Function

Founder Editor G. B. Preston

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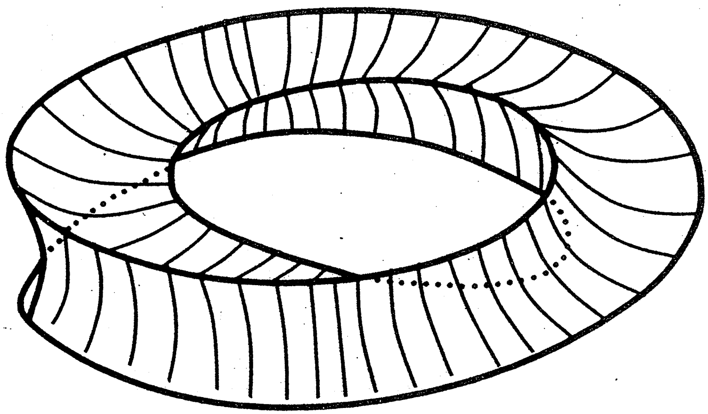
Volume 15 Part 3



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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of secondary schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. Recent issues have carried articles on advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside back cover.

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FUNCTION

Volume 15 *Part 3*

(Founder editor: G.B. Preston)

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THE FRONT COVER

The surface depicted on the front cover is one of a family of generalisations of the familiar Möbius strip (Figure 1). As is well known, the Möbus strip is formed by taking a strip, rotating one end by half a full turn and joining up the ends.

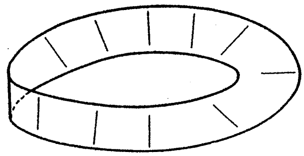

Figure 1

Now look at Figure 2. A small circle rolls inside a large one, the ratio of their diameters being 1 to 2. As this rolling proceeds, the point P on the circumference of the small circle moves backwards and forwards along a diameter of the larger circle. This diameter is taken as the cross-section of the strip from which Möbius strip is formed.

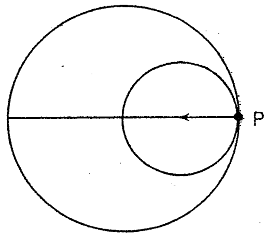

Figure 2

Now go to Figure 3. The situation is much the same except that the ratio of the diameters is now 1 to 3. The point P now moves around the inner, starlike curve, which is called a 3-cusped hypocycloid ("cusp" means "point", as in "bicuspid", a tooth with two points). Now imagine a cylinder with a 3-cusped hypocycloid as its cross section and suppose one end of this to be rotated through one third of a full turn. Now imagine the two ends to be joined. This produces the surface shown on the cover. It is sometimes referred to (for complicated reasons that would take too long to expain) as the *umbilic bracelet.*

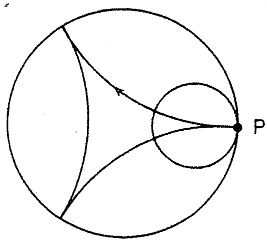

Figure 3

As with the Möbius strip, it is possible to reach any point on the umbilic bracelet's surface from any other without crossing the edge. You may care to explore other properties for yourself.

If (Figure 4) we make the ratio of the circles 1 to 4, the point P travels along a 4-cusped hypocycloid, better known as the *astroid.* For more on the astroid, see the cover story for *Function, Vol.2, Part* 4. By using a cylinder with an astroid as cross-section we can generate, via one quarter of a full turn, another anologue of the Möbius strip. And so it goes.

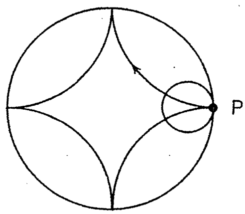

Figure 4

TANAKA'S "SHADOWS"

Michael A.B. Deakin, Monash University

Last year I travelled to Kyoto in Japan, where I attended the International Congress of Mathematicians. Almost 4000 mathematicians, from all around the world, took part, and the programme, running over nine days, was both busy and fruitful. After the manner of these gatherings, it attracted a number of fringe events: book and computer displays, spin-off conferences, informal seminars and the like.

One such was the distribution, in very large numbers, of a pamphlet entitled "On the Shadow of a Parabola and of the Other Curves" by Masakazu Tanaka, whom I take to be a teacher at Kyoto High School.

There are some very nice ideas in this work, and I'll present a few of them here. I do so in my own way, however, because the very personal style and terminology of the original, together with its detailed presentation of alternative approaches, do not make for easy reading.

Essentially, Mr Tanaka is concerned to display complex roots of algebraic equations in a graphical way. He succeeds in doing this, by a variety of techniques, for the complex zeros of both quadratic and cubic functions. I will give the gist of his argument, but in a somewhat simplified form.

Suppose that a quadratic equation has no real roots – or, if you like, that it has two complex roots. To simplify the discussion, assume the coefficient of x^2 to be 1. In this case, the graph is a parabola like that shown in Figure 1.

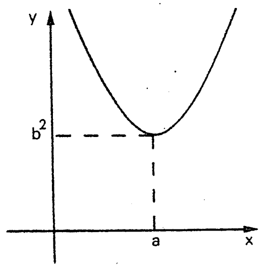

Figure 1

Its equation, found by completing the square, may be written as:

$$y = (x - a)^2 + b^2, \tag{1}$$

and the complex roots of the equation $y = 0$ are

$$x = a \pm ib. \tag{2}$$

Note that a, b can actually be read quite easily off the graph of the quadratic. Thus, if we had the graph of $y = x^2 - 2x + 5$, we could graph it and, by locating the minimum point, discover $a = 1, b^2 = 4$ and hence give the roots $1 \pm 2i$ of the equation $x^2 - 2x + 5 = 0$.

What Mr Tanaka wanted, however, was a way to display a, b as points on the x-axis. This turns out to be quite useful when it comes to visualising the complex roots of cubic equations. In the case of quadratic equations, the procedure is extremely simple.

Look at Figure 2. Through the minimum point of the parabola draw a horizontal line. This will be a tangent to the parabola, and its equation is $y = b^2$. Now draw a second parabola, which is actually a reflection of the first one in this horizontal line, and

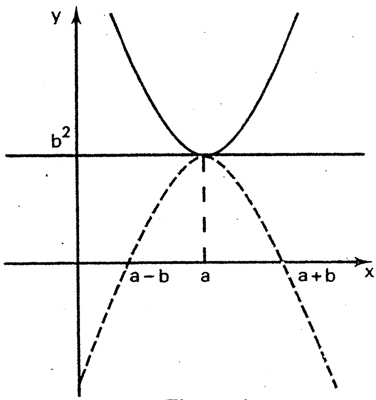

Figure 2

which therefore lies below the "mirror" $y = b^2$. The first parabola is described by Equation (1). This second parabola may be written as:

$$y = 2b^2 - [(x - a)^2 + b^2] \tag{3}$$

or

$$y = 2b^2 - y_1 \tag{4}$$

for short. (It is this second parabola that Mr Tanaka refers to as the "shadow".)

But we may rewrite Equation (3) as

$$y = -[(x - a)^2 - b^2], \tag{5}$$

which, of course, has the solutions

$$x = a \pm b. \tag{6}$$

Thus the complex roots of Equation (1) correspond in a very natural way to the real roots of the "shadow" equation (5).

Let us now move on to the more interesting case of cubic equations. First recall that every cubic equation must have at least one real root. The other two may either be both real or else they form a complex pair. (It can of course happen that two, or even three, roots are equal, but this is a detail you can easily sort out for yourself.) Here we will be concerned with the case in which the remaining two roots are complex.

We are now in a situation where the real and imaginary parts of these complex roots cannot readily be read off the cubic graph. The simple identifications of Figure 1 do not carry over to this more complicated case. Indeed, we could have a graph like any of those shown in Figure 3.

