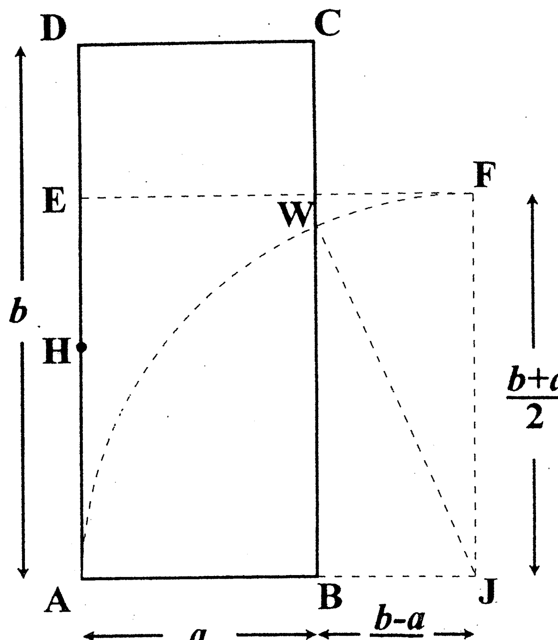
# Function

## A School Mathematics Magazine

Volume 20 Part 5 October 1996




Mathematics Department - Monash University

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# EDITORIAL

This is the 100th issue of *Function* and so we are celebrating with an additional issue which includes the history of *Function* and a complete index of all articles published over the magazine's 20 years of existence.

As we have done over these 20 years, in this issue we have tried to include something for everyone interested in the beauty of mathematics.

The front cover article describes an ancient Indian construction of a square which has the same area as a given rectangle. What is interesting about this construction is that it suggests that Pythagoras' Theorem may have been known before Pythagoras' time.

Alan Roberts expands on the article about anagrams which appeared in the previous issue; he explains why the number of different anagrams generated using the "hopping" technique varies so much with the length of the word. You need some lined paper and a paperclip to read John Shanks' article, which is based on a classical problem formulated by Comte de Buffon in the 18th century. It is about the mysterious role of $\pi$ in the calculation of the probability that a paperclip thrown in the air will cross one of the lines when it lands on the paper. The "Tax on the Tax" article will be of interest to any taxpayer; it describes a dramatic experience suffered by a group of football players when they received an end-of-season bonus.

In the *History of Mathematics* column, Michael Deakin writes about some very recent interpretations of the inscriptions representing Pythagorean triples which appear on the Babylonian tablet Plimpton 322. In the *Computers and Computing* section you will find an introduction to 3D curves described using parametric equations and a very simple program which you could use as a starting point to generate these curves on a computer screen.

This issue also includes contributions from our readers, letters and solutions to problems. Our *Problem Corner* editor has put in some more new problems to challenge you. For the brightest minds, our special correspondent on Olympiads has provided the problems which challenged the participants in the most recent International Mathematical Olympiad.

We hope you enjoy this issue of *Function*.

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# THE FRONT COVER

## An Indian Construction

**Michael A B Deakin**

Our front cover diagram for this issue is a simplification of one given in G G Joseph's *The Crest of the Peacock: Non-European Roots of Mathematics.* It illustrates a construction described in an Indian text known as *Baudhayana's Sulbasutra*, dating from (perhaps) some time before 600 BC.

$ABCD$ is a rectangle with $AB < AD$. It is required to draw a square with the same area as $ABCD$. Let $AB = a$, $AD = b$. Then what we want is a length $s$ such that

$$s^2 = ab$$

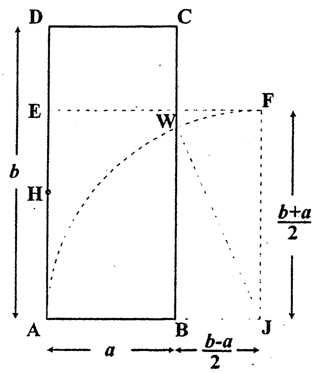


**Figure 1**

The construction goes as follows. From side $\overline{AD}$ cut off $\overline{AH}$ such that $AH = AB = a$. Now let $E$ be the mid-point of $\overline{HD}$. We thus have

$$HE = \frac{1}{2}(b-a)$$

and

$$AE = a + \frac{1}{2}(b-a) = \frac{1}{2}(a+b)$$

Now construct a square $AEFJ$ on the side $\overline{AE}$ in such a way that $\overline{AJ}$ extends the side $\overline{AB}$. Then

$$AJ = FJ = \frac{1}{2}(a+b)$$

and so a circle with centre $J$ and radius $AJ$ will pass through both $A$ and $F$. (One quadrant of this circle is shown in the diagram.) Let this circle cut the side $\overline{BC}$ at $W$. The assertion is that $BW$ is the required length $s$.

This is not hard to see. Fairly clearly the triangle $BJW$ is right-angled with $\overline{JW}$ as the hypotenuse. But by the construction $JW = AJ = \frac{1}{2}(a+b)$ and we may readily find that $BJ = \frac{1}{2}(b-a)$. Thus by Pythagoras' Theorem

$$\begin{aligned} BW &= \sqrt{\left[\frac{1}{2}(a+b)\right]^2 - \left[\frac{1}{2}(b-a)\right]^2} \\ &= \sqrt{ab} \text{ (after a little algebra)} \end{aligned}$$

This proves the assertion. But the interesting point is that all of this may date from a time *before* Pythagoras. This means that what we now know as "Pythagoras' Theorem" may in fact have been known in India before Pythagoras' time. It was also known earlier in China, but that is another story; it very likely was known even earlier yet in ancient Babylon. That too is another story, but you can read it in *Function Vol* 15, *Part* 3, pp. 85-91.

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Just prior to the Atlanta Olympics, *The Bulletin* ran a supplement devoted to them. It included the information that in the course of a weekend, the 40 operators handling a "hot-line" had fielded 5.3 million queries. Even if the "weekend" is supposed to run from Friday lunchtime to Monday lunchtime, this means that each query must have been processed in under two seconds!

That is good old US efficiency for you!

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# HOPPING ANAGRAMS AND CYCLES

**Alan Roberts, Monash University**

Professor Bolton's article on anagram groups (*Function Vol* 20 *Part* 4) gives a mathematical approach to the finding of anagrams when solving cryptic crossword puzzles. I would like to take this a bit further.

In case you're not an addict of cryptics, here is a simple example of an anagram clue:

Seeking large B.A. all mixed-up, to present this subject. (7)

The "7" is the number of letters in the answer. The word "mixed-up" ("confused" or "muddled" would do just as well) warns us that the answer is probably an anagram, made from the letters LARGEBA in some other order. (Always ignore punctuation.)

The answer here is pretty obvious (ALGEBRA), but sometimes it is much harder to find. (E.g.: SATURNALIA is an anagram of AUSTRALIAN, and REREADING is one of GRENADIER.) Professor Bolton examines a "hopping" system used to give rearrangements; even if the actual answer is not among them, one might be close enough to suggest it. The procedure is:

Put the letters in a circle. Keep one of them (the first, say) fixed in position, and make a new word by hopping over the second letter and writing down the third; then hop over the fourth and write down the fifth; and so on. (Check that this turns LARGEBA into LREAAGB.) If the new arrangement isn't helpful enough, repeat the process on the word you have just formed (so that LREAAGB is converted into LEABRAG). Keep doing this until either you have the answer, or your latest word is the one you started with (here, LARGEBA) – there is no point in continuing in either case!

(Notice that, for the method to work, $N$, the number of letters, must be an odd number; otherwise, the hops will not give us all the letters in the original. To see this, try it with $N = 6$, for example.)

