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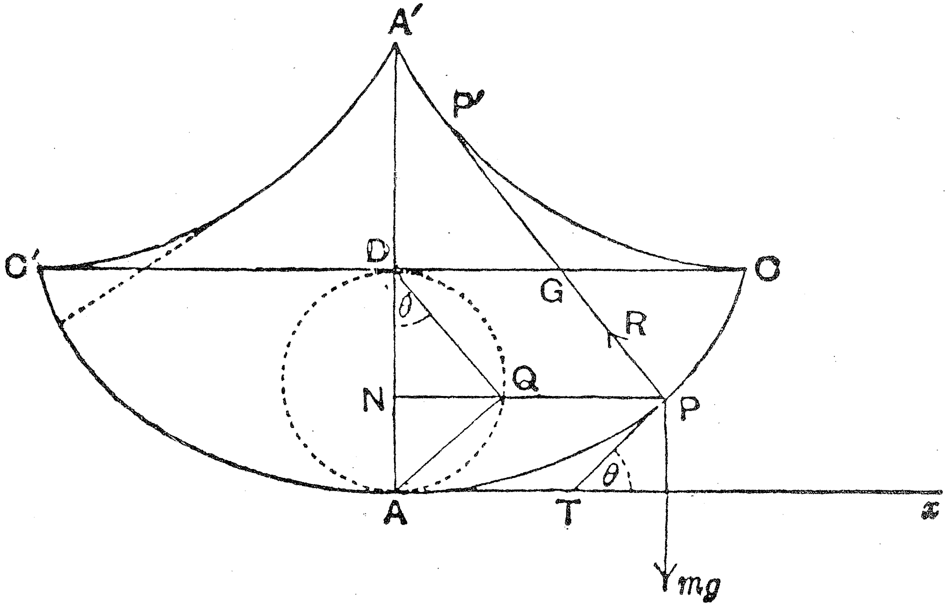

School of Mathematical Sciences – Monash University

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Function is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. It was founded in 1977 by Prof G B Preston, and is addressed principally to students in the upper years of secondary schools, but also more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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THE FRONT COVER

Our Front Cover for this issue is taken from a once-popular textbook: S L Loney's *Dynamics of a Particle and of Rigid Bodies.* This was published by Cambridge University Press in 1909 and was reprinted many times after that. The particular diagram we reproduce comes from page 108 and it revisits material we have used once before (February 1990) but in a different form.

We are concerned with the construction of an accurate pendulum. The regular "simple pendulum" consists of a weight or "bob" suspended by a light string from a fixed point. Provided the amplitude (extent) of its oscillations remains small, then the time taken for each oscillation is approximately independent of that amplitude, but this is only an approximation. (For an account of attempts to improve the approximation, see our cover story for June 2001.)

The cover picture shows how the pendulum may be modified to keep perfect time, even though the amplitude of its swing decreases as time goes by. The point A' is the pivot or point of attachment of the string and the bob is typically at a position P. The string is partially constrained by its contact with a curve $A'C$ along the arc $A'P'$. Thereafter the combined effects of the weight of the bob and its motion constrain the string to follow the straight line $P'P$. The key mathematical question is the determination of the shape of the curve involved.

Look carefully at the diagram and notice that the curve $A'C$ is repeated in mirror image to the left of the diagram as another curve $A'C'$. Both these are congruent to the two halves of the curve $C'AC$ lying below them. This last curve is termed a "cycloid". It is usually described as the path traced out by a point on the rim of a wheel rolling along a straight line. This wheel is shown in a special position in Loney's diagram. It appears as the dotted circle AQD, whose diameter is chosen to be vertical in this special case. Imagine this circle rolling along the line CC'. The point A, initially coincident with C, traces out the curve $C'AC$.

case. Imagine this circle rolling along the line CC'. The point A, initially coincident with C, traces out the curve $C'AC$.

If the wheel has radius a, and it has rolled through an angle φ (in the diagram $\varphi = \pi$), then the position of the point P may be determined. Take A as the origin and draw an x-axis as shown and a y-axis along AD. Then

$$\begin{aligned} x &= a(\varphi - \sin\varphi - \pi) \\ y &= a(1 + \cos\varphi). \end{aligned}$$

One property of the cycloid is that a string wrapped around it and progressively unwound traces out another cycloid congruent to the first. This property is expressed in technical language by saying that *the cycloid is its own involute.* This is one property involved in the construction of an accurate pendulum. The other is that, as the bob travels back and forth along some arc of the curve $C'AC$, it does so in the same time, irrespective of the size of the arc.

Loney's proof of this latter statement depends on two properties that he proves, but which we merely state here. To see what they are, draw AT horizontal through A and PN horizontal through P. PN intersects the circle AQD at Q. PT is the tangent to the cycloid at P. The two properties Loney uses are that PT is parallel to QA, and that the arc PA has twice the length of the distance AQ.

From these, he deduces the property of constant period. That period turns out to be $\pi\sqrt{\dfrac{a}{g}}$ where g is the acceleration due to gravity, and a is as before. That is to say that the bob retraces its path after this time has elapsed. It does this whatever the size of the swing, which could vary from the entire length $C'AC$ to very small oscillations about the point A.

The Department of Mathematics (as it was then called) at Monash once possessed a set of models demonstrating properties of the cycloid. One of these was photographed and featured on our front cover back in February 1990. Even then, it was in rather poor shape, as it had lain neglected for over 10 years, even at this early date. It was at this stage the last of its tribe, and shortly after we took its picture, it joined its relatives on the scrap-heap. Sad, but that is the way of the world!

THE NICENE RULES FOR FINDING EASTER

K C Westfold

[This is the last of the popular expositions from the estate of the late Professor Westfold. For previous articles on the same topic, see *Function*, *Vol* **9**, *Part* **3** and *Vol* **17**, *Part* **4**. Those earlier articles gave straightforward algorithms and detailed historical background. However, neither explains so cogently the Mathematics involved. We have decided to stay with the notation Professor Westfold used, although it is non-standard. However, it is well-explained in the course of the article and it would be clumsy to alter it. For further notes, see our comments at the end of the article. Eds]

In 325 AD the Nicene Council ordained that Easter should be celebrated on the first Sunday after the full moon that happens upon or next after the (northern hemisphere) vernal equinox, but should that full moon happen on a Sunday, then Easter should be deferred to the following Sunday. The equinox was fixed invariably on March 21^{st}, and the fourteenth day of the lunar month was regarded as the day of the full moon although the astronomical full moon generally occurs a day or so later. In consequence, there developed some unbelievably complicated and artificial methods involving Golden Numbers, Dominical Letters, Epacts and other devices to compute the date on which Easter fell in any particular year. Here I will demonstrate that the whole of this complicated apparatus is unnecessary. It may be dispensed with altogether, and replaced by one simple formula which complies strictly with the Nicene rules; as such compliance necessitates, it reconciles three periods which have no common measure: the week, the lunar month and the solar year.

Finding Easter requires the calculation of two non-negative integers:

(a) the number of days (here called F) between the equinox and the relevant ("paschal") full moon; and

(b) the number of days (here called S) between the paschal full moon and the next Sunday.

The calculation of F involves two astronomical constants:

(i) the solar year which has a length of 365 days, 5 hours, 49 minutes and 12 seconds; and

(ii) the lunar month of 29 days, 12 hours, 44 minutes and $3\frac{39}{427}$ seconds.