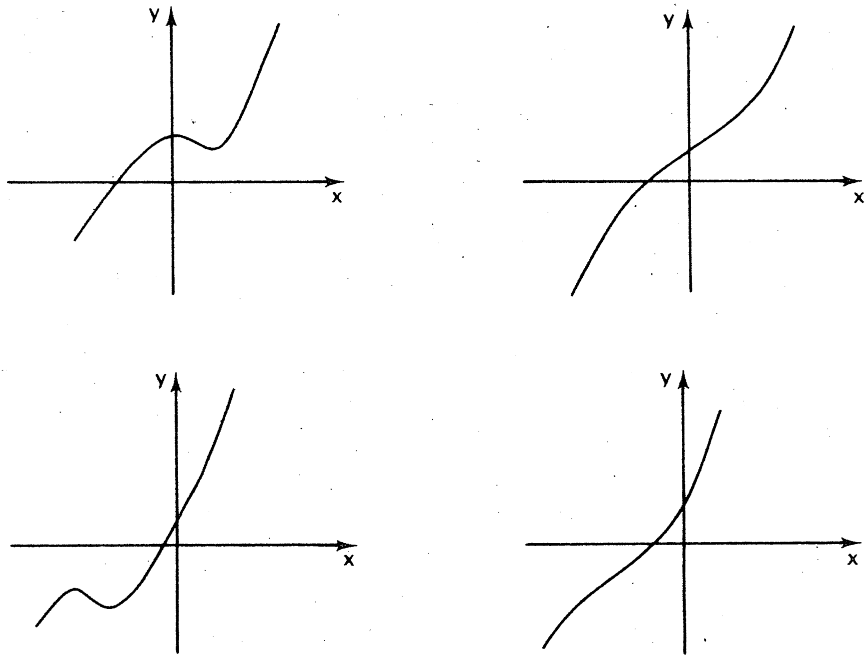

Figure 3

It is this profusion of detail that gives point to Mr Tanaka's construction.

There is, in each case, a unique point of intersection of the curve with the x-axis. Let this point have the coordinates $(c,0)$. Call the equation of the curve

$$y = y_1(x). \tag{7}$$

From the point $(c,0)$, we can draw line which is tangent to the curve at some point other than $(c,0)$. In all four cases, there will be exactly one such tangent. This may be proved by means of calculus or by other means, but it is evident from the four cases of Figure 3.

Call this tangent

$$y = Y(x), \tag{8}$$

and now introduce a second curve, Mr Tanaka's "shadow",

$$y = y_2(x), \tag{9}$$

where we define

$$y_2(x) = 2Y(x) - y_1(x). \tag{10}$$

Now the equation $y_1(x) = 0$ has one real root and two complex roots. Thus its roots may be written as $x = a \pm ib,\ c$. The roots of $y_2(x) = 0$ turn out to be $x = a \pm b,\ c$: three real numbers from which the complex roots of the original equation may easily be determined.

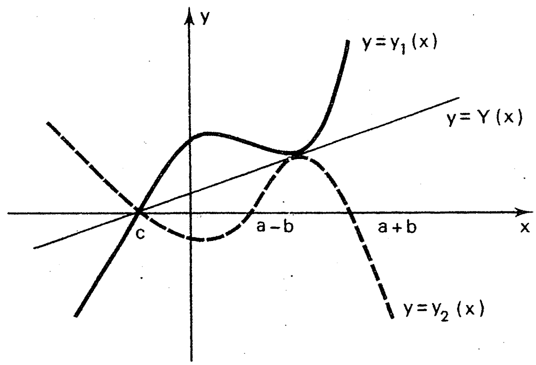

Figure 4

Figure 4 shows what is going on. The line $y = Y(x)$ acts, not exactly like, but rather like, a mirror. The curve $y = y_2(x)$ lies as far below the line $y = Y(x)$ as $y = y_1(x)$ lies above it, or *vice versa* when $x < c$. (We can think of it as the "vertical reflection" of $y = y_1(x)$, but perhaps it's best to stick to Mr Tanaka's word "shadow".)

Take a specific example. Suppose $y_1(x)$ is the cubic $x^3 - 4x^2 - 2x + 20$. Then the equation

$$x^3 - 4x^2 - 2x + 20 = 0$$

has the roots -2, $3 \pm i$. Through the point $(-2,0)$ we can draw a tangent to the curve

$$y = x^3 - 4x^2 - 2x + 20.$$

This tangent has the equation

$$y = x + 2$$

and it touches the curve at $(3,5)$. Thus, in this case, $Y(x) = x + 2$, and so we have

$$y_2(x) = 2(x + 2) - (x^3 - 4x^2 - 2x + 20),$$

by Equation (10). Simplify now to find

$$y_2(x) = -x^3 + 4x^2 + 4x - 16.$$

If we now solve the equation

$$-x^3 + 4x^2 + 4x - 16 = 0,$$

we find $x = -2, 2, 4$, that is to say, -2, 3 ± 1.

To see how this works, suppose the roots of the equation $y_1(x) = 0$ to be $a \pm ib$, c. Then we may write $y_1(x)$ as

$$y_1(x) = (x - a - ib)(x - a + ib)(x - c),$$

which is the same as

$$y_1(x) = x^3 - (2a + c)x^2 + (a^2 + b^2 + 2ac)x - c(a^2 + b^2). \qquad (11)$$

It is a relatively straightforward exercise in calculus (and again there are other ways do to it) to find the tangent through the point $(c,0)$. This is the line $y = Y(x)$ and it may be written as

$$Y(x) = b^2(x - c). \qquad (12)$$

Now use Equation (10) to find

$$y_2(x) = -x^3 + (2a + c)x^2 - (a^2 - b^2 + 2ac)x + c(a^2 - b^2). \qquad (13)$$

We may easily check that

$$y_2(x) = -(x - a - b)(x - a + b)(x - c), \qquad (14)$$

and so a, b may be exhibited explicitly.

For if we graph $y = y_2(x)$, we may read off the roots $a + b$, $a - b$. Then a is the mean of these two, and b is the distance from that mean to either root. This justifies Mr Tanaka's construction and the assertion just above Figure 4.

Mr Tanaka next posed the question of what happens with quartic graphs. Here he was able to give only a partial answer. The case may however be analysed, and I show how in what follows. A quartic graph will have either four real roots, two real roots or no real roots. (Again the case of equal roots is left to the reader.)

If there are four real roots, no question arises, so next take the case of two real roots. The original function will now be

$$y_1(x) = (x - a - ib)(x - a + ib)(x - c)(x - d), \tag{15}$$

which multiplies out to give

$$y_1(x) = x^4 - (2a + c + d)x^3 + (a^2 + b^2 + 2ac + 2ad + cd)x^2$$
$$- (a^2c + a^2d + b^2c + b^2d + 2acd)x + cd(a^2 + b^2).$$

The construction given previously for $Y(x)$ now no longer works, but we may proceed backwards. What we are looking for is a quartic expression with -1 as the coefficient of x^4 and with the factors $(x - a \pm b)$, $(x - c)$, $(x - d)$. This expression is $y_2(x)$ and, if we multiply the factors out, we find

$$y_2(x) = -x^4 + (2a + c + d)x^3 - (a^2 - b^2 + 2ac + 2ad + cd)x^2$$
$$+ (a^2c + a^2d - b^2c - b^2d + 2acd)x - cd(a^2 - b^2).$$

Now use Equation (10) to determine $Y(x)$ in this case. We find

$$Y(x) = b^2x^2 - b^2(c + d)x + b^2cd. \tag{16}$$

This is the "mirror" and it may readily be shown that it passes through the points $(c,0)$, $(d,0)$ and also that it is tangent to the curve $y = y_1(x)$ when $x = a$.

If, to give now the final case, the original quartic has no real roots, there will be two pairs of complex roots $a_1 \pm ib_1$, $a_2 \pm ib_2$. In this case, the "parabolic mirror" is tangent to the original curve both for $x = a_1$ and $x = a_2$. The "shadow equation" has roots $a_1 \pm b_1$, $a_2 \pm b_2$.

This concludes the discussion of quartics. The reader will note that a pattern has now emerged. If we have an equation of degree n

$$y_1(x) = x^n + a_1x^{n-1} + \dots + a_0, \tag{17}$$

the "mirror" will be a curve of degree n-2

$$Y(x) = Kx^{n-2} + \dots + P, \tag{18}$$

for suitable constants $K, \dots, P$. This "mirror" will pass through all the real roots of the equation $y_1(x) = 0$. The complex roots will occur in pairs and, for the values of x

equalling each real part of such a pair, the curves $y = y_1(x)$, $y = Y(x)$ will be tangent.

Then the "shadow curve" $y = y_2(x)$ defined by Equation (10) will pass through each real root of $y_1(x) = 0$ and will replace each pair of complex roots $a \pm ib$ with a pair of corresponding real roots $a \pm b$.

This pattern in fact emerged first in the case $n = 2$, but was not fully evident until the cases $n = 3$, $n = 4$ had been examined. For the quartic and higher graphs, no simple method exists for drawing the "mirror", so that the practical utility of the approach is at its best with the cubic case, where such construction is straightforward. Nonetheless, the higher cases give great insight into the way in which the functions $y_1(x)$, $y_2(x)$ are connected.