The article shows something rather puzzling: that the number $A$ of different anagrams on $N$ letters varies irregularly when we increase $N$. It gives the figures:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $N$: | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 |
| $A$: | 2 | 4 | 3 | 6 | 10 | 12 | 4 | 8 |

Why the sudden drop when $N = 15$, where $A$ changes from 12 to just 4? Is it just an isolated hiccup? Do things smooth out as $N$ gets bigger? No, they don't! I wrote a simple computer program to do the "hopping" for a general number of letters $N$, and here are some of the results:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $N$: | 77 | 79 | 81 | 83 | 85 | 87 | 89 |
| $A$: | 30 | 39 | 54 | 82 | 8 | 28 | 11 |

To understand what is going on, we need a shorthand symbol to describe the hopping process. Here is a standard way to do this, applied to the LARGEBA example:

Give each letter in LARGEBA a serial number – L gets 1, A gets 2 and so on. Write down these numbers (1 to 7) in a row, and below each number write the serial number of the letter that replaces it. Put this double row in brackets and call the whole thing $P$. Thus

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 5 & 7 & 2 & 4 & 6 \end{pmatrix}$$

We can call $P$ an operator, defined by the operation: "Replace each letter in the word by the letter whose serial number is below its own serial number." So, to apply $P$ to the word LARGEBA, we replace its first letter (L) by its first letter (L) (meaning: leave it unchanged), replace its second letter (A) by its third (R), and so on. Thus, when $P$ operates on the word LARGEBA, it gives us the word LREAAGB.

If we carry out the same operation $P$ again on this new word (leave its first letter unchanged, replace its second by its third, and so on), we get the word LEABRAG. We define $P^2$ as the operator whose effect is the same as applying $P$ twice – check that, in the notation used above, we have

$$P^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 2 & 6 & 3 & 7 & 4 \end{pmatrix}$$

We define $P^3, P^4, \ldots$ similarly.

Since $P$ produces a rearrangement, or *permutation*, of the original letters, it is called a permutation operator. Now, there is another way to represent a permutation: by breaking it up into *cycles*. To do this, go systematically through the serial numbers, starting from 1, and keep writing the successive replacements until you get back to 1; then enclose in brackets what you now have, go on to the next serial number not yet dealt with and repeat the process; keep doing this until all the letters have been treated. For example, start with 1 above; since you immediately come back to 1, which is never shifted, write (1). Go on to 2; this is replaced by 3, which itself is replaced by 5, which is replaced by 2 – the serial number we started this cycle with; so write (235). This cycle is always interpreted as though it were written in a circle – that is, it means: $2 \to 3$, $3 \to 5$, $5 \to 2$. Thus, using this cycle notation, we can write

$$P = (1)(235)(476).$$

Try $P^2, P^3, \ldots$; you should find

$$P^2 = (1)(253)(467), \; P^3 = (1)(2)(3)(4)(5)(6)(7).$$

Note that, as the method guarantees, no cycle affects any serial number save its own set, and no two cycles have a serial number in common. Also, if a cycle involves $k$ serial numbers – we will call this a "$k$-cycle" – then repeating it $k$ times must bring back the original order.

This last property shows why doing $P$ three times brings back the original order – or, in the language of group theory, why

$$P^3 = E, \text{ the "identity operator".}$$

So, carrying out a $k$-cycle gives $k$ different anagrams (including the original one). But what if $P$ is a mixture of cycles of different lengths? This can certainly happen – for example, with $n = 9$ we have

$$P' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 3 & 5 & 7 & 9 & 2 & 4 & 6 & 8 \end{pmatrix}$$

and this, in cycle notation, is (1) (235986) (47). So we have both a 2-cycle and a 6-cycle. Now, the 6-cycle will need 6 successive shuffles to restore the letters it works on to their original places; but the 2-cycle will also have restored its letters after 6 shuffles (actually, for the third time!). So the permutation as a whole will restore the original order after 6 shuffles; that is, there will be 6 different anagrams.

The general rule is now pretty clear: to find how many times a permutation must be applied before the original order is restored, and thereby find the number $A$ of different anagrams, break it up into cycles and then take the least common multiple of the cycle lengths.

Here are examples of the $A$-values (i.e. number of permutations) and cycle lengths for larger values of $N$. I have deliberately picked values of $N$ which show how irregular is the way the hopping permutation breaks up into cycles, which is why the values of $A$ jump about so much:

| $N$ | $A$ | Cycle lengths |
|---|---|---|
| 63 | 6 | 1 6 6 6 6 3 6 6 6 2 6 3 6 |
| 65 | 12 | 1 12 12 12 12 12 4 |
| 67 | 66 | 1 66 |
| 69 | 22 | 1 22 11 22 11 2 |

The same approach will give you the solution for *any* value of the "hop step". For example, if you choose to hop over the *two* nearest neighbours instead of only the first (a hop step of 3), carry out the operation once and write down the resulting permutation of the order. Then break the permutation up into cycles (this is straightforward, as described above), and get the least common multiple of the cycle lengths. (Just be careful that your hop step value and $N$ don't have a common factor! If they do, you will find that there are some places never touched by the hop operator – you won't get an anagram containing all the original letters. That is why we took only odd values of $N$ above, where our hop step was 2.)

By the way, in the examples above, we never actually have to work out a least common multiple – conveniently, the largest cycle length is good enough because the smaller lengths all divide into it evenly. Will this always be true, for all $N$ and any hop step? I haven't been able to prove this; perhaps you can.

Note, though, that there are certainly permutations for which it is *not* true. Just choose a set of cycles whose lengths don't all divide into the longest one – e.g., for $N = 7$ you could pick the permutation made up of cycles (1) (2) (34) (567), with lengths 1, 1, 2, 3.

This means that, if the longest cycle length *is* always an exact multiple of the other lengths, this can only be due to the special character of the permutations we are looking at – ones got by the hopping procedure. So the fact that we are dealing only with this special set must come into your proof somehow.

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# BUFFON'S PAPERCLIP

**John Shanks, University of Otago**

"Buffon's Paperclip" is an odd title for an article discussing the classic problem which is usually referred to as Buffon's Needle.[1] But, as we will see later, in this context a paperclip is just as good as a needle!

Comte de Buffon was an 18th century French naturalist and mathematician, who in 1777 formulated the following problem:

> *Suppose that a thin rod is thrown in the air in a room whose floor consists of parallel boards. Of two players, one bets that the rod will not intersect any of the parallel floor joins, while the other bets the opposite, namely that the rod will intersect one of these joins. One may ask which of the two has the higher odds. This game can be played on a checkerboard with a sewing needle, or a headless pin.*

Attempts at solving Buffon's needle problem started the development of geometrical probability, in which concepts of randomness are applied to geometry.

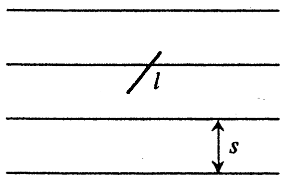


**Figure 1**

To answer Buffon's question, firstly let the needle length be $l$ and let the spacing between the parallel lines be $s$ as in Figure 1, which shows one position of the needle where an intersection has occurred. If we associate a probability $p$ with the event in which the needle crosses a line (we will call this an *intersection*), then it is reasonable that $p$ depends on both $l$ and $s$.

[1] Also see other references in *Function*, *Vol* 3 *Part* 5, *Vol* 4 *Part* 1, and *Vol* 13 *Part* 5.

Later we will consider two methods for deriving a formula for this probability, but for now we quote the result:

$$p = \frac{2l}{\pi s}.$$

This formula applies in the case $l \leq s$. If the needle is longer than the line spacing, then there is a possibility that it crosses two (or more) lines; this case leads to a more complicated formula, which will not be considered here.

Note that $p = 1/2$ when $2l/(\pi s) = 1/2$ or $l = (\pi/4)s \approx 0.785s$. Thus 78.5% of the line spacing is the critical value for the length: this configuration is when the odds are even.

In the Mathematics Department of the University of Otago, we hold a "Hands on Science" programme each year, and Buffon's needle problem is one of the activities. Using a needle length equal to the line spacing, so that $l = s$ and $p = 2/\pi = 0.637$, results from four experiments are shown in the table below. In each experiment a needle was dropped 200 times and the number of drops where an intersection occurred was recorded.

| Experiment | 1 | 2 | 3 | 4 | Combined |
|---|---|---|---|---|---|
| No. of intersections | 123 | 120 | 127 | 133 | 503 |
| Estimate of $p$ | 0.615 | 0.600 | 0.635 | 0.665 | 0.629 |

As in all probabilistic experiments, there can be a good deal of variability which can be reduced by increasing the sample size. Note that the combined estimate from 800 drops gives a fair approximation to the theoretical value 0.637.

