Function

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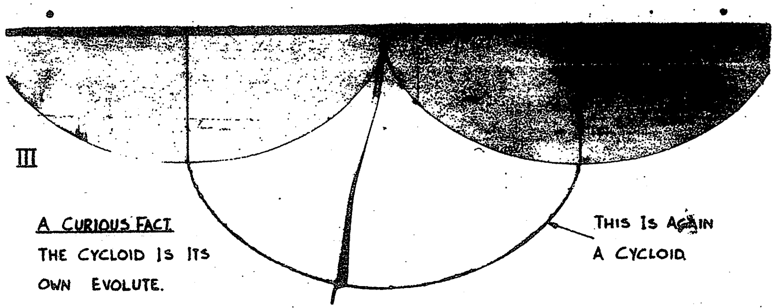

A SCHOOL MATHEMATICS MAGAZINE

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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
FUNCTION,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

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Welcome to new readers of *Function* and welcome back to old ones. *Function* is a mathematics journal meant especially for student readers — most particularly those in the upper forms of secondary schools.

The main article in this issue is Malcolm Cameron's discussion of the paths of the planets. Malcolm Cameron is the author of the book *Heritage Mathematics* (published by Arnold Edward, 1983, price $16.95), a delightful book of human interest and motivation in mathematics. Like the article, it uses the history of mathematics to bring its subject matter to life.

This is also the aim of our new history of mathematics section. The problem section has been revived and we also have a section on competitions. Our regular correspondent, Perdix, is away, but Peter Taylor writes on the Tournament of the Towns. There are also letters to the editor and our cover feature.

Finally, Robyn Arianrhod writes of the Victorian Government's initiatives on girls and mathematics. *Function*'s entire editorial policy can be summed up in two mottoes: Good Mathematics, For Everyone. And that is what Robyn and the Victorian Government are seeking.

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THE FRONT COVER

Michael A.B. Deakin, Monash University

The late Carl Moppert is remembered best for his sundial (*Function, Vol. 5, Part* 5) and his Foucault pendulum (*Vol.* 6, *Part* 2). He also left behind a number of mathematical models, including a set devoted to the properties of the curve named the "cycloid".

The second of this set is shown opposite. A point on the rim of a rolling wheel traces out a cycloid as the wheel revolves. A point on one of the spokes would trace out a different but somewhat similar curve, and if it were fixed to a point on a spoke projecting out beyond the rim, the curve would be different again.

These curves are all examples of "epicycloids" — more general curves generated by rolling the wheel around the outside of a second wheel. If we think of the earth as the centre of our solar system, then the sun travels round the earth in (approximately) a circle, and the planets move in (approximately) circular paths around the sun. Thus, as seen from earth, the planets move in approximately epicycloidal paths. This is the basis of the Ptolemaic system of astronomy, now replaced by the Copernican. See Malcolm Cameron's article in this issue.

Another property of the cycloid is illustrated on the cover which shows the third model in the sequence. If a pendulum is constrained to move between two cycloids, as shown, the path of its bob (known technically as its involute — the original curve being called the evolute) is again a cycloid.

An ordinary pendulum has a period that is nearly independent of the angle through which it swings, provided this is small. With the cycloidal pendulum this relationship is exact. When clocks used the pendulum as the basis of their time-keeping, they sometimes employed cycloidal pendulums to ensure greater accuracy.