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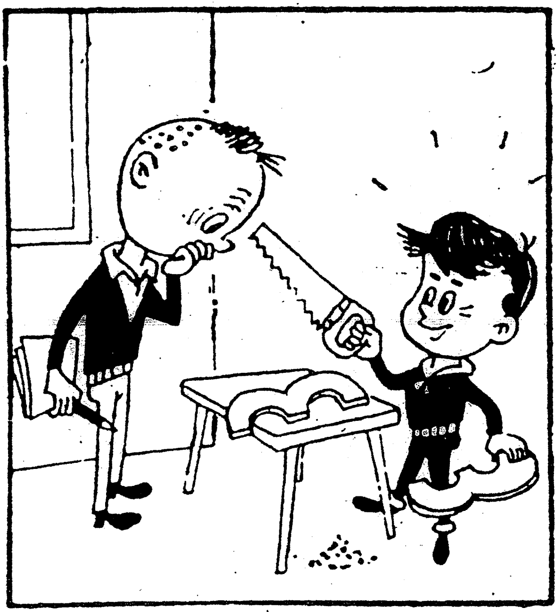


"So you see half of eight really isn't four"

From the *Leibziger Volkszeitung*, courtesy of *Alpha*, a German counterpart of *FUNCTION*.

THREE UPDATES

1. The Monash Sundial

Function, Vol. 14, *Part* 4 contained the story of the Monash sundial and how, after it had been giving the wrong times and dates for several years, it was adjusted on 28 March 1990. This was not an ideal date to do it because the sundial may only be read accurately on the hour and on the 22nd of each month. However, it turned out not to be possible to carry out the adjustment on the 22nd. So we knew, after the adjustment was made, that the sundial had given an accurate reading for *time* at midday on 28 March 1990. The question remained as to how accurate it would prove to be throughout the course of the next year.

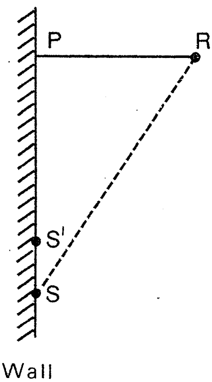
It wasn't very practical to check up on this during the dreary overcast winter months between March (when we adjusted the sundial) and August (when we published our account of that adjustment). However, now that we've had a whole summer to use for that purpose, we can report on how accurate the calculations were.

To give the conclusion first: the sundial is now almost certainly as accurate as we can make it. To the trained eye, it is out still, but not by very much. By amounts in fact that would be quite acceptable in a kitchen clock or a cheap watch. Its accuracy as far as time is concerned may be checked by watching the shadow pass over a time line (the loop technically referred to as an analemma) right on the hour. The maximum error observed since March 28, 1990, has been 40 seconds. This is well within the 5 minutes maximum error claimed by Dr Moppert, the designer.

How about the question of the sundial's accuracy as a calendar? How good is it on dates?

We have had the opportunity to check this on a few occasions and the accuracy is probably about as good as we can expect. The worst discrepancy occurred in the days around the summer solstice, December 22nd, 1990. Refer to the diagram at right.

The ring that casts the shadow on the wall is denoted by R and the shadow it casts is denoted by S. On those dates S lay below the correct position (denoted by S') by a few millimetres. There are several possible reasons for this. First, there are the limits imposed by the accuracy of the original design. Secondly, the wall itself has probably settled slightly; walls do. Thirdly, the adjustment of the ring is probably still not quite perfect. But there's also a fourth factor.

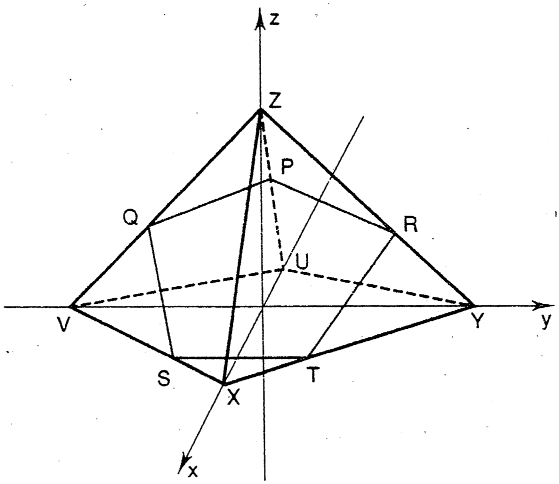


PR, the rod supporting the ring, is metal and expands in the heat and this enlarges the triangle PRS and so causes S to lie below S'. However, there's little to be done about this.

All in all we've decided the sundial is about as accurate as we can get it – and that accuracy is quite impressive.

2. The Beijing Theorem

By the "Beijing Theorem" I mean the result discussed in *Function, Vol.* 15, *Part* 1, pp. 8-12. Refer to this diagram from that article.



The pyramid *ZUYXV* is cut by an oblique plane to make a pentagon *PRSTR*. The base *UYXV* of the pyramid is square. We ask under what circumstances the pentagon *PQSTR* can be regular. The Beijing Theorem states that we require the height OZ $(= a)$ to equal OX, which is taken to be 1 unit long. Thus the pyramid has to be precisely half an octohedron.

First of all, I have myself (MABD) found a quite simple proof of the result. It's a little long to give here, but you could reconstruct it for yourself. Use vectors and express in vectorial form the requirement that $\overrightarrow{ST} \parallel \overrightarrow{QR}$. This establishes that $|XS| = |XT|$ and that the plane cuts the pyramid symmetrically. Next require $\overrightarrow{QS} \parallel \overrightarrow{RT}$. This gives the coordinates of all the points in terms of a. Finally require $|PR| = |ST|$ to find (after some algebra) $a = 1$.

The other matter is that Colin McIntosh (*Function, Vol.* 15, *Part* 2, pp. 48-50) raised the question of whether we could have $a = 0$ or $a = \infty$. The second of these is not possible, as it happens, but the first can occur. In fact there are four cases (whereas $a = 1$ gave two). David Albrecht (Department of Mathematics, Monash University, Clayton) has found them and here they are. Notice that there are two strict *pentagons*, one inside and one outside the square, and two *pentagrams*, again with one inside and one outside. The square is, of course, the pyramid, now squashed quite flat!

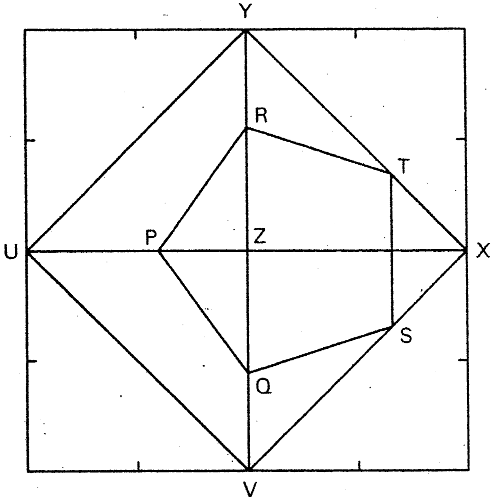


Figure 1

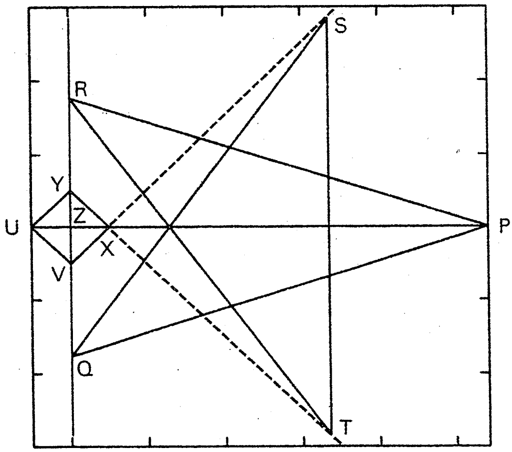


Figure 2

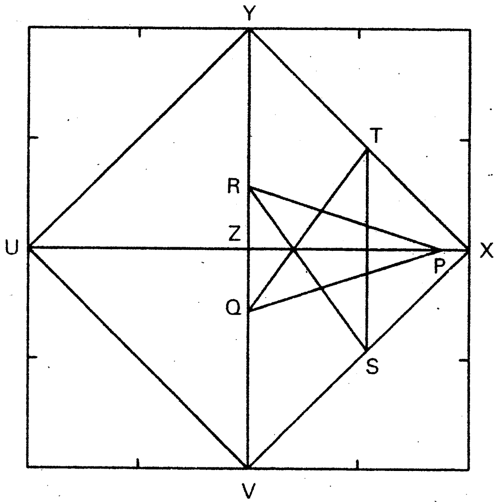


Figure 3

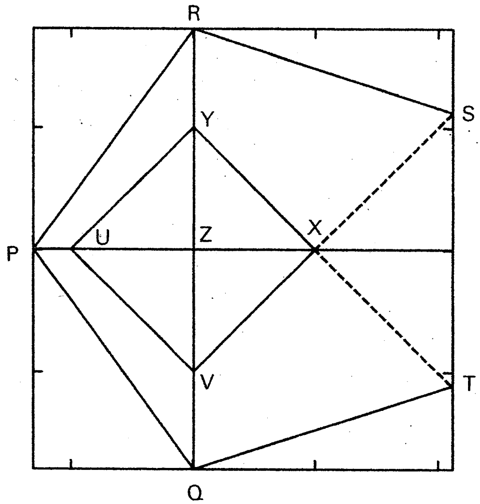
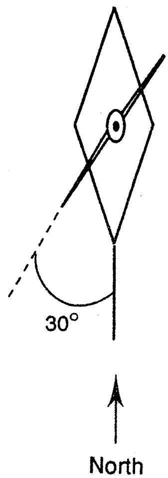


Figure 4

3. Facing Mecca

Following the article on "Facing Mecca" (*Function*, *Vol.* 15, *Part* 2, pp. 51-55) we have had sent to us a special compass enabling the qibla (i.e. the direction of Mecca) to be determined from places in North America. Our thanks to Professor H. Sharif of Shiraz University, Iran.