Now suppose that in the Year 0, the paschal full moon happens on the equinox. Then in the Year 1 it would occur 18 days, 15 hours, 43 minutes and $28\frac{30}{427}$ seconds after the equinox, because 13 lunar months exceed the solar year by that amount. In Year 2, a full moon would happen 37 days, 7 hours, 26 minutes and $56\frac{160}{427}$ seconds after the equinox (i.e. twice 18 days, 15 hours, 43 minutes and $28\frac{30}{427}$ seconds, or 26 lunar months less two solar years). But since the *first* full moon is the one required, one must subtract a lunar month to reach the figure of 7 days, 18 hours, 42 minutes and $53\frac{121}{427}$ seconds. In Year 3, F equals this last-mentioned amount plus the excess of thirteen lunar months over the solar year, so that the paschal full moon then arrives 26 days, 10 hours, 26 minutes and $21\frac{201}{427}$ seconds after the equinox.

Thus, F in each succeeding year is ascertained by adding 18 days, 15 hours, 43 minutes and $28\frac{30}{427}$ seconds to the F of the previous year and subtracting a lunar month whenever the sum exceeds it. This may be represented by the formula:

$$F = \left[\frac{\text{18 days, 15 hours, 43 minutes, } 28\frac{80}{427}\text{ seconds} \times Y}{\text{29 days, 12 hours, 44 minutes, } 3\frac{39}{427}\text{ seconds}}\right]_R$$

where Y is the number of the year and R indicates that only the remainder is to be considered, i.e. what is left over after the maximum number of lunar months have been subtracted. For example for the year 8700 we find

$$F = \left[\frac{18 \text{ days, } 15 \text{ hours, } 43 \text{ minutes, } 28\tfrac{80}{427} \text{ seconds} \times 8700}{29 \text{ days, } 12 \text{ hours, } 44 \text{ minutes, } 3\tfrac{39}{427} \text{ seconds}}\right]_R$$

$$= \left[\frac{162300 \text{ days, } 3 \text{ hours, } 7 \text{ minutes, } 9\tfrac{417}{427} \text{ seconds}}{29 \text{ days, } 12 \text{ hours, } 44 \text{ minutes, } 3\tfrac{39}{427} \text{ seconds}}\right]_R$$

Since there are 5495.995156 whole lunar months in 162300 days, 3 hours, 7 minutes and $9\frac{417}{427}$ seconds, we have, to an excellent approximation, 5496 whole lunar months with nothing left over. So in Year 8700, $F = 0$.

It is plain that the above formula for F with its days, hours, minutes and seconds is quite impracticable. A method of fixing a date must deal in whole numbers of days!

So let the fractional parts of days be disregarded and calculate F in each successive year by adding 18 to the F of the previous year and subtracting 29 whenever the sum exceeds it. That is to say consider the formula

$$F = \left[\frac{18Y}{29}\right]_R$$

This simple formula is not, however, sufficiently accurate, and so a correction is required. We have just seen that in Year 8700, $F = 0$ to excellent accuracy. So now compare

$$[(13 \text{ months} - 1 \text{ year}) \times 8700] - [\text{month} \times 5496] = 0$$

with

$$[18 \times 8700] - [29 \times 5496] = -2784.$$

It is apparent that the simplified formula gives a value of F which is 2784 days in defect after 8700 years.

However if a day is added to F every third year (i.e. 2900 times in 8700 years), one subtracted every sixtieth year (i.e. 145 times in 8700 years) and another added every 300th year (i.e. 29 times in 8700 years), then a nett correction of 2784 days is achieved. The formula, therefore, becomes:

$$F = \left[\frac{18Y + \left(\frac{Y}{3}\right)_Q - \left(\frac{Y}{60}\right)_Q + \left(\frac{Y}{300}\right)_Q}{29}\right]_R$$

where the subscript Q indicates the quotient reached by ignoring remainders.

If we now test this formula by substituting $Y = 8700$, the answer 0 is found, as it should be.

So far, we have been calculating from a purely notional Year 0. A further correction is needed to bring the formula into accord with the present era AD. This can be achieved by adding the number 15 to the top line of the last expression. The formula for F finally becomes:

$$F = \left[\frac{18Y + \left(\frac{Y}{3}\right)_Q - \left(\frac{Y}{60}\right)_Q + \left(\frac{Y}{300}\right)_Q + 15}{29}\right]_R$$

[Here Professor Westfold calculated the value of F for the year 1979, which was when he wrote this piece. We leave it as an exercise to readers to calculate the value for 2003. Eds]

The calculation of S (the number of days between the paschal full moon and the next Sunday) requires us first to calculate the "equinox hebdominal number", here called D, telling us on what day of the week the equinox falls. (If the equinox falls on a Sunday, then $D = 0$, if on a Monday, then $D = 1$, and so on.) Since an ordinary year contains 52 weeks and one day, the equinox advances by one day each year until a leap year occurs. In such a case, the advance is 2 days. Thus D is ascertained by adding to the date of the year the number of leap years that have occurred, and subtracting 7 each time the sum exceeds it. This may be represented by the formula

$$D = \left[\frac{Y + \left(\frac{Y}{4} \right)_Q}{7} \right]_R$$

However, under the Gregorian calendar, leap years do not occur *exactly* every four years. Century years (i.e. those ending in 00) are not leap years unless exactly divisible by 400. Therefore the formula becomes

$$D = \left[\frac{Y + \left(\frac{Y}{4} \right)_Q - \left(\frac{Y}{100} \right)_Q + \left(\frac{Y}{400} \right)_Q}{7} \right]_R$$

But again a further adjustment is needed to bring this formula into line with our present era. This is achieved by adding 2 to the top line of this last formula, and so we reach

$$D = \left[\frac{Y + \left(\frac{Y}{4} \right)_Q - \left(\frac{Y}{100} \right)_Q + \left(\frac{Y}{400} \right)_Q + 2}{7} \right]_R$$

Now that we have a formula for D, it is a simple matter to ascertain the hebdominal number of the date upon which the paschal moon arrives (here called K). Just add D to F and subtract 7 whenever the sum exceeds it.

$$K = \left[\frac{D + F}{7} \right]_R$$

Now if $K = 0$ (indicating a Sunday), the next Sunday is 7 days later, if $K = 1$ (indicating a Monday), the next Sunday is 6 days later, and so on. In other words, $S = 7 - K$. Finally, if E equals the number of days from the equinox to Easter, then $E = F + S$. Thus using the formulae already derived, we have:

$$
\begin{aligned}
E &= F + 7 - K \\
&= F + 7 - \left[\frac{D+F}{7}\right]_R \\
&= F + 7 - \left[\frac{Y + \left(\frac{Y}{4}\right)_Q - \left(\frac{Y}{100}\right)_Q + \left(\frac{Y}{400}\right)_Q + 2 + F}{7}\right]_R
\end{aligned}
$$

Ultimately E may be expressed entirely in terms of Y:

$$
\begin{aligned}
E = {} & \left[\frac{18Y + \left(\frac{Y}{3}\right)_Q - \left(\frac{Y}{60}\right)_Q + \left(\frac{Y}{300}\right)_Q + 15}{29}\right]_R + 7 \\
& - \left[\frac{Y + \left(\frac{Y}{4}\right)_Q - \left(\frac{Y}{100}\right)_Q + \left(\frac{Y}{400}\right)_Q + 2 + \left\{\frac{18Y + \left(\frac{Y}{3}\right)_Q - \left(\frac{Y}{60}\right)_Q + \left(\frac{Y}{300}\right)_Q + 15}{29}\right\}_R}{7}\right]_R
\end{aligned}
$$

If E is less than 11, add 21 for the date of Easter in March; if E is greater than 10, subtract 10 for the date of Easter in April.

