.

In the other extreme case (Figure 8, overleaf), L and C coincide, as do K and H. The extreme cases are treated here as in the general case of Figure 6, but their dissections can be made in one operation (use B as hinge in Figure 7 or C as hinge in Figure 8).

Figures 6, 7 and 8 , showing how the rectangle changes as X moves along BC, suggest there will be one position where the rectangle becomes a square. The side of the square must be $\sqrt{(\text{area})} = \frac{1}{2}(BC)(3)^{1/4}$.

The construction given by Cundy and Rollett states that XM is the length of the side of the square. This extends to the rectangular case. From Figure 6, one side of the rectangle is $XK + XH$ or alternatively $MK + MH$. The equality of these opposite sides follows from the congruence of the triangles NKM and LHX. (They are congruent because $XL = \frac{1}{2}BC = NM$, and have equal angles as XL and NM are parallel.) Hence $XH = KM$, and the length of the side of the rectangle is $XK + XH = XK + KM = XM$. A construction (by compass with a circular arc of radius MX) of X from M can therefore be used to get any rectangle with a given required side, provided this is between the limiting cases in Figures 8 and 7, i.e. $AN \le XM \le BM = BN\sqrt{3}$.

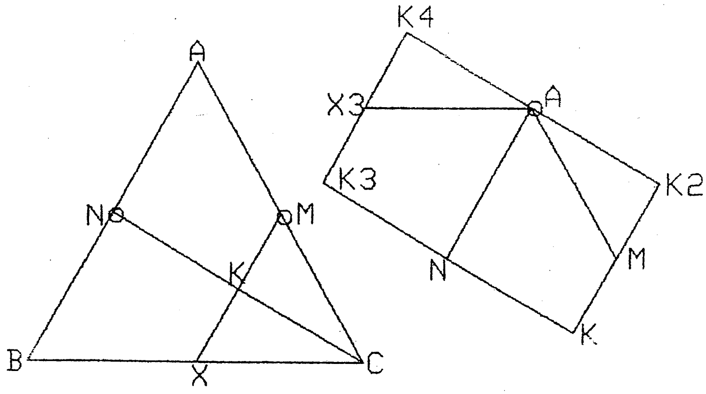


Figure 8

The angle x is related to the size of the rectangle via (see Figure 6) $NK = HL = XL\sin x$, so that one side of the rectangle is $NK + NK3 = 2HL = 2XL\sin x = BC\sin x$. The triangle has area $\frac{1}{4}(BC)^2\sqrt{3}$, so the other side XM is $\left(BC\sqrt{3}\right)/4\sin x$. This also follows by applying the sine formula to triangle XMC: $\dfrac{XM}{\sin 60^\circ} = \dfrac{MC}{\sin x}$, $XM = \dfrac{AC\sqrt{3}}{4\sin x}$.

Equating the sides of the rectangle gives

$$\sin^2 x = \frac{1}{4}\sqrt{3}\text{, or }\cos 2x = 1 - 2\sin^2 x = 1 - \frac{1}{2}\sqrt{3}.$$

This determines $x = 41^\circ 9'$ for the square.

This completes the analysis of the problem illustrated in Figure 1. Many other such problems may be found in the book by Frederickson.

Drawing the figures (by computer package) required calculating coordinates for all points. This is a pleasant exercise in the use of vector algebra.

HISTORY OF MATHEMATICS

Boundary Layer Theory

Michael A B Deakin, Monash University

I want to tell the story of a remarkable breakthrough in Mathematics, one achieved not by a mathematician, but by an engineer. Even today, one hundred years after the event, and when the fruits of his insight have become a standard part of the language and methodology of Applied Mathematics, his biography does not appear on the extensive MacTutor website of mathematical biographies.

Before I get onto the man himself and his insights, however, it will help to look at a few mathematical preliminaries. Start with the quadratic equation

$$\varepsilon x^2 + x + 1 = 0, \tag{1}$$

where ε is a very small number. Because this is a quadratic equation, it has two roots, but because ε is very small, we might expect one of them to lie close to the root of the simpler equation $x + 1 = 0$. That is to say, one root must lie near -1. If we now think about the product of the roots ($1/\varepsilon$), we see that the second root is approximately $-1/\varepsilon$.

As an example, consider the case $\varepsilon = 0.05$. We expect the roots of our quadratic to be near -1 and -20. (The exact values are $-1.05\ldots$ and $-18.94\ldots$.)

The reason for this excursion into elementary Mathematics is to use the result to look at a more complicated equation

$$\varepsilon y'' + y' + y = 0, \tag{2}$$

where now y is a function of x and the prime represents differentiation with respect to x. Equation (2) may be solved with reference to Equation (1). In fact if the roots of Equation (1) are r_1 and r_2, then the solution of Equation (2) is

$$y = Ae^{r_1 x} + Be^{r_2 x}, \tag{3}$$

where A, B are constants. The values of A and B are usually determined from other data supplied, and a very common form is the specification of values for $y(0)$ and $y'(0)$.