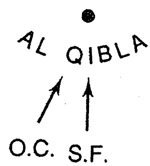
The compass consists of a magnetic needle with a second, smaller needle attached to it at an angle of (as near as I can measure it) 30°. See the figure at right.

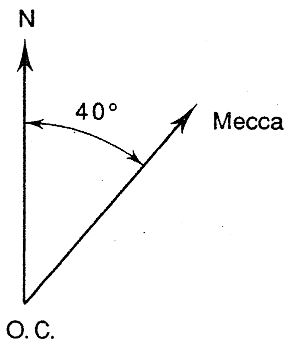


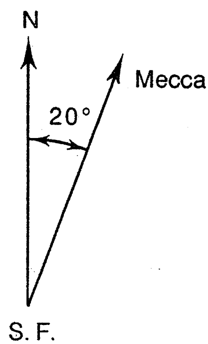
The needle assembly is mounted in a case which has directions assigned to various places in North America. These are printed on the case beneath the needle assembly. A simplified picture appears at lower right. S.F. stands for San Francisco and O.C. for Oklahoma City. These were the two places I chose for further investigation.

I chose Oklahoma City because it is more or less the central one of the various locations listed. San Francisco I chose for comparison and because it gives a relatively easy alignment of the instrument.

Using Formula (1) from the original article, I find that the qibla for Oklahoma City is roughly 40° East of North and for San Francisco roughly 20° East of North. That is, we should have:





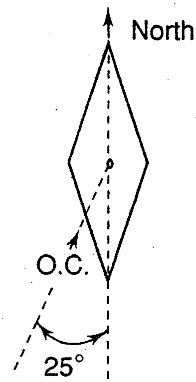


However, there is a complication. Magnetic north is not the same as true north. And this is particularly the case in North America – the North Magnetic Pole is situated in northern Canada.

In the *Encyclopedia Britannica* (*Vol.* 6, p. 29) you will find a map from which the difference can be estimated. A compass needle in San Francisco points about 20° east of true north. Thus to a good approximation the compass needle already and directly indicates the qibla in San Francisco. Check against my diagram of the case (above): the two directions line up.

Now let us come to the more complicated case of Oklahoma City. According to the *Britannica*, magnetic north is about 10° east of true north in Oklahoma City. As previously stated, the qibla for Oklahoma City is about 40° east of north. East of true north, that is to say. So the qibla for Oklahoma City lies about 30° east of magnetic north.

Lining the case up so that the arrow pointing to North is aligned with the north-pointing end of the compass needle means that the arrow for Oklahoma City makes an angle of (as closely as I can measure it) 25°. We have the situation shown in the diagram to the right. 25° and 30° are probably close enough, given all the inaccuracies involved in the calculations and measurements. (If further proof is required for this statement, it comes from the fact that the Oklahoma City arrow also serves for Dallas, Houston and the whole of Mexico.) Discrepancies of less than 10° probably count for very little in daily prayer. If one were, however, building a mosque more accuracy would be needed and proper surveying instruments would be used.



Finally, what of the auxiliary needle attached to the main magnetic needle? My only suggestion here is that it gives a sort of average for the whole region. It's about right for Oklahoma City which is the most central of the places listed. Furthermore, it saves us the labour of aligning the case with the needle and of deciphering the rather hard-to-read print on the case.

* * * * *

That Famous Prediction

The mass of a body is a measure of its energy - content; if the energy changes by L, the mass changes in the same sense by $L/9\times10^{20}$, the energy being measured in ergs, and the mass in grammes.

A. Einstein (1905)

COMPUTER SECTION

EDITOR: R.T. WORLEY

How do calculators and computers calculate functions such as $log(x)$, $sin(x)$ and the like? I have been asked this question a number of times, and the answer I have to give is "it depends on a lot of factors". In this column I cannot cover all the ways that can be used, so I will describe one way - the way used by the BASIC language in an early home computer. The choice of method used in this computer was determined by the requirement that it take as little room as possible in the computer's program memory, since this memory was expensive. Speed was probably of secondary importance.

The first thing to realise is that most computers (from here I will use computer to mean computer or calculator) only work to a certain accuracy, such as 7,9,11 or 17 decimal digits. This means that the value calculated as $log(x)$ need only be calculated to this accuracy. To do this the computer can, instead, calculate the value of a 'simpler' function which agrees with $log(x)$ to the accuracy needed. By 'simpler' is meant a function which, to calculate, uses operations like addition, subtraction, and multiplication. Division may also be allowed, especially if the computer's hardware makes it not too expensive in terms of time.

The typical 'simple' function is the polynomial, for example

$$1 + 0.5x^2 + 0.0416x^4.$$

The small computer can contain a single short program to calculate the value of a polynomial given the value of x and a table of the coefficients. Thus the computer can calculate the value of $sin(0.05)$ using the polynomial program by giving it the value 0.05 of x and a table containing the coefficients of a polynomial that approximates $sin(x)$, and it can calculate the value of $log(0.23)$ by giving the polynomial program the value 0.23 of x and a table containing the coefficients of a polynomial that approximates $log(x)$.

The problem with this approach is that of getting a polynomial to approximate the function to the desired accuracy. Normally a polynomial approximation will have a large error if one takes a large value of x, so the approximation only works well for a particular range of values of x. In addition, the function may not be able to be approximated well by a polynomial - for example, $log(x)$ is very large and negative for small positive x, while no polynomial has this property - and some means of overcoming this must be found.

Consider firstly the example of the function $sin(x)$. If x is the angle measured in degrees it is converted to radians by multiplying by $\pi/180$ (it is more natural in mathematics to measure angles in radians). For angles measured in radians, it can be shown that

$$sin(x) = x - (1/6)x^3 + (1/120)x^5 - (1/5040)x^7 + (1/362880)x^9 + E$$

where the error E is small if x is small. Thus if we ignore E we have the polynomial approximation

$$sin(x) = x - (1/6)x^3 + (1/120)x^5 - (1/5040)x^7 + (1/362880)x^9.$$

For a technical reason, the small computer I looked at used the substitution $x = 2\pi y$, so it used the approximation

$$sin(2\pi y) = 2\pi y - (1/6)(2\pi y)^3 + (1/120)(2\pi y)^5 - (1/5040)(2\pi y)^7 + (1/362880)(2\pi y)^9.$$

In fact, to cope with the error E in a slightly better way, it used the approximation (which has slightly different coefficients)

$$sin(2\pi y) = y(6.283185 + y^2(-41.34165 + y^2(81.602234 + y^2(-76.57498 + y^2(39.71067))))).$$

Note the clever way of writing the polynomial - it means that only the power y^2 of y needs to be calculated.

This approximation is only accurate enough for angles up to $\pi/2$ radians (90°). For other angles the formulae

$$sin(x) = sin(x \pm 2n\pi)$$
$$sin(x) = sin(\pi - x)$$

are first used to reduce the angle to the desired range.