## The paperclip

The result $p = 2l/(\pi s)$ can be derived using calculus similar to Buffon's own solution. We will consider such a derivation later, but first we use a paperclip approach, together with "expected values".

Instead of thinking in terms of probability, we change to consider the *expected number* of intersections. If $X$ is the outcome of some experiment then the expected value of $X$ is found by summing the products of all possible values of $X$ with their probabilities. For example, the expected value of a die toss is

$$1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5$$

For a single drop of the needle, the possible outcomes are 1 intersection with probability $p$ and 0 intersections with probability $1-p$. Therefore the expected number of intersections is $1 \times p + 0 \times (1-p) = p$; that is, the same as the probability of an intersection, *in this case*. The concept of expected number makes most sense when we consider repetitions of the experiment. In 100 drops the expected number will be $100p$; in 1000 drops it will be $1000p$.

An important result from probability theory tells us that the expected value of the sum of two random variables is the sum of the two separate expected values, even if the associated events are dependent. Dependent events occur when the outcome of one affects the outcome of the other.

Now returning to the problem at hand, we have let the expected number of intersections in one drop be $p$. It is reasonable that if we were to halve the length of the needle, then the expected number of intersections would be $p/2$. With this in mind, suppose we take the original needle of length $l$ and bend it to form a square. Consider one side of this square: this has length $l/4$ and thus the expected number of intersections from this side is $p/4$. But the same can be said of all four sides, so the total expected number of intersections is $p/4 + p/4 + p/4 + p/4 = p$, where we have added the individual expected values in accordance with the result mentioned earlier. (Clearly the sides of the square are not independent: if one side crosses a line then one of the other sides must also cross the line; however, as noted the result still holds.)

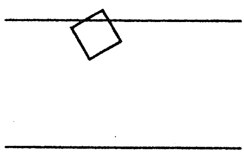

**Figure 2**

Figure 2 shows a case where the square has two intersections with a line. In fact, it will always have 0 or 2 intersections. (The occurrence of the situation where a corner of the square just touches the line can be ignored – it has probability zero.) It is important here to count the total number of intersections, rather than just noting whether the square intersects or not.

In the case of the needle it did not matter which approach we took, as each intersection would always be counted as "one".

You may have noticed that we did not use the fact that our object was a square, only that it was made up of four pieces each of length $l/4$. Indeed, we could have bent the needle into a triangle (Figure 3) with irregular sides $a, b$ and $c$ (where $a+b+c=l$). The total expected number of intersections (adding the contributions from the three sides) would be

$$\frac{a}{l}p + \frac{b}{l}p + \frac{c}{l}p = p$$

once again. In fact, there is no need to use a closed shape: we can bend the needle how we please, as long as it remains a plane curve – that is, is a two-dimensional curve. For example, we could bend it into a paperclip. For such a complex shape (see Figure 3), there could be 0, 1, 2, 3 or 4 intersections from a single drop. The expected number will still be $p$. To see this, we imagine the paperclip made up of a large number, say $n$, of short *straight* sections each of length $l/n$. Conceptually we can let $n$ take larger and larger values so that in the limit we have a smooth curve. With $n$ sections we can apply the previous idea:

Expected number from each short section $= p/n$

Total expected number $= n(p/n) = p$.

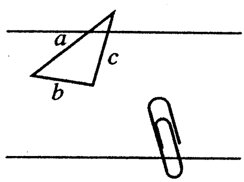


**Figure 3**

So much for the paperclip, whose shape is somewhat complex. Suppose we take the same idea, that of bending the needle into a smooth curve, but try for something simpler; in fact, try for something both simple and symmetric. The circle is the simplest, most symmetric plane figure. If we bend the needle to form a circle, then its diameter will be $l/\pi$. Suppose we drop this circle and it lands as shown in Figure 4 with its centre between two parallel lines. The circle will intersect (twice) if its centre is within $l/2\pi$ of either line. The chance of this happening is

$$\frac{\frac{l}{2\pi}+\frac{l}{2\pi}}{s}=\frac{l}{\pi s}$$

and the expected number of intersections is twice this: $2l/(\pi s)$.

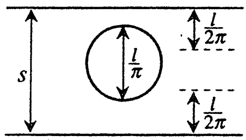


**Figure 4**

But, by a similar argument to that for the square, triangle or paperclip, this expected number remains at $p$ and hence our result:

$$p=\frac{2l}{\pi s}.$$

This example demonstrates that $p$ is not in general equal to the probability of an intersection. In this case, this probability is only $l/(\pi s)$.

## An area derivation

The probability $p$ can also be found by various approaches using calculus, the simplest of these being based on a comparison of areas. Suppose the needle lands as shown in Figure 5, where the parallel lines are drawn horizontally.

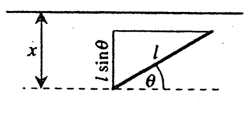


**Figure 5**

While $\theta$ measures the angle between the needle and the horizontal and can take values between 0 and 180° (or $\pi$ radians), $x$ measures the distance between the "lowest" end of the needle and the next "higher" line, and can vary from 0 to $s$. For any given angle $\theta$, the needle will intersect the line if $x < l\sin\theta$. Figure 6 shows the possible values of $x$ and $\theta$ drawn along two axes.

The rectangle represents all possible positions of the needle, since any point in the rectangle, such as the point $A$, corresponds to an angle $\theta$ and distance $x$ which fix the needle precisely (except for its horizontal position, but this does not affect the chance of intersecting a line). Of these possible positions, only those with $x < l\sin\theta$ (shown shaded) correspond to an intersection. Thus

$$p = \frac{\text{area shaded}}{\text{area of rectangle}}.$$

Here is where the calculus is involved, since the area under the curve $l\sin\theta$ requires integration:

$$\text{area shaded } = \int_0^\pi l\sin\theta\,d\theta = -l[\cos\pi - \cos 0] = 2l.$$

Clearly the rectangle has area $\pi s$, so once again we find that $p = 2l/(\pi s)$.

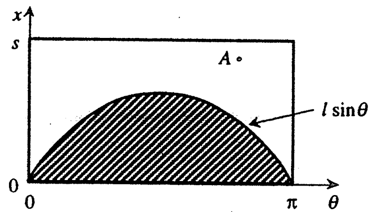


**Figure 6**

## Estimation of $\pi$

By carrying out the needle dropping experiment a large number of times, a good estimate of $p = \frac{2l}{\pi s}$ can be obtained. If $l$ and $s$ are known, then we can turn the situation around and produce an estimate of $\pi$. Buffon's needle problem was never intended as a means of estimating $\pi$ (and we will see why it is inappropriate), but that has not stopped many attempts to use it for that very purpose.

If a needle is dropped $N$ times and $m$ of these intersect a line, then we should have

$$\frac{m}{N} \approx \frac{2l}{\pi s}$$

or

$$\pi \approx \frac{2l}{s}\frac{N}{m}.$$

In the University of Otago experiments mentioned earlier, where $l = s$, we should have

$$\pi \approx \frac{2N}{m}.$$

For $N = 800$ and $m = 503$ we obtain the estimate $1600/503 \approx 3.18$, a value correct to two figures, a somewhat disappointing result from dropping a needle 800 times.

Let us suppose for simplicity that $l = s/2$; then the estimate is simply $\pi \approx N/m$. If $N = 10000$ and $m = 3183$ intersections, then the resulting estimate of $\pi$ would be $10000/3183 = 3.14169$, with an error of about 0.0001. (The value 3183 has been chosen to give the most favourable result; note that values of $m = 3182$ and $m = 3184$ would give estimates with errors about ten times larger.) It is worth considering what happens on the next throw; either we get an intersection (new estimate $= 10001/3184 = 3.14102$) or not (new estimate $= 10001/3183 = 3.14200$). The variation here should warn us that this is a very inefficient method indeed!