In the case under discussion, we have $r_1 \approx -1,\ r_2 \approx -1/\varepsilon$, and I will suppose for illustrative purposes that the further data is

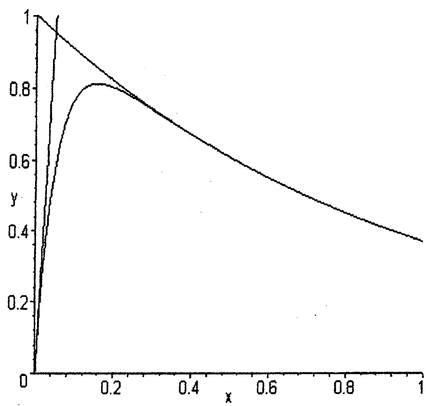
$$y(0) = 0,\ \ y'(0) = 1/\varepsilon.$$

To a good approximation, the solution is then

$$y = e^{-x} - e^{-x/\varepsilon}.$$

Even for moderate values of x, the second term of this expression is minute, and may safely be ignored, but when x is small, the terms are comparable in magnitude, and it is this that applies when $x = 0$, and the extra conditions were applied. For very small values of x, we have $y \approx x/\varepsilon$.

So, we can say that one approximation works when x is very small and another when x is moderate or large. The graph below (for $\varepsilon = 1/20$) shows the full approximate solution (3) and also these two further approximations, all on the one set of axes.



We see that the x-axis may be divided into three parts: (1) an "inner region", for which say $x < 0.01$, where $y \approx x/\varepsilon$, (2) "an outer region", for which say $x > 0.3$, where $y \approx e^{-x}$, and (3) a "transition region" linking the two. A fuller analysis can use better approximations to arrive at details of this third region, but this will not be pursued here.

The point to concentrate on is that if we had simply ignored the first term in Equation (2) and written

$$y' + y = 0, \tag{4}$$

(as we well might be tempted to do), we would reach the solution

$$y = Ae^{-x}, \tag{5}$$

which gives some information on the outer region, but none on the inner, and moreover leaves us with no way to evaluate the unknown constant A.

Something very like this difficulty faced early researchers in the field of Fluid Dynamics. This studies the motion of fluids (i.e. liquids and gases) and proceeds in terms of a set of complicated differential equations called the Navier-Stokes equations. (Production of a major advance toward the solution of these is one of the Clay Challenge Problems for which a prize of US$1 million awaits the successful researcher; see *Function*, April 2001.)

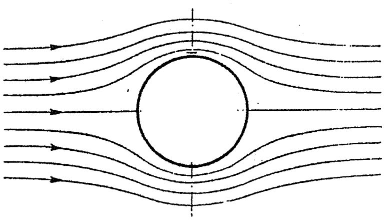
The Navier-Stokes equations are most complicated, and only a very few exact solutions are known. It is even today a major feat to solve them numerically in special cases of outstanding interest. Given this difficulty, it is not surprising that early researchers resorted to simplified versions that were more mathematically tractable. The idea was to neglect two features of real fluids, considering their effect to be negligible. The first of these was the compressibility of the fluid. It is realistic to neglect this for liquids and even for gases in many practical situations. The second is more problematical, but perhaps also more surprising. The idea is to neglect the effects of viscosity ("stickiness") in the fluid. If we think of water or air as the fluid of interest, then we hardly regard these as being "sticky".

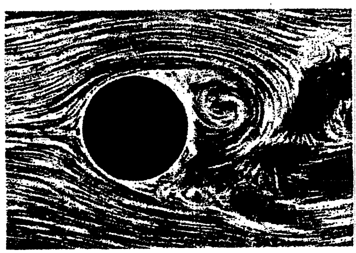
The incompressible, non-viscous fluid resulting from these assumptions is an imaginary construct called an "ideal fluid". Ideal fluids were much studied and there is a large body of successful theory resulting from that study. However, there were also problems.

Just as in the case of the differential equation (2), where the simplified variant (4) could not exhibit the full behaviour found in the more exact equation, so too the neglect of viscosity reduced the complexity of the Navier-Stokes equations by dropping out all the second derivatives. Exactly as in our earlier example, the difficulty arose when it was necessary to apply further conditions at the boundaries of the region being studied.

A real fluid in contact with a solid wall or other object must have the same velocity as that object at any point of contact. The viscosity ("stickiness") means that the layer of fluid immediately in contact with the boundary adheres to it and so shares its velocity. Ideal fluids are not restricted in this way. They may slip over a solid surface and the only condition to be satisfied is that the fluid cannot penetrate the solid object.

Although there were many successes arising from ideal fluid theory, there were also some spectacular failures. The diagrams below show one of these. To the left is the pattern of flow predicted by the theory when a stream of fluid flows round a fixed cylinder; to right is the reality as captured in a photograph. We see immediately that something has gone badly wrong!





Theoretical (left) and actual (right) flow patterns past a cylinder immersed in a stream of fluid.

Summarising this case, a more recent author (Harry L Evans, in *Laminar Boundary-Layer Theory*, 1968) finds six points of disagreement between theory and reality. Three are not visible in the picture on the previous page, but the others are quite evident. They are:

(1) the discrepancy already noted, the ideal fluid slides over the cylinder, whereas the real fluid does not;
(2) there is a pronounced wake behind the cylinder in the real case;
(3) the fluid in the wake is separated from the fluid outside it by means of a well-defined "separation surface".