Now consider the function $ln(x) = log_e(x)$, the 'natural logarithm', or 'logarithm to the base e'. It can be shown that for $-1 < y < 1$,

$$ln(1 + y) = y + (1/2)y^2 + (1/3)y^3 + \ldots + (1/n)y^n + E$$

where the error E depends on n, the number of terms in the polynomial, and on y. If we take $n = 6$, and write the equation for both y and $-y$ we obtain

$$ln(1+y) = y + (1/2)y^2 + (1/3)y^3 + (1/4)y^4 + (1/5)y^5 + (1/6)y^6 + E$$
$$ln(1-y) = -y + (1/2)y^2 - (1/3)y^3 + (1/4)y^4 - (1/5)y^5 + (1/6)y^6 + E^*,$$

where the error in the second equation need not be the same as the error in the first. If we now subtract the two equations, and recall that the logarithm has the property that $ln(a/b) = ln(a) - ln(b)$, we obtain

$$ln((1+y)/(1-y)) = 2y + (2/3)y^3 + (2/5)y^5 + (E - E^*).$$

If we now set

$$y = \frac{x - 1/\sqrt{2}}{x + 1/\sqrt{2}}$$

we find

$$\frac{1 + y}{1 - y} = \frac{(x + \sqrt{2}) + (x - \sqrt{2})}{(x + \sqrt{2}) - (x - \sqrt{2})} = \frac{2x}{\sqrt{2}} = x\sqrt{2}.$$

Taking the natural log of each side and using the polynomial above we find

$$ln(x) + 0.5ln(2) = 2y + (2/3)y^3 + (2/5)y^5 + (E - E^*).$$

To cope with dropping the error term, the coefficients need to be changed slightly, and after rearranging the polynomial, we get the approximation

$$ln(x) = .693147182(-.5 + y(2.885392428 + y^2(.961470663 + y^2(.598978638)))$$

used by my small computer.

To give an example, $ln(.95) = .05129329439$, whereas the approximation given by the polynomial is .05129316143. The error is in the seventh decimal place, which is just about the accuracy required, as the computer calculates with 6 decimal digit accuracy.

Once again, the polynomial approximating $ln(x)$ is only accurate enough for small values of y, that is, values of x close to $1/\sqrt{2}$. For other values of x the computer uses the formula

$$ln(x) = ln((x/2^n).2^n) = n\ ln(2) + ln(x/2^n)$$

The power n is chosen so the polynomial is accurate enough for the logarithm on the right.

The computer provides other functions, such as $exp(x) = e^x$, $cos(x)$, $tan(x)$, $\sqrt{x}$, and the inverse tangent $atan(x)$. The first and last of these are calculated by polynomial approximations, but $cos(x)$ is calculated as $sin(\pi/2 + x)$, $tan(x)$ is calculated as $sin(x)/cos(x)$, and $\sqrt{x}$ is calculated as $exp(0.5 \cdot ln(x))$. These shortcut methods are used, I presume, simply to save expensive computer memory.

There are methods other than using polynomial approximations. For example the inverse tangent $atan(x)$ has approximations

$$atan(x) = x - (1/3)x^3 + (1/5)x^5 - (1/7)x^7 + (1/9)x^9,$$

$$atan(x) = \frac{x + (7/9)x^3 + (64/945)x^5}{1 + (10/9)\ x^2 + (5/21)x^4}.$$

The latter has approximately the same number of multiplications as the former, but in general is more accurate. It could be used in preference to the former if the division is not too expensive. I think division is too expensive in my small computer, and the polynomial was preferred.

Another approximation is

$$atan(x) = \cfrac{.2844.x}{1 - \cfrac{6.8179}{x^2 + 9.9348 - \cfrac{.6304}{x^2 + 1.5947}}}$$

This formula could be used providing that division is not very expensive (it has only one multiplication but three divisions). This form is sometimes called the "continued fraction" form.

In deciding on which approximation to use, one really should be aware of the accuracy to which the computer performs the basic operations of addition, multiplication, subtraction and the like. These operations are not performed completely accurately due to the limitations imposed by the way the computer stores numbers. Because a computer works only with a fixed number of digits, if we add numbers of different sizes some digits of the smaller may be rounded off. For example, if the computer works with only 8 digits and we add .000012345678 to 123.45678, the result is not 123.456792345678 but 123.45679 which is obtained by rounding the result to the 8 digit limit. If the computer works to 8 digits, then no matter what polynomial we use to calculate $sin(x)$ we can clearly expect there to be errors in the last 8th digit due to rounding errors. For this reason, calculators, in particular, sometimes work internally to a greater accuracy than they display. In this case the round-off errors may only affect the undisplayed digit.

We can investigate the way a computer rounds off, and hence the internal accuracy, by performing a few simple tests. We select a small number (call it x for reference later), add 1 to it, and see if the result is 1. Because the display won't necessarily show the result to the internal accuracy, we subtract 1 again, and see what we get. If we get zero displayed, then $1+x$ rounded down to 1. For example, Table 1 gives the results obtained with my calculator

x	1*E*-9	1*E*-10	1*E*-11	2*E*-11	4*E*-11	8*E*-11	5*E*-11
$(1+x)$-1	1*E*-9	1*E*-10	0	0	0	1*E*-10	1*E*-10

x	4.5*E*-11	4.99*E*-11	4.9999999*E*-11
$(1+x)$-1	0	0	0

Table 1.

From Table 1 it seems clear that the smallest number not rounded off when added to 1 is 5*E*-11. The calculator, although it only displays 10 digits, can store internally numbers of 11+ digits (the result 1+5*E*-11 is not stored accurately, though 1+1*E*-10 is). Further investigation shows that (9+5*E*-11)-9 is also nonzero. This suggests that the calculator is actually working with decimal digits (probably in what is called "binary coded decimal"). I tried three other calculators, and they all gave different results, but the results always indicated the calculator used decimal digits. All these calculators worked only to the accuracy of the display, which was 10 digits in two cases and 8 in the third.

I also tried the test on a computer, and obtained the results in Table 2.

x	2*E*-7	1.193*E*-7	1.19209*E*-7	1.1921*E*-7
$(1+x)$-1	1.19209*E*-7	1.19209*E*-7	0	1.19209*E*-7

x	3*E*-7	2*E*-7
$(2+x)$-2	2.38419*E*-7	0

Table 2.

These results indicate that the computer is using binary arithmetic. The number 1.19209*E*-7 is 2^{-23}. This indicates that the computer is using numbers with 24 binary digits of accuracy. When I switched to more accurate arithmetic (so-called "double precision" numbers) in the computer I obtained the results in Table 3.

x	5*E*-16	4.440892*E*-16
$(1+x)$-1	4.44089*E*-16	0

Table 3.

The number 4.44089E-16 is 2^{-51}, indicating that in double precision the computer is using 52 binary digits.

You may care to try to determine whether your calculator or computer uses binary or decimal digits in its numbers, and determine how small a number is not rounded off in arithmetic. If you find a calculator that uses binary arithmetic I'd be interested to have details of it.

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BOOK REVIEW

Computer Ecology and Chaos by E.O. Tuck and N.J. de Mestre (Longman Cheshire 1991 $14.99, 130 pages).

REVIEWED BY:

RORY McAULIFFE†, Wesley College

Computer Ecology and Chaos is subtitled 'An introduction to mathematical computing' but might also be described as 'A rabbit's eye view of mathematical modelling'. The book concentrates on a number of algorithms that model rabbit reproduction and is seasoned with amusing cartoons by Bruce Rankin.

These algorithms begin with the very simple 'double it' rule. The program that accompanies this model is written in QuickBASIC, occupies only four lines and uses the simplest of programming principles, in this case nothing more than a loop and a print statement. Throughout the text, even as the models become more complex, the programs are kept simple and the emphasis is placed on the resulting mathematical models. None of the programs are more than seven lines long and each can easily be translated to other versions of BASIC. I have tried out some of the material with a Year 10 student who was using TurboBASIC on an IBM PC. This student found no difficulty in interpreting the instructions, the attendant programs and the small changes that were required to convert to TurboBASIC. The use of QuickBASIC means that the programs can be implemented on both IBM and Apple machines.

The initial doubling model soon gives way to a discussion of a self-limiting model defined by the formula:

$$x_{n+1} = bx_n(1 - x_n) \quad .$$

In the text, subscripts and other intimidating notations are avoided and the model becomes $y = bx(1-x)$. This allows the authors to develop much more complicated behaviours which, depending upon the choice of the constant b, yield results that are self-limiting, cyclical or chaotic. This last possibility, which is pursued in the later chapters, leads to studies that are subjects of current research and which provide exciting lines of investigation. The final chapter provides an example of a 'strange attractor' related to the population models developed earlier and which provides a highly understandable introduction to the recently developed mathematics of Chaos. Throughout the text students are encouraged to ask 'what if ...' and answer their questions by entering new parameter values into their programs. There are a number of supporting exercises that extend beyond the material covered in the main text. The book is, therefore, highly suited to providing the backbone of an investigative unit of teaching which would lead naturally into the type of project and problem solving work that is required by the VCE.