[This is the end of the theory section of Professor Westfold's article. It continued with the construction of a set of tables, which it is no longer necessary to reproduce. In the years since he wrote, user-friendly software has become available to take their place. The final formula may be expressed in (e.g.) EXCEL with only a small outlay of effort. However, it is perhaps apposite to remark at this stage that the notation now extant differs from that used in the article. If a and b are positive integers, then we may write

$$\frac{a}{b} = q + \frac{r}{b}$$

where q and r are integers and $q \geq 0$ and $0 \leq r < b$. q is termed the quotient and r is referred to as the remainder. It is common to write $q = \left[\frac{a}{b}\right]$ where the square brackets are read as "integral part of". $[x]$ is defined as the largest integer less than or equal to x. This notation today replaces Professor Westfold's use of $\left[\frac{a}{b}\right]_Q$, $\left(\frac{a}{b}\right)_Q$ or $\left\{\frac{a}{b}\right\}_Q$ in the article. However there is no standard notation for his $\left[\frac{a}{b}\right]_R$ which is one reason for our leaving it intact. Since in our division, $\frac{a}{b} - q$ is necessarily less than 1, it is referred to as the "fractional part" of $\frac{a}{b}$; it is sometimes written $\left\{\frac{a}{b}\right\}$. So r is equal to $b\left\{\frac{a}{b}\right\}$ and this replaces Professor Westfold's $\left[\frac{a}{b}\right]_R$. In the EXCEL spreadsheet, $\left[\frac{a}{b}\right]$ is written as INT(a/b), and this gives the quotient, but the remainder must be written as b*(a/b – INT(a/b)), a little clumsy but not impossible. After this note of explanation, we leave further exploration to the reader. Eds]

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THE MONEY OR THE BOX?

Michael A B Deakin, Monash University

My title comes from an old radio game, Pick-a-Box, popular in my youth. Quiz participants, if they succeeded in answering 5 questions correctly, were asked to choose their prize. They could either accept an amount of money nominated by the host or else the unknown contents of a box. One never knew how high the host would go in his bids to induce one to take the money, nor did one know whether the box, when opened, would contain a worthwhile prize like a new Holden or a squib such as a shoelace. The game attracted large audiences, and there was much speculation as to the best strategy to follow, and much rehashing of the course of the various games that took place.

Such games continue to fascinate, although today they are televised and so achieve even greater immediacy. *Function* analysed one such in its issue for August 1992. This one was based on a US television show, and it continues to arouse much controversy. Here, however, I want to look at another, but one which is not so much a real game show, as a "thought experiment" dressed up in this guise.

It is called *Newcomb's Paradox* and it has generated a very large literature. There are many websites devoted to it. One good one to start with is:

http://members.aol.com/kiekeben/newcomb.html

but check out others via your favourite search engine. The one listed above is derived from an article by the columnist Martin Gardner. This first appeared in *Scientific American* in July 1973 and it has been widely reproduced and anthologised since.

The game was invented by the physicist William A Newcomb in 1970. (This Newcomb was the great-great-nephew of Simon Newcomb, who first announced Benford's Law – see *Function*, October 2001). His stimulus was the Prisoner's Dilemma, for which see *Function*, February

1985 and April 1998. It is also related to the Paradox of the Surprise Party (*Function*, February 1981).

It comes in a variety of forms. Here is how one of them goes.

Imagine you are the contestant. You are confronted by two closed boxes, B1 and B2. The game is hosted by a Being of higher intelligence, who claims to be able to read your mind and so to know what choice you will make. You may either:

1. Take both boxes
2. Take only Box B2.

B1 is known to contain $1,000. B2 may contain $1,000,000 or it may contain nothing. The million dollars is placed in B2 if the Being predicts that you will take only B2; there will be nothing at all in B2 if the Being predicts that you will take both boxes.

So! How would you decide? The paradox arises from the fact that excellent arguments can be given for either choice, and it is hard to see where either argument can be faulted.

The Being is in some cases taken to be God, and in this version the paradox is said to have theological repercussions, leading to the alleged conflict between human free will and divine foreknowledge. This path will not be followed here, but we may adopt other suggestions as to how the Being could have such a good grasp of what you are likely to do.

In one version, the Being is a psychic, in another an alien intelligence that has the power to detect and analyse your brainwaves, in yet another the Being is a supercomputer that is wired up to your head! The point is to make plausible the assumption that the Being can predict your choice with great accuracy.

Now think how you would proceed to play the game. You might well argue thus:

Argument 1. Because the Being is such an excellent predictor, it will almost certainly know in advance if you choose both boxes and so will have left B2 empty; if you take only B2, the Being will have anticipated *that*, and the million will be yours. **So, clearly, you should take only B2.** This gives $1,000,000 whereas the other option lands only $1,000.

But now consider the following counterargument.

Argument 2. By the time you get to make your choice, the Being has already acted. The million is either there or it is not. There is nothing the Being can now do to alter this state of affairs. **So why not take both boxes and get the lot?** If the Being put in the million, then you get a total of $1,001,000; if not, at least you get the $1,000.

Gardner put the problem to the philosopher Robert Nozick, who, as he put it, "sharpened" the two arguments. Suppose that the game has been played many times before. There is a history of all these previous plays, and in every case, the Being predicted correctly: that is to say that in every previous play those who chose both boxes got $1,000, while those who chose only B2 got $1,000,000. The lesson of history would almost force us to choose B2 only. This is **Argument 1** put even more forcefully than before.

But now consider the "sharpened" case for **Argument 2**. There is $1,000 sitting in B1. The Being may or may not have put $1,000,000 in B2. If there is nothing in B2, you can at least take home $1,000 by choosing both boxes; if the million is in B2, then you gain a cool $1,001,000, again by choosing both boxes. This way, you are absolutely certain to make a profit!

The situation has been analysed by considering the game as a contest between two players, you (the contestant) and the Being. The "Rules of the Game" may be summarised in terms of a table, known in the terminology of Game Theory as a "payoff matrix".

Here it is.

		Being's Prediction of your Choice	
		B2	*Both*
Your Choice	*B2*	$1,000,000	$0
	Both	$1,001,000	$1,000

The last argument may be summarised by saying that each of the figures in the second row is larger than the corresponding figure in the first. The argument that the choice of both boxes is the wiser one is formalised as the "dominance principle", that says exactly this. Your choice of both boxes gives an advantage over the choice of the single box, whatever the Being might do!

But now consider a more detailed analysis. Begin with the case in which the Being is in fact a con-artist, who can no better predict your choice than a random flip of a fair coin could do. In this case, B2 contains either \$1,000,000 or \$0 with equal likelihood. Choice of both boxes guarantees you \$1,000, and furthermore gives you a 50:50 chance of \$1,001,000. Your expected gain is \$501,000. If on the other hand you chose B2, you have no way to get your hands on the \$1,000 and only the 50:50 chance of the \$1,000,000; your expected gain is \$500,000. So in this case, the calculation of expected gain yields a result in full accord with the dominance principle: you should choose both boxes.

But the paradox arose because it was posited that the Being had very considerable predictive powers. **Argument 1** derives its force from exactly this consideration. The Being is supposed to be much more accurate than mere chance!

In order to analyse this case, suppose that the Being's probability of successfully predicting your choice is p. (We are particularly interested in the case where $p \approx 1$, but for the moment take p to be unrestricted.) It is straightforward to work out the expected returns from the two choices as functions of p.

If both boxes are chosen, that return is

$$[(1-p)\times\$1{,}001{,}000]+[p\times\$1{,}000]=\$[1{,}001{,}000-1{,}000{,}000p].$$

On the other hand, if only B2 is chosen, then your expected return is

$$\$1{,}000{,}000p.$$

So it comes down to a comparison of these two amounts. They are the same in the case when $p = \dfrac{1{,}001{,}000}{2{,}000{,}000} = 0.5005$, or 50.05%. If p is larger than this value, then **Argument 1** has force; if p is less, then we are best to stick with **Argument 2**. This provides a context for the earlier calculation in which, with purely random input from the Being, we had a definite, albeit marginal, advantage if we took both boxes. However, if we move to higher values of p, then we rapidly reach the situation in which the "B2 only" option becomes superior. With $p = 0.9$, for example, the expected return on this choice is $900,000, whereas the "Both boxes" option yields only $101,000.