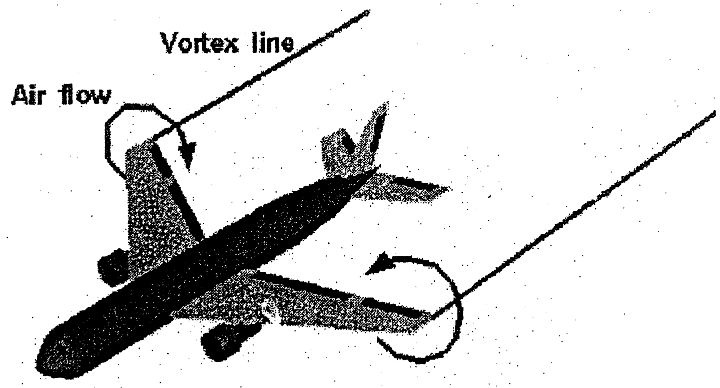
(We might be tempted to list a fourth difference also: the real flow is not symmetric about the centre-line. However, this is a problem with the experimental technique that produced the photo, which dates from about the time of World War I.)

The flow past a cylinder is itself an idealisation, but it was extensively studied because it gives great insight into a problem of great practical importance: the flow over an aircraft wing. Indeed the flow over an aerofoil is studied by means of a transformation of co-ordinates that makes the aerofoil appear circular.

An aeroplane can fly because the pattern of flow over its wings generates a lifting force that keeps the plane aloft. This flow is seen as the sum of two components: the lateral movement of the air past the wing, and a circulation around it. An aerofoil is so shaped that its upper surface is more sharply curved than its lower. The tendency of the air to "stick" to this upper surface means that it has a longer distance to travel in passing over the wing than does the air that passes below the wing. That is to say, it must go faster. This faster flow results (even in the case of an ideal fluid) in a lower pressure on the upper surface than on the lower, and it is this pressure difference that supplies the lift. The faster flow also supplies the circulation component of the total flow.

Now consider the situation of an aeroplane as it begins its take-off. Initially it is standing on the tarmac in air we will suppose to be still. The air thus has no angular momentum. The plane now starts its engines and moves forward, or if you like, the air moves back relative to the plane. As angular momentum is conserved, there should be no circulation around the wing, and hence no lift. And this argument would be quite true if the fluid were an ideal one. This is a result known as "Kelvin's Circulation Theorem" and it is another of Evans' discrepancies between ideal theory and reality. The difference between this prediction and the fact that flight is possible lies in the viscosity of the air. If air were not "sticky", planes could not fly! (The "prediction" that they cannot is known as the "d'Alembert Paradox".)

But this insight does not quite explain the apparent violation of the conservation of angular momentum. The way in which angular momentum comes to be conserved is quite subtle. The diagram below (adapted from the NASA website) shows what happens. At the end of each wing, a vortex is formed and trails behind the aircraft. Furthermore, although the diagram doesn't show it, back at the point of take-off, these two vortices connect up via a vortex that (initially) is equal and opposite to the vortex that surrounds the wings and which supports the plane.



There are thus four vortices associated with the plane in its flight. First there is the circulation over the wings, second and third are the trailing vortices shed at the wingtips, and finally there is the vortex left behind at the airport. This last vortex, however, decays fairly rapidly – destroyed by the very viscosity that established it in the first place! (However, the decay takes time; this is why a second plane, using the same runway, waits for a period until it can be sure of taking off into undisturbed air.)

So three of these four vortices accompany the plane in its flight. They are the circulation around the wings that provides the "lift" and supports the plane, and the two trailing vortices shed by the wingtips. The generation of these vortices and the need to "tow" them along leads to a "drag" on the plane. This drag is called the "induced drag", and it is a necessary evil – an unavoidable consequence of the lift that keeps the plane aloft.

There are thus several different regions of flow in the air around a plane in steady level fligh: a circulation around the wings; the trailing vortices shed from the wingtips; the undisturbed air away from the plane; and also transition regions between these various regimes.

All this was first considered by Ludwig Prandtl, who was born in Freising, Germany on 4 February 1875. An engineer by training, he was noted for his ability to bring physical insights into relatively simple mathematical form. He became professor of mechanics at the University of Hanover in 1901, and later (in 1904) he moved to the University of Göttingen, where he remained till his death in 1953.

It was in 1904 that his new concept of the boundary layer led him to the theory that I have just been summarising. Readers will note that the boundary layer concept applies not only in the immediate vicinity of boundaries, but also (as in the case of the trailing vortices) elsewhere, as a result of the need to satisfy physical laws.

As I remarked, he seems to have been rather ignored by historians of Mathematics. However, you will find some material at

http://www.eng.vt.edu/fluids/msc/prandtl.htm

but even there it is remarked that "Amazingly, there are very few discussions of Prandtl on the web. He is regarded as an engineer and never seems to make it into lists of physicists." To which we might well add "or of mathematicians".

There is also another website that readers may care to look at. I have relied on it in places in my account above. Go to

http://www.allstar.fiu.edu/aero/prandtl.htm

The interesting question to speculate about is quite what breakthrough would satisfy the judges of the Clay awards. Surely Prandtl would have qualified if the awards had been around in his day!

However, if you think that Prandtl has been unjustly neglected by history, then spare a thought for his student Blasius. (This is Paul Richard Heinrich Blasius, not to be confused with Paul Rudolf Heinrich Blasius, who was an ornithologist.) It was "our" Blasius who first produced a mathematical theory of the simplest possible boundary layer: that set up by a flat plate placed in the midst of a uniform fluid flow.