In general, if we claim that $\frac{N}{m}$ is a good estimate of $\pi$, then $\frac{N+1}{m+1}$ and $\frac{N+1}{m}$ must be just as good, otherwise the accuracy will be lost on the very next drop. The difference between these possible estimates gives some guide to the accuracy:

$$\frac{N+1}{m} - \frac{N+1}{m+1} = \frac{N+1}{m(m+1)} \approx \frac{\pi}{m} \approx \frac{\pi^2}{N}.$$

Thus, to be reasonably sure of getting $\pi$ to, say, six figures we would need $\pi^2/N = 0.000005$, or $N \approx 2$ million drops. The situation could be much worse than this, as it is quite possible that the next few drops turn out the same; for example, the next five drops could easily produce five intersections, giving a new estimate of $10005/3188 = 3.13833$.

In a well-publicised experiment in 1901, Lazzerini used a needle with $l = 5/8s$. He made 3408 drops and recorded 1356 intersections. His estimate of $\pi$ was therefore:

$$\frac{5}{4} \times \frac{3408}{1356} = 3.1415929,$$

correct to seven figures. The strange choice for the number of drops raises the suspicion that this was just a fortuitous event, with the experiment being stopped at an optimum point (the true value of $\pi$ being known beforehand, of course). The good fortune is revealed if we consider what would happen on the next drop; the new estimate will be one of the following:

$$\frac{5}{4} \times \frac{3409}{1357} = 3.1401990, \quad \frac{5}{4} \times \frac{3409}{1356} = 3.1425147,$$

both of which are correct only to three figures.

**Your turn**

You may like to perform your own experiment. All you need is a large sheet of paper on which you rule equally spaced parallel lines, and a "needle" which in practice could be any long thin object, such as a piece of stiff wire or a toothpick. You should make the spacing of the lines about the same as or a little larger than the length of the needle, and the lines themselves should be very fine to aid in deciding when an intersection occurs. Drop the needle from a reasonable height and ignore any drops where the needle bounces off or touches the edge of the paper. To minimise any bias that you may have in dropping the needle, the paper should be rotated periodically, say every ten drops. Continue as long as you can, but perhaps note down results every 50 drops so that you can monitor the accuracy of your estimates.

If you have access to the Internet you may want to join in a worldwide experiment. A Web page dedicated to Buffon's needle problem lets you record your results and even has a program that you can download for simulating the experiment on a computer. The location of the page is:

http://www.mste.uiuc.edu/reese/buffon/buffon.html

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Mathematics is not only real, but it is the only reality. That is that the entire universe is made of matter, obviously. And matter is made of particles. It's made of electrons and neutrons and protons. So the entire universe is made out of particles. Now what are the particles made out of? They're not made out of anything. The only thing you can say about the reality of an electron is to cite its mathematical properties. So there's a sense in which matter has completely dissolved and what is left is just a mathematical structure.

– Martin Gardner, 1994

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# THE TAX ON THE TAX

**Michael A B Deakin**

Some years ago, one of the football clubs here in Melbourne gave its players some very generous end-of-season bonuses. This gesture was widely publicised and so came to the attention of the Taxation Department, who declared the bonuses to be taxable income and so demanded their cut. The players were naturally somewhat disappointed (to put it mildly), and in response to their outcry, the club very generously offered to pay the tax on their behalf.

However, this proposition also came to the attention of the Taxation Department, who then said that this further payment would also be taxable. This led me to thinking that here we had an example of the geometric series. For simplicity I will give a purely numerical example.

Suppose, for argument's sake, that the original bonus was \$10000, and that it attracted tax at the rate of 20%. Thus the player had to cough up \$2000 and was out of pocket by this amount. If the club then sought to reimburse this amount, a further tax of 20% $\times$ \$2000, i.e. \$400, would be payable. If matters continued in this way, the process would be very long and drawn out, with more and more demands for ever smaller amounts of money.

After infinitely many exchanges of letters, the club would have outlayed \$12500, of which the player would receive \$10000 and the Taxation Department \$2500.[1] If the various characters in this drama were to have gone down this road, it would have made sense to negotiate all this beforehand, but I rather suspect this would have been too much for the bureaucratic mind to grasp, and that a lengthy process of attrition would have ensued. On my sums there would have been seven demands for tax before the amounts involved became too trivial to bother about. On this scenario, the player would make an extra 10c in uncollected tax. You may care to explore such possibilities.

I was reminded of this episode when I met socially the accountant retained by the club in question. In fact the club did not (overtly at least) pay the first tax bill, and so the scenario discussed above was avoided. Quite how they did reimburse the players he was not prepared to say!

* * * * *

[1]Notice how one fifth of the total is one quarter of the player's share. See the article "Wholesale Profit" in *Function Vol* 20 *Part* 2.

# LETTERS TO THE EDITOR

The cover diagram for *Function Vol* 20 *Part* 4 may be analysed in terms of the Four Circles Theorem, otherwise known as "The Kiss Precise", and concerning the radii of four mutually tangent circles. This theorem was the subject of both the *Front Cover* and the *History of Mathematics* columns in *Function Vol* 15 *Part* 4. Let $r_1$ be the radius of the first circle and so on for each of the four. It will be more convenient to use the reciprocals of the various radii rather than the radii themselves. So put $\epsilon_1 = 1/r_1$, etc. The theorem then states that

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)^2 \tag{1}$$

If we apply this result to the circles of radii 1, 1, $r_n$ and $r_{n+1}$, we obtain a quadratic equation for $r_{n+1}$ in terms of $r_n$. Thus, if the latter is known, the former may be found. We may also use equation (1) to discover $r_1 = 1/4$ by using the two large circles and a third "circle" which is the straight line at the base of the figure (it has infinite radius and so the reciprocal of its radius is zero). I leave the details for readers to explore for themselves.

I also take this opportunity to add a further name to those listed in my earlier article as co-discoverers. A Colonel R S Beard also discovered the result and published it in the journal *Scripta Mathematica* in 1955. However, he missed the elegant and simple form given above.

From the use of the equation connecting $r_{n+1}$ and $r_n$ and from the calculated value of $r_1$, the result $r_n = 1/[2n(n+1)]$ given in the article may also be made to follow, and again the best method is the use of mathematical induction. This method had until recently been thought to be quite modern, the usual names attached to it being those of Francesco Maurolico (1494-1575) and Blaise Pascal (1623-1662). However, last year, Hussein Tahir, a mathematics teacher at Maribyrnong Secondary College, found a much earlier use of the technique in the work of Pappus of Alexandria. This probably dates from some time in the first half of the fourth century.

Mr Tahir's discovery was published in the Australian Mathematical Society's *Gazette* last October and has attracted the attention of the international journal *Mathematical Reviews* (who incidentally did me the honour of asking me to review Mr Tahir's paper). Of interest here is that Pappus employed the technique in the context of mutually tangent circles. Pappus devoted a lot of time to this topic and Mr Tahir has, I believe, a second paper in the pipeline, also on this topic and also due to be published in the *Gazette*.

Among Pappus' results are formulae from which the Four Circles Theorem could be derived. However it would be anachronistic to attribute this result to him as the derivation is neither immediate nor easy.

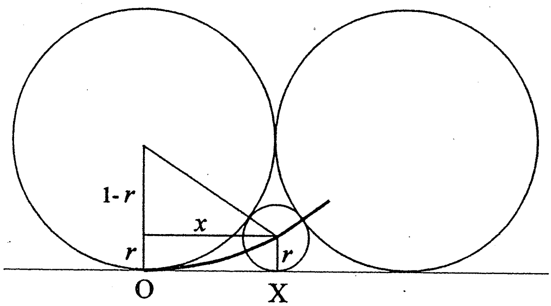
Michael A B Deakin

* * * * *

Dear Editor,

In *Function Vol* 20 *Part* 4, Cristina Varsavsky has examined the family of tangent circles bounded by the two large circles. She assumes (rightly so) that the locus of the centres is a straight line perpendicular to $\overline{OX}$.