The fact that the critical value of p is so nearly 0.5 adds some piquancy to the paradox. We do not have to ascribe great paranormal or supernatural powers to the Being; it will suffice to suppose that it has a shrewd grasp of human nature, and perhaps (as in the case of a parent or close friend) can form a pretty good notion of how you are likely to act. This means that even in quite realistic cases, **Argument 1** is very strong.

Before we go on, we can also consider that the expected value of the return is not the same thing as the *utility* of that return. Suppose, for example, that you were starving and didn't know where your next meal was coming from. Then the assured return of $1,000 arising from the choice of both boxes becomes very attractive; an extra $1,000,000 would of course be nice, but you might miss out completely, and that would be a calamity. On the other hand, if you were very well-to-do, it could well be that $1,000 would mean very little to you, but a chance at $1,000,000 could be most attractive. (This sort of thinking underlies the attractiveness of Tattslotto and other such games; people who do not miss $10 or so per week have an outside chance of extremely large winnings!)

It is this sort of consideration that offers one line of resolution of the paradox. The different strategies available correspond to different "mindsets". If you are the type to stick to assured gains, then a guaranteed return of $1,000 (with the possibility, even a remote possibility, of a lot more) will be very attractive to you. If you enjoy gambling and can afford to come away with nothing, then you may be moved to take the riskier course and so choose only B2. It is this dependence on psychological variables that gives plausibility to the Being's supposed predictive powers.

We may then see the force of the two arguments affecting different players differently. There is no objectively "right" answer. (Indeed, this is the case with a lot of real-life human decision making: legal cases are often decided by a majority verdict of the judges who hear them. Both sides in such disputes present good arguments; it is a matter of which carry more weight with the bench.)

Gardner, in a postscript added later to his analysis, and published in the anthology *The Night is Large*, points to another feature of the paradox. It falls into a category known as "self-referential". The best-known of such paradoxes are now standard fare. There is "Russell's barber"; Bertrand Russell imagined a village where the barber shaved everybody who did not shave himself. Who shaved the barber? We reproduced in the June 1981 issue of *Function* a witty cartoon on this theme. In February 1981, we ran an article on some of the more well-known paradoxes of self-reference.

Here is a version I saw many years ago in a book for school students, *The Argus Students' Practical Notebook, Volume* **5** (1952), p 91.

> "A says to B, 'I will teach you to be a barrister; half fee now, and the other half when you win your first case.' B was taught and called to the Bar, but failed to do anything at all for two years. A then said to himself: 'If I sue him for the instalment of my fee, and win the case, he will have to pay me; if I lose, then he will have won his first case, and will therefore have to pay me.'
>
> "That seems unanswerable until we get B's view: 'If A wins, then I have lost my first case and need not pay him; and if he loses, then by the judgment of the Court I need not pay him.' "

The 'self-referential' nature of these cases arises because the terms of reference contain in the first case a barber who also needs to be shaved and so appears in a dual rôle, and in the second because the court-case involved is itself an aspect of the contract being litigated.

Aidan Sudbury's article on the surprise party (referred to earlier) was based on a technical paper he wrote, and this (in essence) analysed the situation to that of a person who said, "I predict that E will happen, but you have no grounds for thinking that it will!" The interesting point is that when E happens, the predictor has got everything right: E did indeed happen, and the hearer had no grounds for belief that it would, because the predictor's statement made no sense at the time it was uttered. This nonsensical aspect springs immediately from the self-reference: the statement contains alleged information about itself!

Gardner sees the provision of such self-referential information as leading to a logical contradiction, as it certainly does in the case of the barber, but not so clearly in the case of the surprise party. He however dismisses *all* self-referential cases as unacceptable, and so his "solution" is to regard the rules of the game as constituting a hoax, or else a badly designed experiment. On this basis, he would ignore completely the alleged information about the Being's supposed predictive powers and choose both boxes.

This is quite rational if the predictive power is low, but its general application involves Gardner's belief that very high values of p are not in fact possible, even values as low as 51%. At 51%, the expected return if both boxes are taken is \$491,000 versus the expectation of \$510,000 if you choose B2 only. The margin might not be enough to tempt you to take the riskier course of action.

But if the value of p rises to a value near 1, then matters are not so clear-cut, unless, with Gardner, we choose to ignore the information altogether. The case of $p = 0.9$ has already been calculated, and here the expected return is so much greater that you would very possibly take the 10% risk of gaining nothing. As p increases, that risk diminishes and the expected return gets even larger.

However, we do not necessarily "know" the value of p. We must estimate this, and here Nozick's version is subtly different from the original. Nozick assumes that we have access to some previous records of the Being's predictive powers. (Nozick in fact loads the case even further by supposing that the Being's previous record is one of *perfect* prediction.) In such a case, we have evidence on which to base our estimate of p. If, by contrast, we rely only on the Being's word for such abilities, or on hearsay reports or blind trust, we might be inclined to doubt that p in fact had such a high value as claimed.

Finally it should be noted that the game is not a fair one. In the jargon, it is not "zero-sum". The contestant can always win money (by choosing both boxes), and even in a worst case scenario loses nothing. If you had the opportunity to play the game over and over, you could make a nice living by choosing both boxes all the time, or you might be willing to gamble and perhaps become extremely wealthy by changing your strategy from time to time.

Armed with this information, you should now be able to see why no-one plays the game in practice. You will find a simulation on

http://www.bliner.com/scott/newcomb.html

but here it rapidly becomes apparent that the computer is not playing at all; merely cheating!

[Note: The referee indicated that this paradox may also be seen as falling into a category known as "prediction paradoxes". Perhaps the starkest of these is that of an oracle that predicts that one will die on a certain day. ("Beware the Ides of March!") If we were the subject of such a prediction, would we merely accept our fate, or would we try strenuously to avoid it?]

Take two sheets of paper, one lying directly above the other. Crumple the top sheet, and place it on top of the other sheet, then there must be at least one point on the top sheet that is directly above the corresponding point on the bottom sheet! This follows from one of the most useful theorems in Mathematics: an amazing topological result known as the Brouwer Fixed Point Theorem. In dimension three, Brouwer's theorem says that if you take a cup of coffee, and slosh it around, then after the sloshing there must be some point in the coffee which is in the exact spot that it was before you did the sloshing (though it might have moved around in between). Moreover, if you tried to slosh *that* point out of its original position, you can't help but slosh another point back into *its* original position!

Adapted from the Funfacts site of Harvey Mudd College

COMPUTERS AND COMPUTING

Solving Non-Linear Equations: Part 3, Bracketing Methods

J C Lattanzio, Monash University

As stated in the first article in this series, the basis of all bracketing methods is Bolzano's Theorem from Calculus, which states that *if $f(x)$ is a continuous function for all x such that $a \le x \le b$ and $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$ then there is some number x satisfying $a \le x \le b$ and $f(x) = 0$.*

So if we can find such an interval which brackets the root, we can employ some algorithm to reduce the size of this interval. Here I will discuss two such approaches: the Bisection Method and the False-Position Method. The former is conceptually simpler, but the latter is generally preferable in practice.

(a) The Bisection Method

Suppose that we have somehow found that the required root x lies in the interval $a \le x \le b$. Put $c = (a + b)/2$. There are now three possibilities, depending on the sign of c. Either:

(i) $f(a)f(c) < 0$,
(ii) $f(a)f(c) > 0$,
or (iii) $f(a)f(c) = 0$.

In Case (i), $f(a)$, $f(c)$ have opposite signs. This means that $f(x)$ changes sign in the interval $a \le x \le c$, and the required root is now located in this smaller interval. In Case (ii), $f(x)$ must change sign in the other half of the original interval, and so the requited root lies in the interval $c \le x \le b$. In Case (iii), we have been extremely lucky, and c is the value of the root.