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† The reviewer is the Immediate Past President of the Mathematical Association of Victoria.

The content of the book is aimed at students from Years 9 to 11 though the extension material will stretch even the strongest high school students as the difficulty escalates as one progresses. My Year 10 'guinea pig' found that the early parts were easy and that the level of difficulty increased, much as the authors hope. None of the investigations proved too difficult to begin though there are a number of lines of investigation that remain open for further work.

The text is attractively produced and clearly laid out. There are teacher's notes and a number of worksheets that may be photocopied for class use. The book is designed for use by teachers as an investigative unit and as such seems to me to be an important addition in an area that has been under-resourced in the past. There would, however, be no reason why it should not be bought and enjoyed by individual students as a text that will both inform and entertain outside the confines of the traditional curriculum.

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HISTORY OF MATHEMATICS SECTION

EDITOR: M.A.B. DEAKIN

Plimpton 322

Figure 1 depicts a Babylonian clay tablet known as Plimpton 322. This rather strange name derives from the fact that it is housed in the Plimpton Museum of Columbia University (New York City) and is item 322 in their catalogue.

Figure 1

Quite how it got there is not known and it apparently suffered a number of adventures and mishaps on the way. It is believed to have come from Babylon or nearabouts and, because it is written in Old-Babylonian script, to date from between 1900 BC and 1600 BC. The jagged left-hand edge would indicate that it is incomplete – there used to be more of it. In order to assist with the preservation of what remains, it was baked and cleaned in the early 1940s, and this process removed some 20th-century glue which very likely was put there in an attempt to stick together the fragment shown in Figure 1 and a (now missing) left-hand portion.

What remains, however, can be read and (up to a point) translated into English. This task was accomplished by two very distinguished orientalists: O. Neugebauer and A. Sachs. The description that follows relies very heavily on their account, in their book *Mathematical Cuneiform Texts.*

The writing turns out to be almost entirely a list of numbers in four separate columns. Figure 2 (taken from their book) shows Neugebauer and Sachs's transliteration of the text into more familiar numbers and letters from the Roman alphabet. The square brackets indicate places where the original is no longer legible, but the two scholars have been able to supply plausible readings which the brackets enclose. You will also notice that they have suggested corrections to five readings because they believe the original scribe made errors at these points.

	I	II	III	IV
1	[*ta-k*]*i-il-ti ṣi-li-ip-tim*	íb-si$_8$ sag	íb-si$_8$ *ṣi-li-ip-tim*	mu-bi-im
2	[*ša in-*]*na-as-sà-ḫu-ú-*[*m*]*a* sag *i-. . .-ú*			
3	[1,59],15	1,59	2,49	ki-1
4	[1,56,56],58,14,50,6[105],15	56,7	3,12,1[108]	ki-2
5	[1,55,7],41,15,33,45	1,16,41	1,50,49	ki-3
6	[1],5[3,1]0,29,32,52,16	3,31,49	5,9,1	ki-4
7	[1],48,54,1,40	1,5	1,37	ki[-5]
8	[1],47,6,41,40	5,19	8,1	[ki-6]
9	[1],43,11,56,28,26,40	38,11	59,1	ki-7
10	[1],41,33,59,3,45	13,19	20,49	ki-8
11	[1],38,33,36,36	9,1[106]	12,49	ki-9
12	[1],35,10,2,28,27,24,26,40	1,22,41	2,16,1	ki-10
13	[1],33,45	45	1,15	ki-11
14	[1],29,21,54,2,15	27,59	48,49	ki-12
15	[1],27,3,45	7,12,1[107]	4,49	ki-13
16	[1],25,48,51,35,6,40	29,31	53,49	ki-14
17	[1],23,13,46,4[0]	56	53[109]	ki[-15]

105 50,6 written like 56.
106 9,1 error for 8,1.
107 7,12,1 (the square of 2,41) error for 2,41.
108 3,12,1 error for 1,20,25.
109 53 error for 1,46 (i.e., 2·53); cf. below pp. 40 and 41.

Figure 2

But Figure 2 still requires interpretation and translation if we are to know what the tablet is about. Begin with the headings of Columns II, III. *íb-si*$_8$ *sag* is believed to mean "solving number of the width" or something like that and *íb-si*$_8$ *ṣi-li-ip-tim* "solving number of the diagonal" or something similar. Thus the table seems to give the diagonals of rectangles whose widths are given. Now, in order to know the length of a diagonal of a rectangle, we need also to know the height, as well as the width. Neugebauer and Sachs hypothesise that this information was given on the piece that is now missing. (It is not given in Column I, as we shall see.)

So let us begin by thinking of Columns II, III as giving the widths and the diagonals of rectangles. The numbers in these columns are expressed in base 60 and need to be converted into base ten. Thus 1,59 means $1 \times 60 + 59$, i.e. 119 and 2,49 means $2 \times 60 + 49$ or 169. Similarly, 3,31,49 means $3 \times 60^2 + 31 \times 60 + 49$, i.e. 41270. We can thus build up a Table (Figure 3) whose left-hand columns are values of b, the base, and d, the diagonal, of a rectangle. The original figures are given in the columns. Neugebauer and Sachs's corrections are in parentheses. (The final column is merely a count. It corresponds to Column IV of the tablet; see Figure 2.)

b	d	h	No.
119	169	120	1
3367	11521 (4825)	3456	2
4601	6649	4800	3
12709	18541	13500	4
65	97	72	5
319	481	360	6
2291	3541	2700	7
799	1249	960	8
541 (481)	769	600	9
4961	8161	6480	10
45	75	60	11
1679	2929	2400	12
25921 (161)	289	240	13
1771	3229	2700	14
56	53 (106)	90	15

Figure 3

Knowing b, d we can calculate h, the height of the rectangle by Pythagoras' Theorem

$$h^2 + b^2 = d^2. \tag{1}$$

This enables us to supply the third column of Figure 3. But immediately we look at this, we see something very striking indeed: all the values of h are integral. (It is this that justifies the corrections which are also plausible on other grounds, e.g. $106 = 2 \times 53$ and $25921 = 161^2$.)

Triplets of integers satisfying Equation (1) are known as Pythagorean triples. (See *Function, Vol.* 6 *Part* 3, pp. 20-24.) It looks very much as if the ancient Babylonians had some means of generating Pythagorean triples. They can hardly have found such large numbers as 12709 merely by trial and error.

Now there *is* a systematic way to generate Pythagorean triples. It was described in the *Function* article referred to above. These triples come in two varieties. First there are the primitive triples. To generate primitive triples, let u, v be positive integers such that:

(a) u, v are relatively prime

(b) one is even and the other odd

(c) $u > v$.

Then

$$b \text{ (or } h) = u^2 - v^2;\ h \text{ (or } b) = 2uv;\ d = u^2 + v^2 \qquad (2)$$

is a primitive Pythagorean triple. All other Pythagorean triples are multiples of primitive triples.

E.g. take $u = 2, v = 1$; then $b = 3, h = 4, d = 5$

or take $u = 3, v = 2$; then $b = 5, h = 12, d = 13$

etc.

These are primitive triples. $b = 6$ $(= 2 \times 3)$, $h = 8$ $(= 2 \times 4)$, $d = 10$ $(= 2 \times 5)$ is a non-primitive triple derived from the simpler (3,4,5).

This out of the way, let us come back and look at the first column of our clay tablet. These numbers Neugebauer and Sachs interpret as fractions. Babylonian fractions are expressed, remember, in base 60. Thus, for example, look at No. 2 on the list in Column 1. After correction, this reads

1, 56, 56, 58, 14, 56, 15

which equals

$$1 + \frac{56}{60} + \frac{56}{3600} + \frac{58}{216000} + \frac{14}{12960000} + \frac{56}{777600000} + \frac{15}{46656000000}$$

or 1·94915856.

Now look at Figure 4.

No.	Fraction	$(d/h)^2$
1	1·9875	1·983402778
2	1·949158560	1·949158552
3	1·918802126	1·918802127
4	1·886247906	1·886247907
5	1·815007716	1·815007716
6	1·785192901	1·785192901
7	1·719983676	1·719983676
8	1·692773437	1·692709418
9	1·642669444	1·642669444
10	1·586122566	1·586122566
11	1·5625	1·5625
12	1·48941684	1·48941684
13	1·451041666	1·450017361
14	1·43023882	1·43023882
15	1·387160493	1·387160494

Figure 4

The second column here gives the value of the fraction in our notation. I used my calculator to get an answer to 9 decimal places. If we now look at $(d/h)^2$ we get the third column of Figure 4 which, except for a few instances, agrees very well with the previous column.