I wish to examine the family bounded by the left hand circle and the line $\overline{OX}$. Let the radius of the given circle equal 1; draw a circle of radius $r$, touching the left hand circle and the line $\overline{OX}$. Applying Pythagoras we find that $x^2 = (1+r)^2 - (1-r^2)$, so $x^2 = 4r$, where $x$ is the abscissa of the centre of the circle. Therefore the locus of the centre of the circle is a parabola. This is illustrated in the following diagram.




Garnet J Greenbury
Brisbane

* * * * *

# HISTORY OF MATHEMATICS

## Yet More on Babylon

**Michael A B Deakin**

Following my discussion of Babylonian square roots in the last issue, I take this opportunity to add a few remarks to another earlier column (*Function Vol* 15 *Part* 3) on the Babylonian tablet Plimpton 322. Since that paper was published, I have become aware of an absolute mass of literature on this topic. In particular, Jens Høyrup wrote an article in the 1994 *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences* that draws attention to a very detailed and careful study by another Scandinavian researcher, Jöran Friberg, who in his turn lists many other discussions. There is also an article in *American Mathematical Monthly* (1980) by R Creighton Buck, and indeed there are others, but mostly in less accessible places.

Eleanor Robson, whose email postings I used extensively in my previous column, also makes some interesting comments on this matter, which I will summarise briefly. What is clear is that Plimpton 322 is a listing of Pythagorean triples, that is to say, numbers $b, h, d$ such that $b^2 + h^2 = d^2$, as in the problems discussed in the last issue. (Actually only $b$ and $d$ appear. It is usually assumed that the values of $h$ were on a piece that has broken off and is now missing.) Pythagorean triples, however, are those triples $(b, h, d)$ in which the three entries are all *integers*.[1]

Some of the triples listed on Plimpton 322 involve quite large numbers. The largest value of $d$ is 18541. They are also listed in decreasing order of $d/h$, starting with 1.40833... and ending with 1.1777.... There is clearly some system behind all this, but it is unclear quite why *these* particular triples were listed, and not others. One commonly accepted interpretation has it that the ancient Babylonians had discovered the formula for generating Pythagorean triples, and this was the suggestion I followed in my earlier column.

However, there have been some developments since then. In the first place, a previously untranslated word has been understood by Jens Høyrup as a technical term in cut-and-paste geometry. Second, there has been further debate on one assumption made by Neugebauer and Sachs. In the first column of numbers (neither $b$, $d$, nor $h$ but some other quantity), they had

[1]See the article by Mark Eid in *Function Vol* 20 *Part* 3 and the earlier articles referred to there.

assumed that an initial 1 had been broken off each entry with the left-hand portion. Eleanor Robson doubts that this was so.

She begins with a rectangle of sides $n$ and $1/n$ (and hence of area 1). By means of cut-and-paste geometry she converts this to a shape like that shown in Figure 1, whose area (still 1, of course) is the difference of two squares. Notice that $n$ and $1/n$ make a reciprocal pair, and remember that the Babylonians were interested in reciprocal pairs.

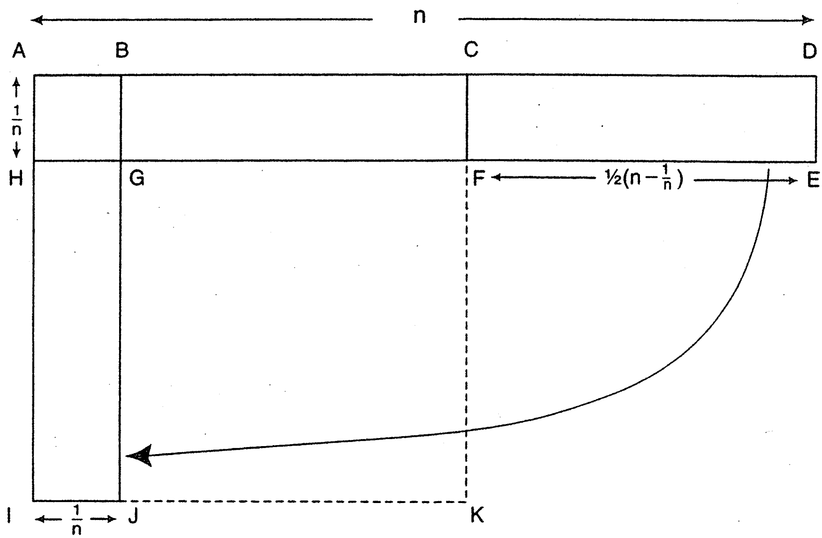



**Figure 1:** The difference of two squares produced by cut-and-paste geometry from a rectangle of length $n$ and breadth $\frac{1}{n}$. The original rectangle is $ADEH$. Put $AB = HG = AH = \frac{1}{n}$, and let $C$ bisect $BD$ and $F$ bisect $GE$. The rectangle $CDEF$ is then moved to the new position $GHIJ$. The new figure is $ACFGJI$, and this is the difference of the two squares $ACKI$ and $GFKJ$. This is one of several ways in which the cut and paste operation could be performed.

Now consider a right-angled triangle with $b = 1$. Then $d^2 - h^2 = 1$, and the left-hand side is the difference of two squares, so we can set it up as in the previous paragraph. Now put $d = \frac{1}{2}(n + \frac{1}{n})$ and $h = \frac{1}{2}(n - \frac{1}{n})$. This means that given any value of $n$, we can generate a triad 1, $d, h$ such that $d^2 - h^2 = 1$. Now put $n = \frac{u}{v}$. Then $d = \frac{1}{2}(\frac{u}{v} + \frac{v}{u})$ and $h = \frac{1}{2}(\frac{u}{v} - \frac{v}{u})$. Next take the equation $d^2 - h^2 = 1$ and multiply throughout by $2uv$. This produces another right-angled triangle, which is an exact scale model of the first triangle, and all of whose sides are integral. The diagonal will be $u^2+v^2$, the height $u^2 - v^2$ and the breadth $2uv$. These three expressions are referred to as the *generating formulae* for Pythagorean triples.

Dr Robson believes that this was the way in which the numbers on Plimpton 322 were produced. She calculated the value of $n$ for each of the pairs of numbers $b, d$ listed on it and found each to be a relatively simple fraction in base sixty, beginning with (2; 24) and ending with (1; 48). The values of $n$ decrease steadily and each has a simple reciprocal in base sixty, beginning with (0, 25) and ending with (0; 33, 20). These values, of course, *increase* steadily.

It is possible to make up a list of all the possible values of $n$ that could have been included. Three are in fact missing. Dr Robson writes: "I could think up reasons for [their] having been omitted – but they are not part of the usual [Old Babylonian] reciprocal repertoire, and I think it's OK just to say that the scribe missed them."

On this interpretation, the first column (without its extra 1) becomes merely the value of $h^2$ in the case $b = 1$. This is consistent with the new understanding of the column heading.

Note that on this interpretation, the generating formulae for Pythagorean triples need not have been explicitly known (as the usual accounts assume, and as I assumed in my column on the matter). We, looking back, see the connection; the scribe who wrote out the figures need not have.

Dr Robson concludes: "I see Plimpton 322 not as the culturally anomalous product of a far-sighted genius ... but as the outcome of two central concerns of [Old Babylonian] maths: reciprocal pairs, and cut and paste geometry. ... I presume it was composed or copied ... by a teacher who was preparing a list of parameters for right-angle problems."

**Acknowledgement:** It is a pleasure to thank Dr Robson for her constructive contributions to this article.

* * * * *

# COMPUTERS AND COMPUTING

## Plotting 3D Curves

**Cristina Varsavsky**

A curve in the plane is often described by a relationship between two variables, usually called $x$ and $y$; it is the collection of points $(x, y)$ in the cartesian plane which satisfy a mathematical equation. For example, the circle centred at $(a, b)$ and with radius $r$ is the graph of the equation $(x-a)^2 + (y-b)^2 = r^2$; the graph of $y = (x-a)^2 + b$ is a parabola with its vertex at $(a, b)$, etc.