These three cases are illustrated in Figure 1, which shows the situation if $f(a)>0$. (If $f(a)<0$, the figure is much the same; the graphs of the various functions $f(x)$ being replaced by their reflections in the x-axis.)

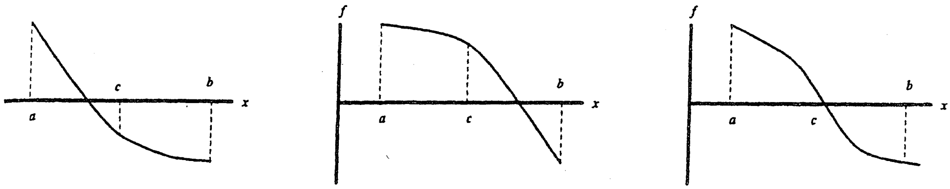


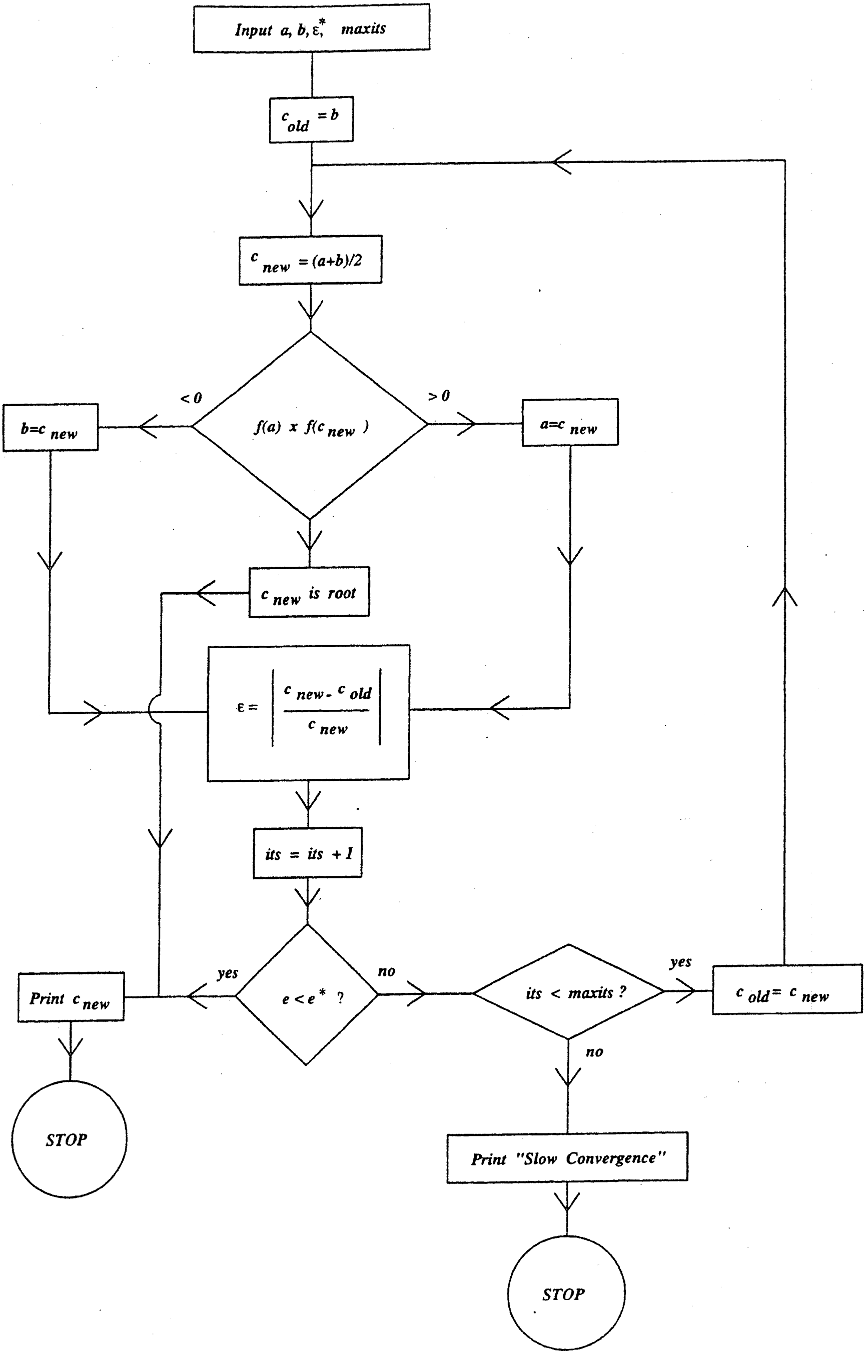
Figure 1

Now:

In Case (i), rename c as b and repeat the process;
In Case (ii), rename c as a and repeat the process;
In Case (iii), we have found the root already.

We continue with this process until an accurate answer is obtained. It is usual to continue until the relative error is less than some predetermined small amount ε. That is to say, the length of the confining interval, divided by the estimate of the root is less than amount ε. This is in line with the remarks in Part 1 of this series. However, also in line with those remarks, it is advisable as a check to examine the value of $f(c)$, where c is our final estimate of the value of the root. This value should normally be small.

A very nice property of the Bisection Method is that it will **always** converge, albeit slowly. In fact, because each iteration halves the interval, after n iterations, the root is constrained to lie in an interval of width $2^{-n}(b-a)$, where our initial interval is $a \le x \le b$. Indeed we can tell *in advance* how many iterations will be needed.


Input a, b, ε*, maxits
c old = b
c new = (a+b)/2
f(a) x f(c new)
< 0
> 0
b=c new
a=c new
c new is root
ε = | (c new - c old) / c new |
its = its + 1
e < e* ?
yes
no
Print c new
STOP
its < maxits ?
yes
no
c old = c new
Print "Slow Convergence"
STOP

The table below illustrates the method in the case of $f(x) = x - e^{-x}$. You should easily determine that there is a root in the interval $0 \le x \le 1$. We shall seek to evaluate its value to within 1%. We work to three decimal places to avoid round-off errors (even though only two decimal places are called for). Follow the computations in the table for yourself; the final column is the relative error – the computation ceases when the value entered in this column becomes less than 0.01. We thus find $x \approx 0.57$.

iteration	a	$f(a)$	c	$f(c)$	b	$f(b)$	ϵ
1	0.000	−1.000	0.500	−0.107	1.000	0.632	——
2	0.500	−0.107	0.750	0.278	1.000	0.632	0.33
3	0.500	−0.107	0.625	0.090	0.750	0.278	0.20
4	0.500	−0.107	0.562	−0.007	0.625	0.090	0.11
5	0.562	−0.007	0.594	0.041	0.625	0.090	0.05
6	0.562	−0.007	0.578	0.017	0.594	0.041	0.03
7	0.562	−0.007	0.570	0.005	0.578	0.017	0.01
8	0.562	−0.007	0.566	−0.001	0.570	0.005	0.007

Table 1

Page 20 gives a flowchart for the Bisection Method. Using this, you should be able to write a program which implements it.

(b) The False Position Method (*Regula Falsi*)

The bisection method uses only the information as to the signs of $f(a)$, $f(b)$, as we improve on our initial approximations. A little thought suggests that we might use their *magnitudes* as well. Consider Figure 2 overleaf.

The method envisages a straight line drawn between the points $(a, f(a))$, $(b, f(b))$. The equation of the straight line joining these two points is

$$\frac{f(x)-f(a)}{x-a}=\frac{f(b)-f(a)}{b-a}$$

The root we seek is c, where $f(c)=0$, and so

$$c=\frac{af(b)-bf(a)}{f(b)-f(a)}$$

This value is our new guess for the root.