This leads to a quite complicated differential equation, now named after Blasius. There are plenty of informative websites about the equation and the flow, but they say little of the researcher himself. I have managed to learn that he was born in 1883, and was still alive at the time of Prandtl's death. He received his doctorate in 1917, for the work he did with Prandtl at Göttingen. However, that is about all I have been able to discover.

COMPUTERS AND COMPUTING

Algorithmic Complexity and beyond

Cristina Varsavsky

How to decide which is the best algorithm to perform a task? How do we measure the efficiency of an algorithm? What computer memory is required to implement an algorithm? These questions belong to the well develop field of "computational complexity". When we look at the time required by an algorithm to perform a task, we are analysing its "time complexity", and when we look at the computer memory required to run the algorithm, we are analysing its "space complexity". These two complexity dimensions are essential when algorithms are implemented, because it is important to know whether the algorithm will perform the task in one millisecond, a few hours, or many years, and at the same time, we need to know the computing facilities required to run the program.

In a previous *Function* article[1], I discussed the time complexity of an algorithm, which is directly related to its running time. The same algorithm will most likely run faster on a faster machine. Also, the speed of an algorithm may depend heavily on the particular programming language in which it is written. We saw that the running time is directly related to the number of operations performed during its execution and we illustrated this with an algorithm to search for a word in a list of words (a very common task when say, a telephone number is to be found in a list of bank customers). We showed two algorithms with different time complexities: naive search and binary search.

The naive search algorithm performs a sequential search: it simply goes through the list starting from the beginning, comparing each element with the search word until that word is found. How may operations are needed to perform this search? See the programs below. The operations involved are assignments (Steps 1 and 3), additions (Step 3), and comparisons (Steps 2 and 4). We will assume that assignments happen instantaneously, so we will only count the number of sums and comparisons. Obviously that number will very much depend on the position of the word we are searching for. Computer scientists and mathematicians usually look at the worst-case situation; in other words, they are mainly interested in an upper bound on the number of operations for a fixed input size. For this algorithm, the worst case occurs when we need to go to the end of the list (the search word was the last one, or was not in the list at all). By the time we get to the end of the list, we have made k comparisons in Step 2, k additions in Step 3, and k comparisons in step 4: a total of $3k$ operations.

Algorithm NaiveSearch
(Search for X in the list n(1), n(2), n(3), ..., n(k))

Step 1.	`Set i=1.`
Step 2.	`If n(i)=X then output i and stop.`
Step 3.	`Set i to i+1.`
Step 4.	`If i>k then output "X is not in the list" and stop.`
Step 5.	`Go to step 2.`

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[1] See *Function Vol* 19 *Part* 5.

On the other hand, the binary search algorithm performs this same search task more efficiently:

Algorithm BinarySearch
(Search for X in an ordered list n(1), n(2), ..., n(k))

Step 1. `Set first = 1 and last=k.`
Step 2. `Set mid =` *`floor`*`[(first+last)/2].`
Step 3. `If n(mid)=X then output mid and stop.`
Step 4. `If mid=first then go to step 7.`
Step 5. `If n(mid) precedes X then set first=mid+1 and go to step 2.`
Step 6. `Set last=mid-1 and go to step 2.`
Step 7. `Set mid=last.`
Step 8. `If n(mid)=X then output mid and stop.`
Step 9. `Output "X is not in the list" and stop.`

This algorithm compares the search word with the word in the middle of the list, and discards the half of the list not containing the search word. This technique is called binary search and works only on a sorted list (NaiveSearch doesn't need the list to be arranged in alphabetical order; but lists are usually built up in such a way). The *`floor`* function gives the highest integer less than or equal to its argument. Steps 2 to 6 define a loop. Each time the loop is executed we have, in the worst situation, a sum and a division in Step 2, a comparison in each of Steps 3, 4, and 5, and either an addition in Step 5 or a subtraction in Step 6: a maximum of 6 operations per loop. Next we need to determine the number of times the loop is executed. Each time through the loop we reduce the length of the search interval to a half of its previous value; therefore after the first pass we have at the most $k/2$ words left, after the second pass we have $k/4$, after the third $k/8$, and so on. So the maximum number of passes is a number j such that $k/2^j \leq 1$ or equivalently, $k \leq 2^j$. Taking $\log_2$ of both sides of this last inequality, we have $\log_2 k \leq j$. Thus, we make at most $\lceil \log_2 k \rceil$ passes through the loop ($\lceil \; \rceil$denotes the ceiling function, i.e. the smallest integer greater than or equal to its argument); with the comparison made outside the loop (step 8) the worst-case count of operations comes to $6\log_2 k + 1$.

We say that NaiveSearch has *linear* time complexity, while the complexity of BinarySearch is *logarithmic*. The time required to perform a search within a short list will not be very different, but if the list is large, then the difference will be significant. For example, if $k = 2{,}000{,}000$ (quite possible for when you need to find a telephone number in a bank customer list), the upper bound for NaiveSearch is 6,000,000 operations while for BinarySearch it is only 127 operations.

In the mid 1960's, Greg Chaitin, proposed to look at the computational complexity from a different perspective. His idea was not to analyse the *time* it takes for a computer program to produce a particular output, but to consider the *size* of the computer program, that is, the amount of information one needs to give a computer to perform a given task. His theory is known as *algorithmic information theory*. He developed it while he was still a teenager, inspired by a teacher of the programming course for talented students he attended while in high school, who encouraged the students to find the shortest programs to perform routine tasks.