Now

$$\left(\frac{d}{h}\right)^2 = \frac{d^2}{d^2-b^2} \tag{3}$$

and Neugebauer and Sachs believe this to be the import of the heading to Column I. This, they think, should translate as "The *takiltum* of the diagonal which has been subtracted such that the width ...", which may mean $d^2/(d^2-b^2)$, but we do not know the meaning of the word *takiltum* and it's not sure quite that this was what was written – other parts are quite illegible. "Life wasn't meant to be easy." However, the correspondence between the columns of Table 4 is hardly coincidence. Even where the agreement isn't perfect, it is quite good except for Nos. 1, 8 and 13.

My suggestion for No. 1 is that both the scribe and Neugebauer and Sachs omitted a zero. The fraction should read, in more modern notation,

[1, 59, 0], 15

in Figure 2. This gives 1·983402777, in very good agreement. Similarly with No. 13, where [1], 27, 0, 3, 45 gives 1·450017361 which is perfect agreement to eight decimal places.

The discrepancy in No. 8 would seem to be due to error: [1], 41, 33, 59, 3, 45 should be [1], 41, 33, 45, 14, 3, 35 to six sexagesimal places. The others (Nos. 1, 13) are not classed as errors as the Babylonians seem not to have recorded zeros. These are a much later invention.

Thus Column I of the tablet records values of $(d/h)^2$. Look again at Figure 3 and pay particular heed to the values of h. Except for Nos. 2, 5 and 15 (in fact) all are "round numbers" in base 60, and even No. 15 is "reasonably round" and the other two "fairly reasonably round".

This imprecise statement can be made precise as follows. The Babylonian base 60 has prime factors 2, 3, 5. In fact

$$60 = 2^2 \times 3 \times 5.$$

Every value of h in Figure 3 may be found to be of the form

$$h = 2^{\alpha}\, 3^{\beta}\, 5^{\gamma} \tag{4}$$

and indeed every value is even (i.e. $\alpha \geq 1$).

Thus $h = 2uv$ and $b = u^2 - v^2$ (not the other way around). This resolves the ambiguity in Equation (2). Now if

$$2uv = 2^{\alpha}\, 3^{\beta}\, 5^{\gamma}, \tag{5}$$

then each of u, v must also be of the form (4). That is to say, in base 60, both $\frac{1}{u}, \frac{1}{v}$ will be terminating sexagesimals.

[Compare our base, ten, which has factors 2, 5. $\frac{1}{2} = 0{\cdot}5$ and $\frac{1}{5} = 0{\cdot}2$. These *deci*mal expressions terminate as do the expressions for the reciprocals of all numbers of the form $2^{\alpha}\, 5^{\beta}$. All other reciprocals repeat *ad infinitum*. The case is similar in base 60 except that the reciprocal terminates if and only if the original number is of the form $2^{\alpha}\, 3^{\beta}\, 5^{\gamma}$.]

If we now combine Equations (2) and (3) we find

$$\frac{d}{h} = d \,/\, \sqrt{(d^2 - b^2)} = \frac{1}{2}\left[\frac{u}{v} + \frac{v}{u}\right] \tag{6}$$

and if u, v are nice "round" numbers in base 60, then this expression can be evaluated exactly.

Let us take a specific case. Choose No. 3 from Figures 3, 4. From Figure 4, $h = 4800$, $b = 4601$. From this data we find

$$u = 75 \qquad\qquad v = 32.$$

Thus for this case

$$\frac{d}{h} = \frac{1}{2}\left[2{\cdot}34375 + 0{\cdot}42666. \ldots\right]$$

and this is exact. (Note that the recurring decimal in this last expression converts into a terminating expression in base 60. In fact, in base 60

$$\frac{d}{h} = (0;\ 30) \times \{(2;\ 20,\ 37,\ 30) + (0;\ 25,\ 36)\}$$

and this is exact.)

So we have

$$\frac{d}{h} = 1{\cdot}385208333. \ldots$$

and this is exact. Taking the trouble to square this exactly gives

$$\left[\frac{d}{h}\right]^2 = 1{\cdot}9188021267361111. \ldots . \tag{7}$$

Compare Figure 4.

If we now convert the number given by Equation (7) into base 60, the result is

$$\left[\frac{d}{h}\right]^2 = (1;\ 55,\ 7,\ 41,\ 15,\ 33,\ 45)$$

and this is not only exact, but precisely what Figure 2 gives. In other words, a Babylonian scribe living over 3500 years ago achieved better accuracy than my CASIO fx-570!

Finally, look again at Figure 4. Notice how the fractions decrease regularly. The entries in the table are listed in a very deliberate order.

Another feature I leave to the reader to discover. If, for each instance of Figure 3, we deduce the values of u, v, we find that in every case bar Nos. 11, 15 the Pythagorean triple is primitive. No. 11 is in fact the familiar (3,4,5) case and No. 15 is rather more complicated. There are in fact good reasons, if we express these cases in base 60, to put the numbers in this form rather than the more standard ones and Neugebauer and Sachs go into these.

And much else besides. But enough has been said to show the level of mathematical attainment in ancient Babylon. (Equations (2), for example, were clearly known.) Finally, three cheers for Neugebauer and Sachs who had the necessary historical, linguistic and mathematical expertise to tell us about all this.

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PROBLEMS AND SOLUTIONS

EDITOR: H. LAUSCH

This section contains contributions from Australia, Austria, Ethiopia and Germany. Moreover, we inform our readers of the latest olympiad news from Australia and the Pacific Rim.

SOLUTIONS

Problem 12.4.3 (proposed by a puckish Lewis Carroll in The Monthly Packet beginning in April, 1880). Place twenty-four pigs in four sties so that, as you go round and round, you may always find the number in each sty nearer to ten than the number in the last.

One solution was received. Here is the

Official solution. Place 8 pigs in the first sty, 10 in the second, nothing in the third, and 6 in the fourth: 10 is nearer ten than 8; nothing is nearer ten that 10; 6 is nearer ten than nothing; and 8 is nearer ten than 6.

Problem 14.5.5 Between three towns, A, B, C there is a continual migration of families, so that the number of families in each town in unaltered, while the whole number of families migrating at any specified time is always even. Show that, if by the end of any time an even number of families left A for B, then by the end of the same time the number of families that have left B for A is also even.

David Shaw (Newtown, Geelong, Victoria), who also sent in an elaborate solution to Problem 14.5.3, presented us with a very brief response to Problem 14.5.5:

Solution. Let e_{XY} represent the number of families leaving X for Y. Then $e_{AB} + e_{BA} + e_{AC} + e_{CA} + e_{BC} + e_{CB}$ is an even number, but

$$e_{AC} - e_{CA} + e_{BC} - e_{CB} = 0.$$

because the population of C is unchanged.

By addition, $e_{AB} + e_{BA} + 2e_{AC} + 2e_{BC}$ is even.
Therefore if e_{AB} is even, e_{BA} must also be even.

Apologies for the double-numbering of four problems in Volume 15, Issue 1, the New Year's issue. We let one pair linger on as a challenge to FUNCTION readers:

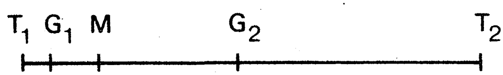
Problem 15.1.5 If the bisectors of two angles of a triangle are equal. the triangle is isosceles. We want a *Euclidean proof.*

Problem 15.1.6 Each day a wife leaves home by car to collect her husband at the station when it is 18.00 hours. Today her husband arrives at 17.00 hours and sets out walking at 4 kilometres per hour. The wife sets out at the usual time, meets him on the road, and they get home 20 minutes earlier than usual. Find the average speed of the car.

– and present solutions to the second pair by John Barton, North Carlton, Victoria. Many thanks!

Problem 15.1.5 A young man has two girl friends who live in diametrically opposite directions from his house and whom he normally sees about an equal number of times each month. The bus route which goes to both their houses passes the front of his house. He decides to leave his house at random times and to take the first bus which comes along. Since the buses run with the utmost regularity he thinks he will see his girl friends an approximately equal number of times. He follows this regime for a few months and finds that he has been to see one girl friend five times as often as he has been to see the other! Why is this?

Solution. The young man lives at M, the girl friends reside at G_1, G_1. The bus terminals are T_1, T_2.



Let the time travel from T_1 to T_2 be 1 unit. Assuming no creation or disappearance of buses, the time from T_2 to T_1 is 1 unit, whether or not any waiting is done by a bus at T_1 or T_2.