We have also seen in *Function Vol* 18 *Part* 4 that some curves can be better described using polar coordinates; in this case the corresponding equation relates the radius and the angle. Many nice curves come from simple polar expressions.

Curves can also be expressed using parametric equations. We can think of a curve as being the path traced out by a moving point: at any time $t$, the position of the point is given by $(x(t), y(t))$. The $x-$ and $y-$ coordinates of each point on the curve are expressed in terms of the parameter $t$. For example, to represent a circle using parametric equations, we choose as parameter the angle corresponding to each point on the circle; we write

$$\begin{aligned} x(t) &= a + r\cos t \\ y(t) &= b + r\sin t, \quad 0 \leq t < 2\pi \end{aligned}$$

As another example, we consider the *cycloid*[1], the curve traced out by a point $P$ on the circumference of a circle as this circle rolls along a straight line (see Figure 1). This curve is best described by choosing as parameter the angle of rotation of the circle; the equations are:

$$\begin{aligned} x(t) &= r(t - \sin t) \\ y(t) &= r(1 - \cos t) \quad , \quad t \geq 0 \end{aligned}$$

Parametric curves are widely used in computer aided design, in computer graphics, and in specifying the shapes of letters and symbols in laser printing. A pioneer in this area was the mathematician Bézier who worked in the automotive industry. The *Bézier curves* are parametric curves which use

[1] See *Function Vol* 14 Part 1.

control points to obtain a mathematical expression of a curve which outlines the shape of the object being designed. As a matter of fact, the curve in Figure 1 was produced using the Bézier facilities of a drawing program. You can find more about Bézier curves in *Function Vol* 15 *Part* 4.

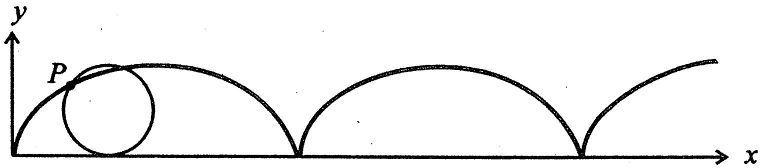



**Figure 1**

These three ways of representing a plane curve by means of a mathematical expression can be extended to curves in three-dimensional space. However, adding a third dimension makes the drawing and representation of curves a bit more complicated. A curve in three dimensions is a collection of points $(x, y, z)$ defined by some mathematical expressions. It could no longer be a single equation linking the variables $x$, $y$, and $z$; such equations represent surfaces in three dimensions. The intersection of two surfaces determines a curve, so two equations involving $x$, $y$, and $z$ should be provided to describe a curve in three dimensions.

The natural extension of polar coordinates in the plane is the use of polar coordinates in three dimensions, that is, the radius (distance to the origin), and two angles, representing the latitude and the longitude. Again, an equation involving these three variables will correspond to a surface, so two such equations are needed to express the curve.

Parametric equations are the simplest natural extension from the two-dimensional plane to represent space curves. Just in the same way as in the plane, a curve in three dimensions can be interpreted as being traced out by a moving point. The three coordinates of a point on the curve are thus expressed using parametric equations:

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

with $t$ – the parameter – varying over an interval.

Space curves are more difficult to draw by hand than plane curves. Fortunately, there are many readily available computer programs that can produce them on the screen. You can also write your own computer program to plot them.

In the computer column of *Function Vol* 19 *Part* 3 we discussed the underlying principles of representing surfaces on the flat computer screen, using a system of three axes, $x$, $y$, and $z$, as shown in Figure 2(a). Each point $(x, y, z)$ corresponds to the point $(a, b + z)$ on the screen (Figure 2(b)), where

$$a = -\frac{\sqrt{3}}{2}(x - y), \quad b = \frac{1}{2}(x + y)$$

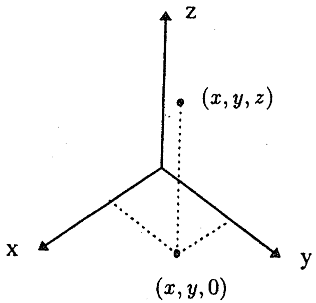



**Figure 2(a)**

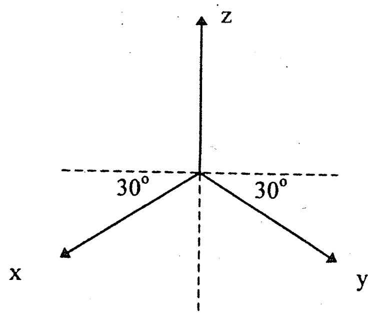



**Figure 2(b)**

The curve in three dimensions is then plotted by varying the value of the parameter $t$ on a discrete set of values, to plot a set of points belonging to the curve, and then joining these points with line segments. The basic program follows. You can make many modifications to it, such as labelling the axes, or enclosing the curve in a frame to give a better idea of its shape.

```
REM 3D Curves

   SCREEN 9: WINDOW (-12, -9)-(12, 9)
   INPUT "initial value of t:    ", initial
   INPUT "last value of t:       ", last
   INPUT "step size:             ", jump

REM Draw the 3 axes
   LINE (0, 0)-(-SQR(3) * 6, -6)
   LINE (0, 0)-(SQR(3) * 6, -6)
   LINE (0, 0)-(0, 8)
```

```
REM Draw the first point
   t = initial
   x = (3 + SIN(30 * t)) * COS(t)
   y = (3 + SIN(30 * t)) * SIN(t)
   z = COS(20 * t)

   a = -SQR(3) / 2 * (x - y)
   b = -1 / 2 * (x + y)

   PSET (a, b + z)

REM Draw the curve
   FOR t = initial + jump TO last STEP jump
      x = (3 + SIN(30 * t)) * COS(t)
      y = (3 + SIN(30 * t)) * SIN(t)
      z = COS(30 * t)

      a = -SQR(3) / 2 * (x - y)
      b = -1 / 2 * (x + y)

      LINE -(a, b + z)
   NEXT t
```

The program is written to display the *toroidal spiral* – that is, a spiral contained in a torus; its output, corresponding to $0 \le t \le 2\pi$, appears in Figure 3. You can vary the different numbers involved in the equation to see the effect on the shape of the curve.

The number of curves defined using parametric equations is limitless; here is just a small sample of them:

1. $x = t + \sin t + \sqrt{2/3}\cos t$, $y = t - \sin t + \sqrt{2/3}\cos t$, $z = 2t - \sqrt{2/3}\cos t$
2. $x = t\cos t$, $y = t\sin t$, $z = t$
3. $x = t$, $y = \frac{1}{1+t^2}$, $z = t^2$
4. $x = \cos t$, $y = \sin t$, $z = \ln t$
5. $x = t$, $y = \cos(2t)$, $z = \sin(2t)$

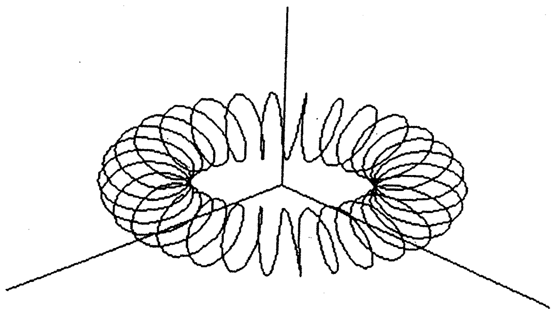


**Figure 3**

* * * * *

# $10^{100}$

The number $10^{100}$ is known as *googol*. It has no particular mathematical significance, but it seems to hold some interest for the kinds of people who are fascinated by large numbers. The word "googol" arose when the American mathematician Edward Kasner asked his 9-year-old nephew to suggest a name for the number. (It would appear to be unlikely that the nephew gave the matter a great deal of thought before answering.)

Googol is certainly large compared with the numbers we use to describe the physical world. For example, googol is much larger than the estimated number of subatomic particles in the universe. However, googol is not large when compared with some of the numbers that mathematicians encounter routinely in their work – it is very much smaller than the largest known prime number, for example.