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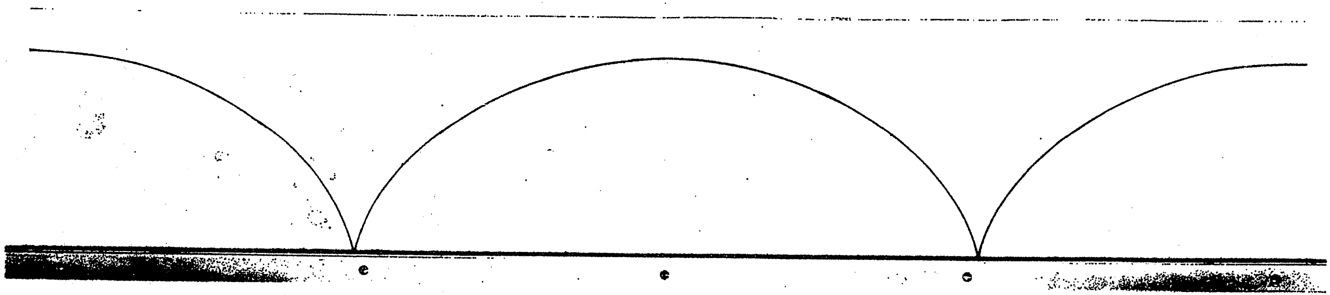
THE CYCLOID HAS BEEN INVESTIGATED
BEFORE. IT IS THE PATH OF A POINT
ON THE RIM OF A ROLLING WHEEL.

II



THE PATHS OF THE PLANETS

Malcolm J. Cameron

The Paths of the Planets

From simple to complex. That's how most people see progress. Not so in mathematics. In the case of finding the mathematical paths of the planets it was the opposite. The whole thing simplified when the problem was viewed from the right perspective.

This is the story that bewildered the ancient Greeks; how they added complexity upon complexity trying to save the situation. And ended up nowhere. Of the iconoclastic Anaxagoras; of Eudoxus and his interesting curves; of Ptolemy and his beautiful complex epicycles. Then of Niklas Koppernigk who put the problem into perspective and Johannes Kepler who cracked the problem.

The Great Meteorite of 467 B.C.

Anaxagoras claimed that he had predicted the great meteorite which fell at Egos Potamoi in the year 467 B.C. This linked him to others in ancient Greece — one who foretold a storm which saved the life of Croesus, one who predicted an earthquake, and one who made money by predicting a good olive harvest.

This makes difficult the prediction of Thales. He predicted an eclipse of the sun. Or did he?

Anyway, Anaxagoras was one of the first to consider things astronomical. He supposed the meteorite to have fallen from the sun, so concluded the sun was a mass of red-hot iron greater than Peloponnesus, and therefore not at a very great distance from the earth. He was the first to think that the seven planets were arranged as follows: the Moon, the Sun, Venus, Mercury, Mars, Jupiter and Saturn. The stars were stony particles torn away from the circumference of a flat earth. He taught that the moon was illuminated by the sun and hence explained the phases of the moon; he showed that eclipses of the Moon or Sun were caused by the interposition of the Earth or Moon.

Anaxagoras even had an explanation for the Milky Way. He thought that owing to the small size of the sun, the earth's shadow stretches far through space, and as the light of the stars seen through it is not overpowered by the sun, we see far more stars in the part of the sky covered by the shadow than outside it. An ingenious idea, but Anaxagoras ought to have seen that if it was correct the Milky Way should change its position among the stars in the course of a year.

That was not his only worry. His speculations in astronomy were proving too daring for the locals. So much so that Anaxagoras ended up in an Athens prison, where he pondered the problem of finding a value for π in the sand on the cell floor. His crime was "impiety".

Worse followed. The citizens of Athens democratically decided to banish him. On being taken to the city wall Anaxagoras turned to the citizens and defiantly claimed: "It is not I who have lost the Athenians, but the Athenians who have lost me". That was Anaxagoras!

The Problem

The problem was the seven "wandering stars" or planets; in particular, how their paths loop back on occasions. The aim was to find a mathematical curve which would account for this retrograde motion (see Figure 1).