Chaitin's theory gives a new way to grasp the mathematics of information used to describe the structures and objects of the world. The theory is applicable beyond the computer domain; it is also used to describe structures and phenomena of the physical world, in Biology, Music, Art, Business, etc.

As Chaitin himself recognises, his theory is hard to explain. In an attempt to do so, let us think of objects that can be described by binary strings. For example, think of the number π. In algorithmic information theory, the program-length complexity of π is defined as the shortest program (in bits) that will produce π as the output without any additional information. It turns out that this number is pretty small. There are short programs which will produce any required number of digits of π; the number π is an infinite sequence of seemingly random digits, but it contains only a few bits of information. The numbers e and $\sqrt{2}$ also contain only a few bits of information, because they can be produced by short programs. Chaitin showed that not all numbers are computable in the same sense as the numbers π, e and $\sqrt{2}$. He showed this with a number Ω which is a real number between 0 and 1. This number represents the halting probability of a universal Turing machine[2].

According to Chaitin, the most important application of his algorithmic information theory is not to measure the efficiency of algorithms, but to show the limits of mathematical reasoning. He proved that there is no algorithm for testing whether any algorithm is the shortest way to compress a piece of information. He argues that once you have settled on a programming language, there must be a program that does the job that is the shortest one; there might be several of the same minimum length, but surely there is one with a minimum length. However, you can never be sure whether a program that does the job *is* the most concise one, the shortest one; you know there is one, but you cannot point to it. He relates this finding to Gödel's *Incompleteness Theorem.* Gödel proved that any consistent set of axioms to describe arithmetic is incomplete, that is, it must contain statements that can neither be proved nor disproved within the system. Chaitin also relates his theory to Turing's *Halting Problem*, which in simple terms says that no computer program can say in advance if another computer program will eventually halt or not. Moreover, he uses his number Ω to show that there is randomness in pure mathematics!

Chaitin has published several books on his theory, its relation to the other two important incompleteness theorems (Gödel's and Turing's), and its mathematical implications. Some of them are very involved and require a sound background in Mathematics and Computer Science. But he also published for the non-experts; I found his book *Conversations with a Mathematician – Maths, Arts, Science and the Limits of Reason*[3] very stimulating. The book is a collection of lectures on his work on complexity, information, randomness and irreducibility. In particular, I recommend his lecture *A Century of Controversy over the Foundations of Mathematics* where he takes the audience through the several crises experimented by mathematicians during the 20th century – seeing his findings as building upon the work of Cantor, Russel, Hilbert, Gödel, Turing and Boltzmann. His enthusiasm and passion for what he does radiates from every page. Great read!

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[2] A Turing machine is a general model of a computing machine invented by Alan Turing. An introduction to Turing machines can be found in *Function* Vol 22, 2, pp. 63–65.
[3] Published by Springer in 2002.

LETTER TO THE EDITOR

Regarding the article on Columella's Formula in *Function, Vol* 27, *Part* 4, some serious gremlins seem to have crept in. The formula should read

$$A(\theta)=\frac{1}{2}(\mathrm{vs}\,\theta+\mathrm{ch}\,\theta)\mathrm{vs}\,\theta+\frac{1}{56}\mathrm{ch}^2\,\theta.$$

[There are two changes: the insertion of the factor $\frac{1}{2}$ before the first term and the replacement of vs by ch in the last.]

The website

http://www.hpm-americas.org/nl48/nl48frm.html

contains what is to me a more plausible derivation of Columella's Formula, but I concede that plausibility is in the eye of the beholder.

The webpage derives

$$A(\theta)=\frac{1}{2}(\mathrm{vs}\,\theta+\mathrm{ch}\,\theta)\mathrm{vs}\,\theta$$

from various approximation formulae that were used in antiquity and notes that this gives an exact value for $A(\theta)$ when $\theta=\pi$ if π were equal to 3.

The webpage then makes the (arbitrary?) assumption that, in order to account for the true value of π, a correction term of the form

$$k\left(\frac{\mathrm{ch}\,\theta}{2}\right)^2$$

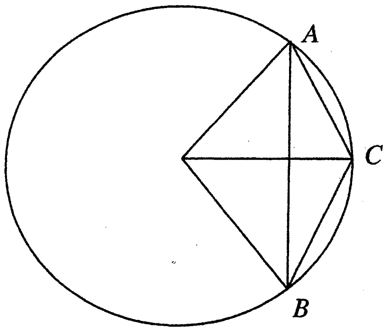
should be added and notes that Columella uses $k=\frac{1}{14}$, which (as Lévy-Leblond notes) is consistent with the value $\pi=\frac{22}{7}$.

A completely different approach for finding an approximation for the area in terms of the chord and the versine is possible.

Let A, B be the ends of the chord and let C be the midpoint of the circular arc joining them. The unknown area is now seen to be equal to the area of the triangle ABC plus twice the area of a new region shaped like the original. The area of the triangle is $\frac{1}{2}\mathrm{ch}\,\theta\,\mathrm{vs}\,\theta$. And finding the area of the new region is just the same as the original problem again, but with half the value of θ. We have

$$A(\theta) = \frac{1}{2}\mathrm{ch}\,\theta\,\mathrm{vs}\,\theta + 2A\left(\frac{\theta}{2}\right)$$

and it is possible to proceed iteratively by repeating the process.