Just in case you really want to know, $10^{\text{googol}}$ (1 followed by googol zeros) is called *googolplex*.

* * * * *

# PROBLEM CORNER

## SOLUTIONS

PROBLEM 20.3.1 (modified from a problem in *Alpha*, August 1995)

Find a straightedge and compass construction of a triangle $ABC$, given three points $X, Y, Z$ where the circumscribed circle intersects respectively the extension of the median through $C$, the extension of the altitude through $C$, and the extension of the angle bisector at $C$.

SOLUTION by Keith Anker, Monash University

The situation is depicted in Figure 1.

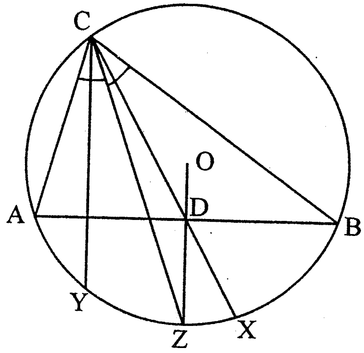



**Figure 1**

The centre of the circle, $O$, can be found by constructing the perpendicular bisectors of $\overline{XZ}$ and $\overline{YZ}$ and finding their point of intersection. Thus the circle can be constructed. The arcs $AZ$ and $ZB$ are equal, since they subtend equal angles at $C$. Therefore $\overline{ZO}$ is perpendicular to $\overline{AB}$. Since $\overline{CY}$ is also perpendicular to $\overline{AB}$ (by the definition of $Y$), $\overline{CY}$ is parallel to $\overline{ZO}$. Construct $\overline{CY}$ through $Y$ and parallel to $\overline{ZO}$, and locate $C$ where $\overline{CY}$ intersects the circle. Now, since the midpoint $D$ of $\overline{AB}$ lies on both $\overline{ZO}$ and $\overline{CX}$, it must lie at their intersection. Thus $A$ and $B$ can be located by constructing $\overline{AB}$ through $D$ perpendicular to $\overline{ZO}$, and finding the points where this line cuts the circle.

Also solved by Claudio Arconcher (São Pablo, Brazil).

PROBLEM 20.3.2 (based on a problem on the Internet)

Ten girls and ten boys are at a party. All the girls prefer cakes, and all the boys prefer ice creams. The children sit around a round table in no particular order, and each of them is served either a cake or an ice cream. Show that it is possible to rotate the table in such a way that at least ten children get what they prefer.

SOLUTION

Call a child "satisfied" if the child gets the dessert he or she prefers. Consider all 20 rotations of the table. Each dessert makes 10 children satisfied, so the total "amount of satisfaction" is 200. If each rotation led to at most 9 satisfied children, there would be at most 180 units of satisfaction.

Also solved by Derek Garson (Lane Cove, NSW).

PROBLEM 20.3.3 (1995 Old Mutual Mathematics Olympiad, South Africa; reprinted from *Mathematical Digest*, October 1995, University of Cape Town)

Suppose that $a_1, a_2, a_3, \ldots, a_n$ are the numbers $1, 2, 3, \ldots, n$ but written in any order. Prove that

$$(a_1 - 1)^2 + (a_2 - 2)^2 + (a_3 - 3)^2 + \ldots + (a_n - n)^2$$

is always even.

SOLUTION from *Mathematical Digest*

We note that it is equivalent to add up the parities (1 if odd, 0 if even) and then see if the result is even. But $x^2$ always has the same parity as $x$. So it is sufficient to prove that

$$(a_1 - 1) + (a_2 - 2) + (a_3 - 3) + \ldots + (a_n - n)$$

is always even. But this is merely

$$(a_1 + a_2 + \ldots + a_n) - (1 + 2 + 3 + \ldots + n) = 0$$

since $a_1, a_2, a_3, \ldots, a_n$ is a rearrangement of $1, 2, 3, \ldots, n$. Since 0 is even, we are done.

Also solved by Derek Garson (Lane Cove, NSW).

PROBLEM 20.3.4 (from *Alpha*, July 1995)

Prove that for every natural number $n$ such that $n \geq 2$ and every natural number $k$, the number $(1+k+k^2+\ldots+k^n)^2 - k^n$ is not prime.

SOLUTION

There are two cases to consider.

**Case 1:** $k = 1$. Then $(1+k+k^2+\ldots+k^n)^2 - k^n = (n+1)^2 - 1 = (n+2)n$, and both factors are greater than 1 because $n > 1$.

**Case 2:** $k > 1$. Using the formula for the sum of a geometric series, we obtain:

$$\begin{aligned}(1+k+k^2+\ldots+k^n)^2 - k^n &= \left(\frac{k^{n+1}-1}{k-1}\right)^2 - k^n \\ &= \frac{k^{2n+2} - 2k^{n+1} + 1 - k^{n+2} + 2k^{n+1} - k^n}{(k-1)^2} \\ &= \frac{k^{2n+2} - k^{n+2} - k^n + 1}{(k-1)^2} \\ &= \left(\frac{k^{n+2}-1}{k-1}\right)\left(\frac{k^n-1}{k-1}\right)\end{aligned}$$

Since each factor in this last expression is the sum of a geometric series, we can write the expression in the form

$$(1+k+k^2+\ldots+k^{n+1})(1+k+k^2+\ldots+k^{n-1})$$

in which the two factors are clearly both greater than 1. Hence the expression is a composite number, as required.

PROBLEM 20.3.5 (posted on the Internet by Bill Taylor, University of Canterbury, New Zealand)

Given any four points on the circumference of a circle, mark the four midpoints of the arcs between adjacent pairs of points. Form two chords by joining the opposite pairs of midpoints. Prove that these two chords cross at right angles.

SOLUTION based on a solution posted on the Internet by Geoff Bailey

In Figure 2, the four given points are indicated by larger dots, and the four midpoints of the arcs are indicated by smaller dots. The pairs of equal arcs are marked. The two chords are also shown, intersecting at $X$. Choose a pair of adjacent midpoints, $A$ and $B$, and draw the chord $\overline{AB}$. We want to show that the angle $AXB$ is a right angle. This will be true if the other two angles in the triangle $AXB$ add up to 90 degrees. But these two angles are subtended by arcs the sum of whose lengths is half the circumference of the circle. Hence their sum is 90 degrees.

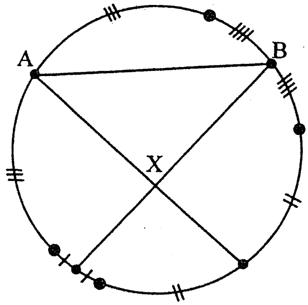



**Figure 2**

Also solved by David Shaw (Geelong, Vic).

## Solution to an earlier problem

In this issue we present the solution to the following amusing puzzle, which appeared in the August 1992 issue of *Function*.

PROBLEM 16.4.1 (by Lewis Carroll; submitted by G J Greenbury, Upper Mt Gravatt, Qld)

The Governor wished to give a small dinner party and invited his father's brother-in-law, his brother's father-in-law, his father-in-law's brother, and his brother-in-law's father.

How many guests attended?

SOLUTION

While up to four guests could have been invited, the word "small" implies that we are being challenged to find the smallest possible number. It turns out that there could have been just one guest! In order to arrive at this highly unlikely scenario, we must assume that some people have married their cousins. It works like this:

The Governor's father, Arthur (say), has two sons, the Governor and Bill, and a daughter, Clare. Arthur's sister, Deirdre, is married to Edward, who is the guest. This makes Edward the Governor's father's brother-in-law.

Deirdre and Edward have a daughter, Faye, and a son, Graham. It so happens that Faye is married to her cousin Bill, which makes Edward the Governor's brother's father-in-law. Also, Graham is married to his cousin Clare, making Edward the Governor's brother-in-law's father.

Finally, Edward has a brother, Henry, whose daughter Irene just happens to be married to the Governor! (Irene and the Governor are not blood relations.) This makes Edward the Governor's father-in-law's brother.