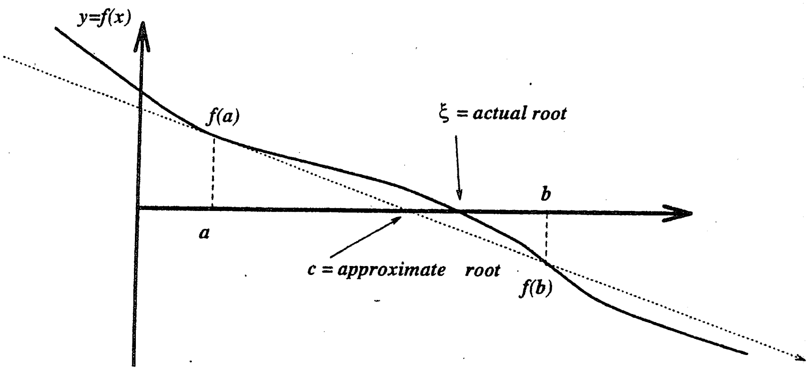


Figure 2

The program written for the Bisection Method will work also for this method because in every other respect the two are the same. The only change is that the line which assigns $c = (a + b)/2$ is replaced by the formula given above.

The convergence rate for the False-Position Method is usually (but not always) faster than that for the Bisection Method. In the case of the example studied before this feature *is* observed. Compare the table given earlier with that opposite.

The same answer, $x \approx 0.57$, is reached, but this time after three iterations instead of eight.

iteration	a	$f(a)$	c	$f(c)$	b	$f(b)$	ϵ
1	0.000	−1.000	0.613	0.071	1.000	0.632	----
2	0.000	−1.000	0.572	0.008	0.613	0.071	0.07
3	0.000	−1.000	0.568	0.001	0.572	0.008	0.008

Table 2

(c) Pitfalls of Bracketing Methods

Although bracketing methods will always converge, given appropriate initial conditions, there are nevertheless some subtleties to watch out for.

The conditions of the Bolzano Theorem mean that there must be an odd number of roots between a, b. But look at Figure 3. The bracketing methods will find one or another of these roots, but working out which one might be difficult!

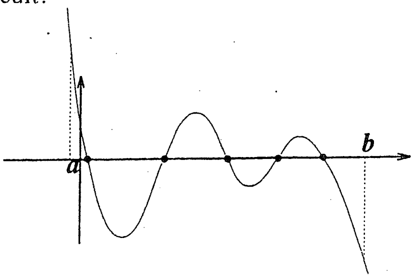


Figure 3

Figure 4 (overleaf) shows another possibility. The left-hand root is a double one, so that on some counts there would be two roots between a and b (although it is usual to count a double root as two equal roots). Bracketing methods are not well adapted to the evaluation of multiple roots.

Furthermore note that the continuity condition on $f(x)$ is absolutely vital. Figure 5 (also overleaf) shows what can happen if it is violated. Here, although $f(a) < 0 < f(b)$, there is no root between a and b.

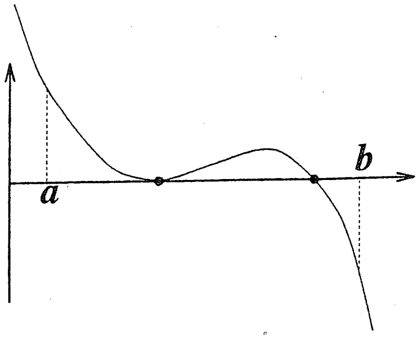


Figure 4

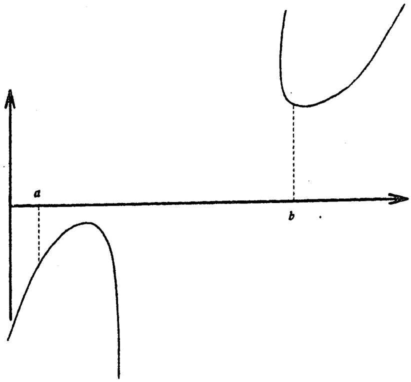


Figure 5

These complications serve to emphasize the usefulness of a preliminary graphical exploration as outlined in my previous article.

Finally it should be noted that there are cases in which the Bisection Method is faster than the False-Position Method. One such is the determination of the root of $x^{10}-1$ in the interval $0 \leq x \leq 1.3$. After five iterations, the bisection method is in error by 1.6%, but the False-Position Method is out by 59%!

∞∞

LETTER TO THE EDITORS

I read with interest the news item "Primality is *P*" in the October 2002 issue of *Function*. However, the article contains an error: the statement that the computational times of the Sieve of Eratosthenes increase exponentially with n is incorrect. For the purpose of analysing the computational time of an algorithm, the "size" of a problem is the number of symbols (specifically, in the case of a digital computer, the number of bits) needed to represent the input. In particular, the size of the primality problem is the *number of digits* in n, rather than n itself. Now from an order of magnitude point of view, the number of digits in n can be taken to be proportional to $\log n$. The logarithm can be to any base, because different bases merely yield different constants of proportionality.) The Sieve of Eratosthenes is not in *P*, because the computational time of the algorithm increases exponentially as a function of log n. The new algorithm is claimed to have a computational time of order $(\log n)^{12}$, i.e., the computational time is of the order of (at most) the twelfth power of the size of the problem. Since the computational time is of the order of a polynomial function of the problem size, the new algorithm is in *P*.

I also took the trouble to look up the algorithm on the Web at the address given. On doing so, I noted that there is a misprint in the version in *Function*: in line 7, n should be raised to the power of, rather than multiplied by, $\frac{r-1}{q}$.

Peter Grossman
Gerald St, Murrumbeena

[Our thanks for these corrections. Eds]

HISTORY OF MATHEMATICS

Tides in Lakes

Michael A B Deakin, Monash University

In 1755, *Scots Magazine* published a news story from which I quote.

> "On the first of November last, Loch Lomond all of a sudden, and without the least gust of wind, rose against its banks with great rapidity; and immediately retiring, in about five minutes subsided as low, in appearance, as ever it used to be in the greatest drought of summer. In about five minutes after it returned again, as high and with as great rapidity as before. The agitation continued in the same manner, from half past nine till fifteen minutes after ten in the morning; the waters taking five minutes to rise and as many to rise again. From ten to eleven, the agitation was not so great, and every rise was somewhat less than the immediately preceding one, but taking the same time, viz. five minutes to flow and five to ebb as before. At eleven the agitation ceased. The height the waters rose was measured immediately after, and found to be 2 feet 6 inches [75 cm] perpendicular.
>
> "The same day, at the same hour, Loch Lung and Loch Keatrin were agitated in much the same manner; and we are informed from Inverness, that the agitation in Loch Ness was so violent as to threaten destruction to some houses built on the side of it."

That "same day" was also the day of the great Lisbon earthquake, in which Lisbon was destroyed, and it may be taken as established that this disturbance was what set the lochs of Scotland sloshing around in their beds.

Such disturbances are known as *seiches* and in fact they are quite common, although not often as spectacular as these. Even before this extraordinary manifestation, it was known that Le Léman (Lake Geneva) underwent systematic rises and falls in its level, especially at its Western end, where the tapering of the lake induced a magnification of the effect.

Later on, these were the subject of extensive study by F A Forel, who set up an observation post on the shore of the harbour at Morges in 1869. Forel also initiated the mathematical discussion of the phenomenon, and published his first theoretical study in 1873.

However it was the later work of Chrystal that allowed much more accurate comparison with observation. George Chrystal (1851-1911) was a Scot, who came rather late to Mathematics, encountering it first at University, but mastering it well enough to receive a first-class degree with majors in Mathematics and Natural Philosophy (i.e., Physics). By 1877, he had become professor of Mathematics at the University of St Andrews, but shortly afterwards, he moved to the larger University of Edinburgh. He is best remembered today for his textbook on Algebra, published in 1886.

[This made a brief earlier appearance in *Function*. In April 1999, I discussed its (geometric) proof of the irrationality of $\sqrt{2}$.]