Intuition would tell us that when this is done, then the true area could be approximated to whatever level of accuracy is required (and that all such estimates will lie below the true value). The resulting formulae are more accurate than Columella's, but more cumbersome. Moreover, the iteration suffers from a loss of accuracy when carried too far.

Derek Garson
(by email)

[Our thanks for the corrections. Readers may care to explore further the later points of this letter. In particular, look at the formulae when expressed in terms of the familiar sine and cosine functions. *Eds*]

NEWS ITEMS

Top Maths Honour to Expatriate Aussie

Terence Tao, an Australian now living in the US, has won one of the prestigious Clay Awards from the Clay Mathematics Institute (CMI). The 2003 Clay Research Awards were presented at its Annual Meeting held on Friday, November 14 at MIT (Massachusetts Institute of Technology). The awards, which recognise extraordinary achievement in mathematics, went to Richard Hamilton, of Columbia University, and Terence Tao, now of UCLA (University of California at Los Angeles).

The annual Clay Research Award is the institute's highest recognition of general achievement in mathematical research. The Clay Research Award takes the form of the elegant bronze sculpture "Figureight Knot Complement vii/CMI" by sculptor Helaman Ferguson.

Richard Hamilton has made significant advances toward the proof of Thurston's "geometrisation conjecture", of which the celebrated Poincaré conjecture is a special case. Recent work by Grigori Perelman of St. Petersburg has spectacularly advanced these ideas and brought us much closer to an understanding of the conjectures. (See *Function*, June 2003, p 82.)

Terence Tao was recognised for his contributions to several areas of Mathematics, but principally for "ground-breaking work in analysis [advanced calculus]".

Award recipients were named Clay Research Scholars for one year, and received a bronze replica of the CMI icon by sculptor Helaman Ferguson. Former recipients of the Clay Research Award include several Fields Medallists.

Terence Tao (b. 1975), a native of Adelaide, attended Blackwood High School and later graduated from Flinders University at the age of 16 with a B.Sc. in Mathematics. He received his Ph.D. from Princeton University in June 1996 and then took a teaching position at UCLA where he was assistant professor until 2000 when he was appointed full professor. He began a 3-year appointment as a CMI Long Term Prize Fellow in March 2001.

Back here in Australia, he first came to notice in 1986, when, at the age of 11, he won a bronze medal in the International Mathematical Olympiad. The next year he won silver and the next gold.

Clearly he has built on these early successes to achieve his present eminence!

For more on the Clay Awards and the recipients, visit the website

http://claymath.org

The Doubly Golden Euro

In March of last year, an article in *The Mathematical Gazette* (UK) carried the title "The golden euro". Here is the background. Most of the countries of the European Union have now abandoned their old currencies and have embraced a unified system, the Euro (€). However, there are a few renegades, among them the United Kingdom, which (probably for sentimental, not to say jingoistic, reasons) clings to the Pound Sterling (£).

Douglas Quadling, the author of *The Gazette's* article noted that the exchange-rate between the pound and the euro was such that

£1 = €1.62.

Now the number 1.62 is very close to the number $1.618\ldots = \tau$, the Golden Ratio. This number has the property that

$$\frac{1}{\tau} = \tau - 1,$$

and so we have the approximate equation

€1 = £0.62.

Suppose now that we get the Australian dollar ($) into the act. This has shown considerable variability lately, but for a time, we had the approximate equation

$$€1 \approx \$1.62.$$

What now can we say about the relative values of the dollar and the pound?

Well, suppose we had *exactly*

$$£1 = €\tau,$$

and also

$$€1 = \$\tau.$$

Then we would have

$$£1 = \$\tau^2.$$

Because of the equation previously noted, we then have

$$\tau^2 = 1 + \tau$$

and so we can set up the approximate equation

$$£1 = \$2.62.$$

Or we can do the calculation in reverse.

$$\$1 = £\tau^{-2}.$$

But $\tau^{-2} = 2 - \tau$. (Check this as an exercise!) So

$$\$1 = £(2 - \tau).$$

In terms of practical conversion, this last equation becomes

$$\$1 \approx £0.38.$$

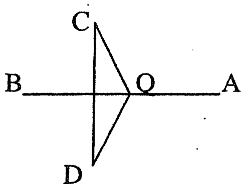
In real life, of course, things are more complicated than this. Currency conversion costs money and so our simple equations are crude approximations at best. Furthermore, as currencies float, equations that were once valid cease to hold. Who knows what the rates will be when this issue of *Function* goes to press!

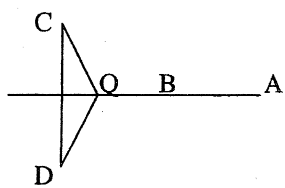
PROBLEMS AND SOLUTIONS

Solution to Problem 27.3.1 (related to the earlier Problem 26.5.4)

> It is desired to site the hub Q of a cabling network serving four outlets at A, $(a, 0)$; B, $(b, 0)$; C, $(0, c)$ and D, $(0, -c)$ in such a way as to minimise the total length of cable needed. Find the co-ordinates of Q.