Interested (or confused) readers are encouraged to draw the family tree, which in this case might be better described as a family web.

## PROBLEMS

*Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received in sufficient time will be acknowledged in the next issue but one, and the best solutions will be published.*

### PROBLEM 20.5.1 (from *Mathematical Mayhem*, University of Toronto)

Three distinct integers $a, b$ and $c$ form an arithmetic progression, and $a+b$, $b+c$ and $c+a$ form a geometric progression (not necessarily in that order). If $a+b+c = -1197$, find the triplet $(a, b, c)$ which satisfies these conditions.

### PROBLEM 20.5.2

Let $\alpha, \beta$ and $\gamma$ be the radian measures of the angles of a spherical triangle (a figure on the surface of a sphere whose sides are parts of three great circles). If the sphere has radius 1, prove that the area of the triangle is $\alpha+\beta+\gamma-\pi$. (Hint: There is a simple and elegant proof which begins by extending the sides to form three great circles. Consider the areas of the pairs of "orange slices" formed by the great circles taken in pairs.)

### PROBLEM 20.5.3

Each point in the plane is coloured red or blue, and not all points are the same colour. Prove that it is always possible to find:

(a) an interval of unit length joining two points of the same colour;

(b) an interval of unit length joining two points of different colours.

PROBLEM 20.5.4 (from *Mathematical Spectrum*)

Find all natural numbers $n$ such that $2^n + n^2$ is a perfect square.

PROBLEM 20.5.5 (based on a problem on the Internet)

Find the unique natural number composed of the digits 1 to 9 (used once only) with the property that, for each $n$ $(1 \leq n \leq 9)$, the number formed by the first $n$ digits is divisible by $n$.

PROBLEM 20.5.6 (based on a problem on the Internet)

A coin is tossed a large number of times. Before each toss, you guess the outcome of the toss.

(a) If the coin is biased so that it lands heads 75% of the time, and you guess heads 50% of the time, what percentage of guesses will be correct in the long run?

(b) If the coin lands heads 50% of the time, and you guess heads 75% of the time, what percentage of guesses will be correct in the long run?

(c) Resolve the apparent paradox in parts (a) and (b).

* * * * *

# OLYMPIAD NEWS

## The XXXVII International Mathematical Olympiad

*Mumbai (Bombay) was this year's venue for the IMO. Teams, most comprising six members, from 75 countries had to contend with six problems during nine hours spread equally over two days in succession.*

*Here are the two papers:*

### First day: 10 July 1996

1. Let $ABCD$ be a rectangular board with $|AB| = 20$, $|BC| = 12$. The board is divided into $20 \times 12$ unit squares. Let $r$ be a given positive integer. A coin can be moved from one square to another if and only if the distance between the centres of the squares is $\sqrt{r}$. The task is to find a sequence of moves taking the coin from the square which has $A$ as a vertex to the square which has $B$ as a vertex.

(a) Show that the task cannot be done if $r$ is divisible by 2 or 3.

(b) Show that the task can be done if $r = 73$.

(c) Can the task be done when $r = 97$?

2. Let $P$ be a point inside triangle $ABC$ such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let $D, E$ be the incentres of $APB, APC$ respectively. Show that $AP, BD$ and $CE$ meet at a point.

3. Let $S = \{0, 1, 2, 3, \ldots\}$ be the set of non-negative integers. Find all functions $f$ defined on $S$ and taking their values in $S$ such that

$$f(m + f(n)) = f(f(m)) + f(n) \text{ for all } m, n \text{ in } S.$$

**Second day: 11 July 1996**

4. The positive integers $a$ and $b$ are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. Find the least possible value that can be taken by the minimum of these two squares.

5. Let $ABCDEF$ be a convex hexagon such that

$AB$ is parallel to $ED$,
$BC$ is parallel to $FE$ and
$CD$ is parallel to $AF$.

Let $R_A, R_C, R_E$ denote the circumradii of triangles $FAB, BCD, DEF$ respectively, and let $p$ denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{p}{2}.$$

6. Let $n, p, q$ be positive integers with $n > p + q$. Let $x_0, x_1, \ldots, x_n$ be integers satisfying the following conditions:

(a) $x_0 = x_n = 0$;

(b) for each integer $i$ with $1 \leq i \leq n$,

either $x_i - x_{i-1} = p$
or $x_i - x_{i-1} = -q$.

Show that there exists a pair $(i, j)$ of indices with $i < j$ and $(i, j) \neq (0, n)$ such that $x_i = x_j$.

*The paper turned out to be relatively difficult; only one student, a Romanian, finished with the full score of 42 points. By far the hardest problem on the papers was Question 5 (try it!).*

*The Australian team finished in place 23. The top five teams (in descending order of their unofficial scores) were: Romania, the United States of America, Hungary, Russia and the United Kingdom.*

*The other countries of the Asian Pacific Mathematics Olympiad ranked as follows: 7. Vietnam, 8. the Republic of Korea, 16. Canada, 20. the Republic of China (Taiwan), 26. Singapore, 27. Hong Kong, 37. New Zealand, 45. Colombia, 47. Thailand, 50. Macau, 70. Indonesia, 71. Chile, 72. Malaysia, 74. the Philippines.*

*Members of the Australian team received awards as follows:*

Jian He, Victoria, silver medal

Brett Parker, Victoria, silver medal

Alexandre Mah, New South Wales, bronze medal

Daniel Ford, New South Wales, bronze medal

Daniel Mathews, Victoria, bronze medal.

*During the IMO, the Annual General Meeting of the Asian Pacific Mathematics Olympiad (APMO) was held, at which APMO Chairman Carlos Bosch-Giral handed over Peter O'Halloran Medals, named after the Australian Founding Chairman of the APMO and sponsored by the Mexican Academy of Scientific Research, to the representatives of those countries from where the eight top-scoring papers (all with perfect scores) had originated. Among those medal winners is Jian He of University High School, Melbourne.*

*Congratulations to all winners!*

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How can it be that mathematics, a product of human thought independent of experience, is so admirably adapted to the objects of reality?

– Albert Einstein

# Is there life on Mars?

C.R. [*Cogent Reasonist*]: Tell me, Mr I. R., what in your opinion is the probability of life, in some form or other, on the planet Mars?

I.R. [*Insufficient Reasonist*]: H'm, let me see. Well, since I am totally ignorant of the answer, I shall have to assume that the possibilities of life and no life are equally likely. Therefore my answer is $1/2$.

C.R.: Very good. But now let us look at the problem from another angle. What would you say is the probability of no horses on Mars?

I.R.: Again I confess total ignorance, so again I must conclude $1/2$.

C.R.: And the probability of no cows?

I.R.: Again, $1/2$.

[*This sort of thing goes on for several minutes, while C. R. names, let us say, 17 more specific forms of life.*]

C.R.: Very well. But now we must conclude that the probability of all these things occurring at once – no horses *and* no cows *and* no dogs *and* none of the other 17 forms of life which I specified – is the product of the individual probabilities, or $(1/2).(1/2).(1/2)...$ to twenty terms. In other words, the probability that none of these twenty forms of life exists is $(1/2)^{20}$, or $1/1048576$. Am I right so far?

I.R. (*beginning to understand the trouble for which he is heading*): Why, yes, I am afraid you are.

C.R.: Thank you. But if the probability that *none* of these forms of life exists is $1/1048576$, what, may I ask, is the probability that *at least one* of them exists?

I.R.: Unfortunately, I must confess that this probability is the difference between your result and 1 – that is to say, $1048575/1048576$.

C.R.: And so, Mr I. R., we are led to two results concerning the probability of life on Mars. One of these is .5, and the other is about .999999 – very near to certainty. Surely one of the two must be wrong. Can it be that your principle of insufficient reason is at fault?

– from E P Northrop *Riddles in Mathematics*
Pelican/Penguin, 1960

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*Function* deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:


The Editors, *Function*
Department of Mathematics, Monash University
900 Dandenong Rd
Caulfield East VIC 3145, Australia
Fax: +61 (03) 9903 2227
e-mail: function@maths.monash.edu.au



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