Late in his career, Chrystal turned to the analysis of seiches. By a "seiche" [the word is French, and pronounced *Saysh*], Chrystal meant a general slopping of the entire water of the lake. It is to be distinguished from surface waves, although in its own way it *is* a wave, but of a quite different character. Look at Figure 1.

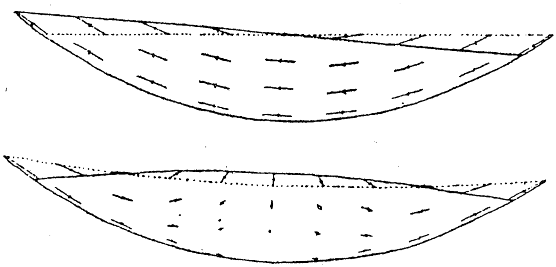


Figure 1

This shows two different ways in which the water of a lake might oscillate. In the top illustration, the entire body of water has displaced to the left, and will subsequently move right and later back again, etc. The lower illustration shows a more complicated motion, in which water has bunched up in the middle, and will slosh outwards to the sides and then back again, and so on.

This reminded Chrystal of the vibrations of a stretched string (as on a guitar for example). This moves up and down in various ways, the two simplest of which are illustrated in Figure 2. The oscillation at the right has all the parts of the string moving in unison, up and down, but the second shows a motion whereby the right- and left-hand halves of the string oscillate in opposite directions.

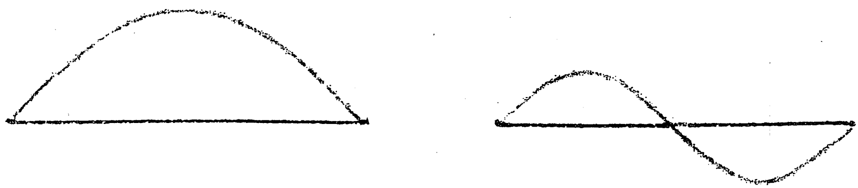


Figure 2

Before getting onto Chrystal's analysis of seiches, let me briefly recap the theory of the vibrating string. If the displacement of the string from its resting position is given by u, then u will depend on the time t and the position x of the piece of string undergoing the displacement. It also depends on three physical quantities: T, the tension in the string, ρ, its mass per unit length (density), and l, its length, all supposed to be constants.

The two ends of the string are fixed and do not move, and under these assumptions the displacement u is a sum of terms of the form

$$u = \left(A_n \cos \frac{n\pi t}{l} \sqrt{\frac{T}{\rho}} + B_n \sin \frac{n\pi t}{l} \sqrt{\frac{T}{\rho}} \right) \sin\left(\frac{n\pi x}{l} \right) \qquad (1)$$

where n is an integer and the values A_n and B_n are constants depending on n.

Chrystal set up his analysis as shown in Figure 3. It works best if the lake is a long this lake with steep sides, and this is what is illustrated.

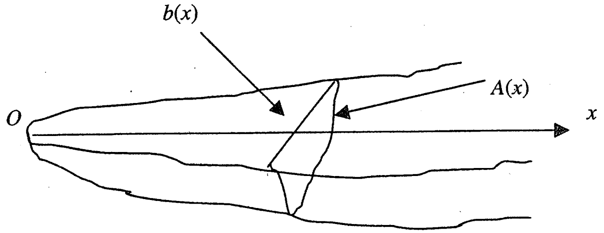


Figure 3

Take one end of the lake as the origin, and set up an x-axis horizontally along the length of the lake. The y-axis is also horizontal and is perpendicular to the x-axis and the z-axis is vertical. As before, t represents the time. Chrystal supposed that the width of the lake at its surface was $b(x)$ and that if we took a vertical plane at x, then the cross-section of the water in the lake would have an area $A(x)$ at that position. He supposed that the water in these vertical planes moved all together in the x-direction, and that at a time t its position was displaced a distance he called ξ.

This enabled him to set up a complicated equation for this displacement, but then he did a very clever thing. He replaced ξ by the product $A(x)\xi$, which he called u, and then he used a different co-ordinate instead of x. This co-ordinate v was the *area of surface* behind (in the diagram, to the left of) the line drawn across the surface at x.

This led Chrystal to an equation that is in fact the same as that for the vibrating string except that there is the big difference that the term that previously involved the tension and the density now involved not a constant but a function $\sigma(v) = A(x)b(x)$, a combination that came to be called the "lake function", although Chrystal's term was "normal curve". Chrystal also made the assumption that u was zero at the two ends of the lake. This is true if the lake ends in vertical walls, and also gives a good approximation for more general cases, as we can use the fact that $A(x) = 0$ at the ends of the lake. (Remember that $u = A(x)\xi$.)

The situation was exactly the same as for the vibrating string if we generalized the analysis of this latter case to allow ρ to vary instead of being constant. This does mean that the solutions are not as simple as those shown in Equation (1), and must be separately calculated for each different lake-function. Generally, this must be done numerically using a computer.

Back in those days, there were no computers, of course, and so Chrystal was forced into the detailed analysis of many special cases, where the lake-function was given a form based on some idealised geometry. This led him to generalisations of the sine and cosine functions of Equation (1). Nowadays, we do not need to take this route, because (assuming sufficient information about the shape of the lake under discussion), we may simulate all this on a computer.

Chrystal's studies stimulated much further work. This continued well into the twentieth century and involved one of the greatest mathematicians of that era: the Italian Vito Volterra, whose life and work I hope to discuss in a later column.

More on HRT and Breast Cancer

Our News Item on this topic was only one of several appearing at about the same time. Monash Statistician Aidan Sudbury had a discussion in *Financial Review* (August 24, 2002), and the ABC's Norman Swan discussed it with Sue Lockwood of the Breast Cancer Action Group in his *Health Report* program for August 5. A transcript of this should still be available on the web via the ABC's site.

PROBLEMS AND SOLUTIONS

Before getting on with the solutions to the problems set last June, we acknowledge a letter from our regular correspondent Keith Anker, who sent us solutions to Problems 25.5.3 and 25.5.4. It seems that these must have gone astray in the mail, as we never received them. The solution to Problem 25.5.4 was essentially the same as the one we published, but it compressed the argument elegantly into a single line.

With this behind us, let us now proceed with the solutions to last June's set. (Alert readers will notice that these problems were misnumbered – we here use the numbering that should have been used back then!)

SOLUTION TO PROBLEM 26.3.1 (the third "Professor Cherry" problem)

The challenge was to prove the identity

$$\frac{d^m(a-b)(b-c)+b^m(a-d)(c-d)}{c^m(a-b)(a-d)+a^m(b-c)(c-d)}=\frac{b-d}{a-c}$$

in the cases $m = 1$ and 2.

Solutions were received from Keith Anker, Šefket Arslanagić (Bosnia), J C Barton, J A Deakin, Carlos Victor (Brazil) and Colin Wilson.

Most began with a cross-multiplication, and this path will be followed here, with the possibility of zero denominators postponed for the while. This means that the identity to be proved becomes

$$(a-c)\left[d^m(a-b)(b-c)+b^m(a-d)(c-d)\right]$$
$$-(b-d)\left[c^m(a-b)(a-d)+a^m(b-c)(c-d)\right]=0.$$

Wilson saw this expression as a function of a involving three parameters b, c, d. If $m = 0$, 1 or 2, then this expression is quadratic in a. It is now a simple matter to show that this second identity is indeed true in the three special cases $a = b$, $a = c$, and $a = d$. It follows by the principle of pseudoinduction (*Function*, June 2001) that it always holds when $m = 0$, 1 or 2. (It may in fact also be proved that it holds for no other values of m.)