(For convenience, the notation has been slightly altered from that printed previously.) We received a detailed discussion from Keith Anker, and some of this is incorporated in the summary given here. First note that without loss of generality we may take $a > 0$ and $c > 0$. This produces the following figures, the first of which has $b < 0$ and the second $b > 0$.





By symmetry, we expect Q to lie on the x-axis and this may also be formally proved by very simple arguments. Therefore the co-ordinates of Q are $(q, 0)$, say.

In the first case therefore, the function to be minimised is

$$(a-q)+(q-b)+2\sqrt{q^2+c^2}$$

and this is clearly minimised when $q = 0$. In other words, the hub is to be placed at the intersection of the diagonals of the quadrilateral $ACBD$.

In the second case, two possibilities arise. B may lie either to the left or to the right of A. No real difference exists between the two possibilities. Our figure illustrates the second in which $a > b$. We assume without loss of generality that this is so.

However the case $b > 0$ is also more subtle than that just dealt with, as it can be interpreted in either of two ways. One is to require that each of the points A, B, C, D be separately connected to Q. In such a case, three subcases arise. In the first, Q lies to the left of B as in the figure; in the second, it lies to the right of A; in the third, it lies between the two. If Q lies to the left of B, then the total distance will be greater than that involved if it is made to coincide with B (as may be proved by use of the triangle inequality), and if it lies to the right of A, then the distance will be greater than that involved if it coincides with A (by a slightly more complicated argument).

We are thus led to examine the remaining possibility. The function to be minimised is $2\sqrt{q^2+c^2}+a-b$, and the minimum must lie in the range $b \le q \le a$. The minimum is clearly achieved when $q = b$, which is to say when Q coincides with B.

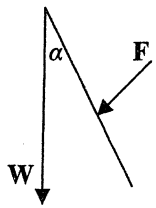
If we allow another interpretation in which the cable connecting A to B to automatically connect A to Q then we need only so site Q as to minimise the total distance $BQ + CQ + DQ$ and this is exactly the Steiner problem mentioned in our discussion of Problem 26.5.4. If $b > c/2$, there exists a point P such that $\angle CPB = \angle DPB = \angle CPD = 120°$. In that case, Q is to coincide with P. Otherwise choose $Q = B$.

Solution to Problem 27.3.2

> A cone whose base radius is a and whose base-to-vertex height is h rests with its base on a horizontal surface. It is desired to pick up the cone by grasping it about its curved surface. Under what conditions can this be done?

We received analyses from Julius Guest and also from Keith Anker, who distinguished two cases. In the first, the "grasping" is symmetric; in the other, not. Anker goes on to outlaw this second interpretation, but we include a brief discussion below.

Below is a diagram of the symmetric case showing a typical cross-section. A force **F** is exerted on the side of the cone, and this may be resolved into two components: **N**, perpendicular to the oblique side of the cone, and a frictional force **G** along that side. The maximal value of $|\mathbf{G}|$ is $\mu|\mathbf{N}|$, where μ is a constant known as the coefficient of friction. The only possible upward force on the cone is supplied by this frictional component.



Let the apical angle of the cone be α as shown in the diagram. Then the total upward force is $\mu|\mathbf{N}|\cos\alpha$ and the total downward force is $|\mathbf{N}|\sin\alpha + \mathbf{W}$, where **W** is the weight of the cone. We thus require

$$\mu|\mathbf{N}|\cos\alpha > |\mathbf{N}|\sin\alpha + \mathbf{W}.$$

This inequality may be recast as

$$|\mathbf{N}|(\mu - \tan\alpha) > |\mathbf{W}|\sec\alpha.$$

If now $\mu < \tan\alpha$, then the inequality cannot possibly be satisfied. Therefore, as $\tan\alpha = a/h$, a *necessary* condition for success is $\mu > a/h$. This condition will also be sufficient provided we can make $|\mathbf{N}|$ large enough.

Another strategy might be to attempt to topple the cone, so that it falls over and may then be "scooped up" in the inverted position. This depends on the coefficient of friction between the cone and the surface on which it rests. We leave the details to the reader.

Anker points out that the problem may be posed as "Can you pick up a wet bowl from the dish drainer?"

Solution to Problem 27.3.3 (from the 1962 Beijing Mathematical Olympiad, further discussed in *American Mathematical MONTHLY*, Jan 2003, pp 25 …)

> A number of students sit in a circle while their teacher gives them candy. Each student initially has an even number of pieces of candy. When the teacher blows a whistle, each student simultaneously gives half of his or her candy to the neighbour on the right. Any student who ends up with an odd number of pieces of candy gets one more piece from the teacher. Show that no matter how many pieces of candy each student has at the beginning, after a finite number of iterations of this process all the students have the same number of pieces of candy.

Anker also solved this problem. His analysis was similar to that given in *American Mathematical MONTHLY*, which we tend to follow here.

Unless the students already have the same number of pieces of candy each, some will have more than others (and in consequence, others less). If k is the number of pieces some student has, then $m \le k \le M$, for some positive integers m (for "minimum") and M (for "maximum").

A student with M pieces passes on $M/2$ to the right and receives no more than this from the left. Note that M is even, so that if this student receives $M/2$ from the left, his/her total remains at M. Otherwise (even if the teacher supplies an extra piece), the total can never exceed M. Thus no-one in the group can ever hold more than M pieces.