If we now assume that, for whatever reasons, out of each 1 unit of time, the time for the bus to travel from T_1 to M is $\frac{1}{6}$ unit, and the time taken for the bus to travel from T_2 to M is $\frac{5}{6}$ unit, then M will, probably, visit G_1 five time as often as he will visit G_2.

Problem 15.1.6 Let O and I be the circumcentre and incentre, respectively, of a triangle with circumradius R and inradius r; let d be the distance OI. Show that $d^2 = R^2 - 2rR$.

John Barton refers to the recommendable book "Modern Geometry" (London 1943), where the problem is given as a theorem (Th. 23, p.35).

Solution. I abbreviate it here. Let P be the intersection of AI with the circumcircle of ΔABC.

Then

$$\begin{aligned} R^2 - OI^2 &= (R - OI)(R + OI) \\ &= AI.IP \quad \text{(intersecting chord theorem)} \\ &= AI.PB \end{aligned}$$

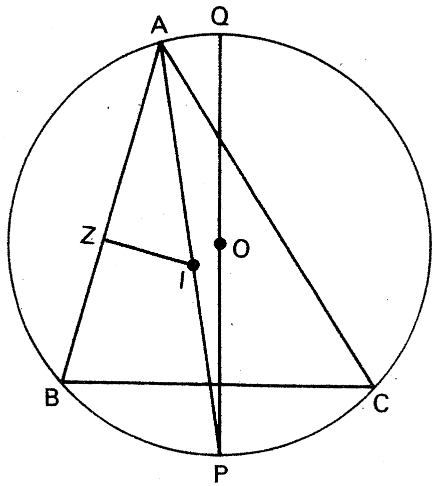
by a theorem based on the fact that the triangle of the ex-centres of ΔABC has ΔABC as its pedal triangle.

Let Z be the point on AB such that AB and IZ are perpendicular, and Q be the point on the circumcircle of ΔABC opposite to P. Then

$$AI = IZ \operatorname{cosec} \frac{A}{2} = r \operatorname{cosec} \frac{A}{2};$$

$$PB = PQ \sin \frac{A}{2} = 2R \sin \frac{A}{2} \quad (\angle BQP = \angle BAP = \frac{A}{2}).$$

Hence $d^2 = R^2 - 2rR$.



PROBLEMS

The first problem was proposed by Dieter Bennewitz, Koblenz, Germany, at the confluence of Rivers Rhine and Moselle. Danke schön!

Problem 15.3.1 Let $p > 5$ be a prime number. Prove that the equation $x^4 + 4^x = p$ has no solution in integers if the last digit of x is different from 5.

Readers wishing to solve a harder problem may try to prove: Let $p > 5$ be a prime number. Prove that the equation $x^4 + 4^x = p$ has no solution in integers. In this form the problem was stated at the 1977 Kürschak Competition (Hungary).

Here is another problem by K.R.S. Sastry, Addis Ababa, Ethiopia:

Problem 15.3.2 The Euler line of a triangle is the line containing its orthocentre, circumcentre and centroid [Euler has shown that these three points do lie on one line]. Determine all triangles ABC in which the Euler line bisects an angle subtended by a side at its orthocentre.

YEAR TWELVE INTERNATIONAL

Here are two problems from Salzburg, where Mozart grew up without worries about any year-twelve examinations. Now many Salzburg year-twelve students wish these exams, – called "Matura" (!) – "entry ticket" to an Austrian university: 1. didn't exist or, 2. aren't too difficult. Thanks to Professor Fritz Schweiger, a former rector of Paris Lodron University Salzburg, and Professor Erwin Niese, mathematics master at Akademisches Gymnasium Salzburg – both firm supporters of Mozart – the mathematics component of the Salzburg Matura has become available in Australia. FUNCTION takes pride in being first to publish these samples.

Problem 15.3.3 Inscribe a rectangle of maximal area into the ellipse given by $x^2 + 2y^2 = 18$. The four vertices of this rectangle lie on the hyperbola with the equation $x^2 - y^2 = a^2$.

(a) Determine these four vertices, the area of the rectangle and the value of a.

(b) Show that the foci of the ellipse and the hyperbola coincide. Show that the two curves intersect in right angles.

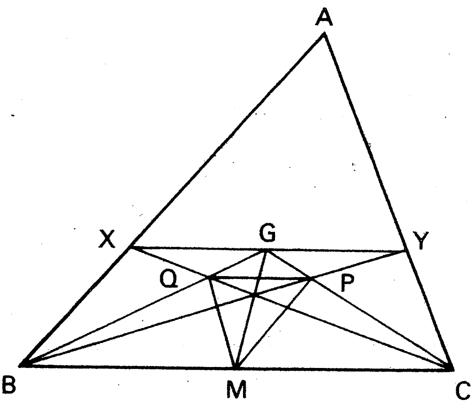
Problem 15.3.4 Given are two circles K_1 and K_2, with centres C_1 and C_2 respectively, same radius R and a common tangent t such that K_1 and K_2 lie on the same side of t. Let t touch K_1 at T_1 and K_2 at T_2. Let K_2 be another circle, with centre C_3, radius R and tangent to K_1 and K_2 such that K_1 and K_2 lie in the exterior of K_3. How should the circles be placed in the plane such that the area of the pentagon $C_1T_1T_2C_2C_3$ is as large as possible? Express area and perimeter of this pentagon as functions of R.

MATHEMATICAL OLYMPIADS

1. The 1991 Asian Pacific Mathematics Olympiad

The Asian Pacific Mathematics Olympiad (APMO), an annual competition, was started in 1989 by Australia, Canada, Hong Kong and Singapore. Since then the number of participating Pacific Rim countries has trebled. Beside students from the founding countries, participants of the 1991 APMO were from Colombia, Malaysia, Mexico, New Zealand, the Phillippines, the Republic of China, the Republic of Korea and Thailand. Here are the questions of this four-hour examination:

1. Given ΔABC, let G be the centroid and M be the mid-point of BC. Let X be on AB and Y on AC such that the points X,G and Y are collinear and XGY and BC are parallel.



Suppose that XC and GB intersect at Q and that YB and GC intersect at P. Show that ΔMPQ is similar to ΔABC.

2. Suppose there are 997 points given on a plane. If every two points are joined by a line segment with its mid-point coloured in red, show that there are at least 1991 red points on the plane. Can you find a special case with exactly 1991 red points?

3. Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be positive real numbers such that

$$a_1 + a_2 + \ldots + a_n = b_1 + b_2 + \ldots + b_n$$

Show that

$$\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \ldots + \frac{a_n^2}{a_n + b_n} \geq \frac{1}{2}(a_1 + a_2 + \ldots + a_n).$$

4. During a break n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule; he selects one child and gives him a candy, then he skips the next child and give a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on.

Determine the values of n for which eventually (perhaps after many rounds) all children will have at least one candy each.

5. Given two tangent circles, C_1, C_2, and a point P on their radical axis, i.e. on the common tangent of C_1 and C_2 that is perpendicular to the line joining the centres of C_1 and C_2. Construct with compass and ruler all the circles C that are tangent to C_1 and C_2 and pass through the point P.

Australian medal winners (school year in parenthesis) were:

Gold:	Anthony Henderson (10), NSW, Sydney Grammar School
Silver:	Luke Kameron (12), NSW, Knox Grammar School
	Meng Tan (12), Qld, Brisbane Grammar School
Bronze:	Joanna Masel (12), Vic, Methodist Ladies' College
	Angelo di Pasquale (12), Vic, Eltham College
	Martin Roberts (12), Tas, Rosny College
	Justin Sawon (12), SA, Heathfield High
Honorable Mention:	Tom Brennan (12), NSW, Know Grammar School
	Stuart Sellner (12), WA, Rossmoyne SHS
	Weiben Yuan (12), NSW, Cabramatta HS.

2. The XXXII International Mathematical Olympiad (IMO)

In April, the ten-day IBM Mathematics School took place at Melbourne Church of England Grammar School. Candidates for the team that is to represent Australia at this year's IMO and other highly-gifted students who look forward to at least one more year of secondary education were there to undergo a day-and-evening-filling programme consisting of tests and examinations, problem sessions and lectures by mathematicians. Communication with the outside world was largely possible only by a Telecom Australia portable phone.

Sigtuna, a place halfway between the Swedish capital Stockholm and the old university city of Uppsala, where the scientists Swedenborg and Linnaeus (Linné) lie buried, is this year's venue of the IMO. There the Australian team will have to contend with six problems during 9 hours spread equally over two days in succession. The following students were selected as team members:

Anthony Henderson, Luke Kameron, Joanna Masel, Angelo di Pasquale, Justin Sawon, Meng Tan, Reserve: Tom Brennan.

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