It remains to discuss the possibility of zero denominators in the original expression. Consider first what happens if $a = c$. Barton showed that the denominator of the left-hand side of the original identity is divisible by $(a - c)$ in the cases $m = 1, 2$, and it is not difficult to show this also in the case $m = 0$. Thus the two expressions will both be infinite in this case, unless we also have $b = d$. This leads to a few very special cases that we leave to the reader to explore. The other cases of zero denominators are:

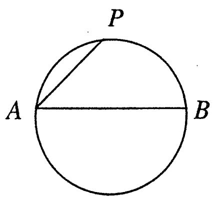
For $m = 0$, $a + c = b + d$,
for $m = 1$, $ac = bd$,
for $m = 2$, $abc + acd = abd + bcd$, i.e. $ac(b + d) = bd(a + c)$.

Again, we leave the full exploration to the reader.

SOLUTION TO PROBLEM 26.3.2

This problem, a hardy perennial, read: "Let a circular field of unit radius be fenced in, and tie a goat in its interior to a point on the fence with a chain of length r. What length of chain must be used in order to allow the goat to graze exactly one half the area of the field?"

Solutions were received from Keith Anker, Šefket Arslanagić (Bosnia), Jim Cleary, Julius Guest and Carlos Victor (Brazil). All answers were substantially the same, and so we here print a composite.



In the diagram above, let A be the point to which the goat is tethered, and let AB be the diameter through A. Let AP represent the chain, and suppose its length to be r. If we also drew the line segment PB (which, to avoid cluttering up the diagram, is not done here), then angle APB, being the angle inscribed in a semi-circle, would be a right angle, and the triangle APB would be a right-angled triangle with AB as the hypotenuse.

Because the field has unit radius, it follows that $AB = 2$, and if we let angle $PAB = \theta$, then $r = 2\cos\theta$.

The situation is symmetric about the diameter AB, and so we can confine attention to the top half of the diagram. The total area of the upper semicircle is $\pi/2$, and so the area the goat can graze must be $\pi/4$. This area comprises two parts: (a) a *sector* of a circle of radius r, and apical angle θ, (b) a *segment* of a circle lying above the line-segment AP.

Now consider the two areas separately. Area (a) is simply $\frac{1}{2}r^2\theta$ but Area (b) is more complicated. We have $\angle PBA = \frac{\pi}{2} - \theta$. Let O be the center of the circle. Then angle POA will be twice angle PBA, so $\angle POA = \pi - 2\theta$. The area of the *triangle* POA is then $\frac{1}{2}\sin(\pi - 2\theta) = \frac{1}{2}\sin 2\theta$. The area of the *sector* POA is $\frac{1}{2}(\pi - 2\theta)$, so that the total of Area (b) is $\frac{1}{2}(\pi - 2\theta - \sin 2\theta)$.

So now put all this together to find that the area the goat can graze is $\frac{1}{2}(\pi - 2\theta - \sin 2\theta + r^2\theta)$ from which it follows that we require

$$\frac{1}{2}(\pi - 2\theta - \sin 2\theta + r^2\theta) = \frac{\pi}{4}$$

Now substitute $r = 2\cos\theta$ into this equation to find (after a little simplification)

$$\theta + \frac{1}{2}\sin 2\theta - 2\theta\cos^2\theta = \frac{\pi}{4}.$$

This may be rewritten as

$$\sin 2\theta - 2\theta \cos 2\theta = \pi / 2.$$

This equation can only be solved numerically, but it has a solution of $\theta = 0.95285$ (radians), or 54.6°. Thus $r = 2\cos 54.6° = 1.158$.

SOLUTION TO PROBLEM 26.3.3 (from the US syndicated column *Ask Marilyn*)

This was a problem in probability. Mr and Mrs Smith are in the habit of dining two nights each (seven-day) week at the Taste-e-Bite Café. It is quite random which two nights they choose, but about three quarters of the time they notice that Mr and Mrs Brown are also there. They conclude that the Browns eat there more frequently than they do themselves. Is their conclusion justified?

We received solutions from Keith Anker, Joseph Kupka and Carlos Victor. Kupka, in particular sent a very detailed analysis that showed that as with "many 'casual' problems in probability, the stated assumptions are insufficient to produce a definite answer".

Victor confined himself to showing that in certain circumstances, the conclusion would not be justified. Anker made the remark that the *pattern* of the Browns' dining made no difference. They could go on regular nights or at random like the Smiths. Because the Smiths go on random nights, they are sampling the Browns' behavioral patterns, and are able to form a conclusion about the frequency of the Browns' visits to the café.

More to the point, however, is the question of whether the Smiths' and the Browns' attendance at the café are independent events. Suppose that they are, and further suppose that the probability of the Browns going to the café is p. The probability that the Smiths go is 2/7. Thus the probability that the two couples both attend is $2p/7$. But this is to equal about ¾ of 2/7. I.e.

$$\frac{2p}{7} = \frac{3}{4} \times \frac{2}{7}$$

This leads us to the conclusion that p = 3/4, so that the Browns dine at the café about 21/4, or 5+ times per week.

If the events are not, however, independent, this conclusion needs to be modified. If the Browns seek to avoid meeting the Smiths, then (assuming they have some success in this endeavour) they dine at the café *even more often* than the figure we have just deduced. However, if the Browns *actively seek* the Smiths' company, then the conclusion might not be valid.

For more on this problem and for an analysis considering other interpretations of the data, see pp 84-85 of Edward Barbeau's *Mathematical Fallacies, Flaws and Flimflam* (published by the Mathematical Association of America). This reproduces an analysis by Elliot Weinstein of Baltimore. It generally agrees with the Smiths' deduction, as did Marilyn in the original column. It was an attempt to sort out perceived flaws in Marilyn's reasoning that led Weinstein to his rather complicated discussion.

SOLUTION TO PROBLEM 26.3.4

This asked for the value of $\log_3 169.\log_{13} 243$.

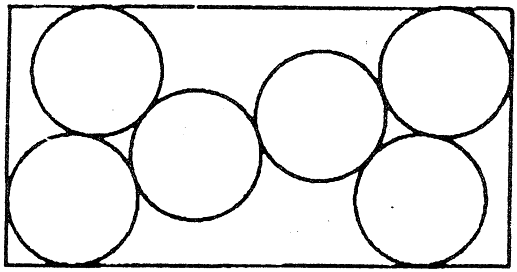
Solutions were received from Keith Anker, Šefket Arslanagić (Bosnia), J C Barton, J A Deakin, Julius Guest and Carlos Victor (Brazil). All were essentially the same.

$$\log_3 169.\log_{13} 243 = \frac{\log 169}{\log 3} \times \frac{\log 243}{\log 13} = \frac{\log 13^2}{\log 3} \times \frac{\log 3^5}{\log 13} = \frac{2\log 13}{\log 3} \times \frac{5\log 3}{\log 13} = 10.$$

Here are this issue's new problems.

Problem 27.1.1 (from the Wasan, traditional Japanese Mathematics, reproduced in *History in Mathematics Education*, ed J Fauvel and J van Maanen)

The diagram overleaf shows six circles packed as arranged inside a rectangle. The circles are all equal and the radius of each is 1. Find the dimensions of the rectangle.



Problem 27.1.2

Volume 2 of Arthur Mee's *Children's Encyclopaedia* shows a set of five cards, each containing 30 numbers between 1 and 60 (inclusive). The idea is to ask a friend to choose a number in this range and to identify those cards on which it appears. From this information it is possible to identify the number the friend chose.

How are such puzzles constructed and what is so special about the number 60?

Problem 27.1.3 (based on a problem in *Mathematical Bafflers*, ed Angela Dunn)

x, y, z are positive integers such that $x + y + z = xyz$. Find all solutions of this equation.

Problem 27.1.4 (from the same source)

A rower is moving upstream when his cap falls into the water. He does not realize this until 10 minutes later. Then he instantly reverses direction, and chases the cap as it floats downstream. He finally retrieves it one kilometer downstream from the point where it entered the water. What is the speed of the stream?

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