Correspondingly a student with m pieces will receive at least $m/2$ pieces from his/her left in return for the $m/2$ passed on to the right. Only if the student on the left also has m pieces, will the value of k fail to increase. But if there are n students all next to one another and each with m pieces, the leftmost individual will receive an increase. (The only exception can occur if the students all have m pieces, in which case, we are done.) Thus there will be at most $n-1$ students in the row of students with m lollies. The number of students possessing the minimal amount thus decreases with each redistribution.

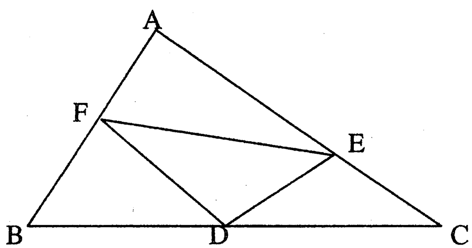
This means that the total number of lollies increases as the redistributions continue. But that total cannot exceed MN, where N is the total number of students. This forces the situation to stabilise with each student holding k pieces of candy, and with $k \leq M$.

The analysis in *American Mathematical MONTHLY* is by Glenn Iba, of MIT (Massachusetts Institute of Technology), and James Tanton, an Australian now resident in the USA. They note that it is as yet unsolved what the final value of k will be and how many redistributions are needed to achieve it.

Solution to Problem 27.3.4 (proposed by Dan Buchnick, Israel)

> Let ABC be a triangle and let D be a point on AB, E a point on BC, and F a point on CA. Join DE, EF, FD. We have now divided the original triangle into four smaller triangles: ADF, BED, CFE and DEF. Show that of these four triangles, DEF can never have the smallest area.

The proposer sent a solution, but we preferred that sent in by Keith Anker. Consult the diagram below.



Let $AF = \alpha(AB)$, $BD = \beta(BC)$, $CE = \gamma(CA)$. If at least one of the points D, E, F is not a mid-point of its side, there are two cases:

(1)` Two adjacent segments with a common vertex (without loss of generality, B) are less than (or equal to) half the side they lie in with one of these actually less. Again without loss of generality take it that

$$\frac{FB}{AB} < \frac{1}{2} \quad \text{and} \quad \frac{BD}{BC} \leq \frac{1}{2}$$

(i.e. $1-\alpha \leq \beta \leq \frac{1}{2}$ with one strict inequality).

Through C draw CH parallel to DF, meeting AB in H, and let ED meet CH in J. Join JF and CF. Then consider the areas of the various triangles.

$$\begin{aligned} \Delta EDF &> \Delta JDF, \text{ because of the extra } \Delta EJF \\ &= \Delta CDF, \text{ with the same base and height} \\ &\geq \Delta DBF, \text{ same base and equal or lesser height.} \end{aligned}$$

As one or other of the inequalities in the above chain is a strict one, we are done.

(2) α, β, γ are all greater than or equal to a half (or equivalently less than or equal to a half). Then

$$\frac{\Delta DBF}{\Delta CBA} = \frac{\frac{1}{2}FB \times DB \times \sin B}{\frac{1}{2}CB \times AB \times \sin B} = (1-\alpha)\beta,$$

with similar expressions for ΔDCE and ΔEAF. Then

$$\frac{\Delta DEF}{\Delta ABC} = 1-(1-\alpha)-(1-\beta)-(1-\gamma) = R, \text{ say.}$$

Now let $\alpha = \frac{1}{2}+x$, $\beta = \frac{1}{2}+y$, $\gamma = \frac{1}{2}+z$, where x, y, z are all greater than or equal to 0 and less than a half. It follows that

$$\begin{aligned} R &= 1 - \left\{\left(\tfrac{1}{2} - x\right)\left(\tfrac{1}{2} + y\right) + \left(\tfrac{1}{2} - y\right)\left(\tfrac{1}{2} + z\right) + \left(\tfrac{1}{2} - z\right)\left(\tfrac{1}{2} + x\right)\right\} \\ &= 1 - \tfrac{3}{4} + \tfrac{1}{2}(x + y + z) - \tfrac{1}{2}(x + y + z) + (xy + yz + zx) \\ &= \tfrac{1}{4} + (xy + yz + zx) \\ &> \tfrac{1}{4} \text{ by the constraints on } x, y, z. \end{aligned}$$

So in this case also one of the other triangles will have an area less than one quarter of ΔABC, and hence less than ΔDEF.

It only remains to say that the one case not so far considered is that in which each of the points D, E, F is the mid-point of its corresponding side. In that case all the triangles have equal area, and so we might agree with the proposer that none of them can be described as minimal.

And now for the next crop of problems.

Problem 28.1.1 (submitted by Julius Guest)

Prove that $4 \times 6^n + 5^{n+1} - 9$ is divisible by 20 for all positive integers n.

Problem 28.1.2 (submitted by Šefket Arslangić, Bosnia)

Let ABC be a triangle with sides a, b, c. Let

$$p = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \quad \text{and} \quad q = \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$$

Prove that $|p - q| < 1$.

Problem 28.1.3 (from *School Science and Mathematics*)

Show that for all natural numbers n, $n^9 - 6n^7 + 9n^5 - 4n^3$ is divisible by 8640.

Problem 28.1.4 (from *School Science and Mathematics*)

A fair coin is tossed n times. What is the probability that the outcome sequence does not contain two successive heads?

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