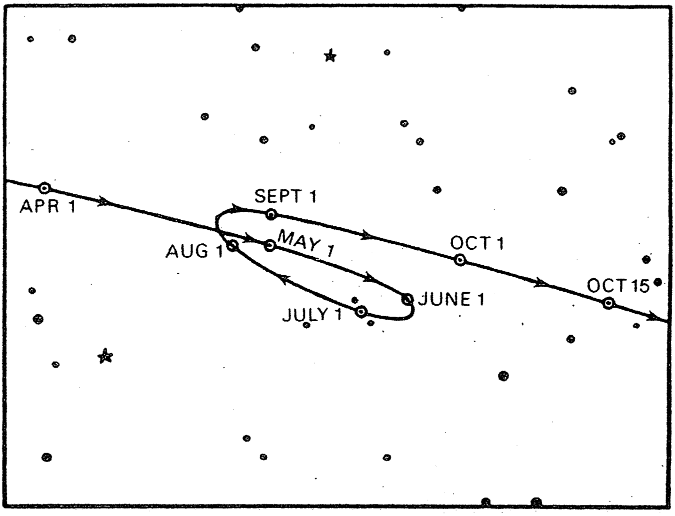


Figure 1

The path of Mars through the sky from April to October

The subsequent early history of astronomy was a search for a mathematical curve — the path of a planet.

Eudoxus – The First Attempt

An interesting attempt at the problem was that of Eudoxus, a Greek mathematician 408 B.C. to 355 B.C.. Most of the fifth book of Euclid is a summary of his work; as well he discovered a method of approximation. He was also the first to propose the system of 3 years of 365 days followed by one of 366 days.

His entry for the path of the planets is the "figure of eight" curve, or the "lemniscate", projected onto the sphere (Figure 2). Its equation in modern form is

$$x^2 + y^2 = a\surd(x^2 - y^2).$$

The curve has a colourful history. Eudoxus called it the "hippopede" curve because it was a favourite practice in the riding schools to let the horses describe this figure in cantering. It was discovered in modern form in 1694 by Jacob Bernoulli in a paper devoted to the theory of the tides. He named it the "lemniscate of Bernoulli" from the word "lemniscus", meaning a pendant ribbon. The curve subsequently became popular in mathematical challenges.

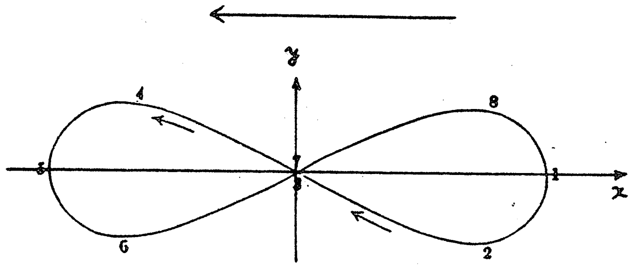


Figure 2

The path according to Eudoxus

The lemniscate has a parametric form

$$x = a\left[\frac{u+u^3}{1+u^4}\right] \qquad y = a\left[\frac{u - u^3}{1+u^4}\right]$$

which makes it easy to plot the ribbon bow, e.g. with $a = 1$ it is best drawn for $u = -1$ to $+1$ in steps of 0.1 noting that it is symmetrical about both the x and y axis. The connoisseur may draw the curve from its polar form

$$r^2 = a^2 \cos 2\theta.$$

The beautiful system of Eudoxus was well-nigh forgotten even in ancient Greece. The system assumed that the planets moved in circular orbits by being attached to a rotation sphere. Each planet was on the equator of a sphere revolving with uniform speed round its two poles. In order to explain the loops or retrogressions in the paths of the planets, as well as their motion in latitude, Eudoxus assumed that the poles of the sphere wobbled. In fact the poles were attached to another sphere which also rotated.

The net result of the system of rotating spheres was the movement of the planet along the lemniscate in the direction of the arrow passing over the arcs 1–2, 2–3, 3–4, 4–5 etc. in equal times. The motion of the planet through the constellations is completed by allowing the whole curve to move along the x axis.

This was the first attempt to tie down the apparently lawless motion of the planets. For Saturn and Jupiter, and practically also for Mercury, the system accounted well for the motion in longitude, while it was unsatisfactory in the case of Venus, and broke down completely only when dealing with the motion of Mars.

Kalippus – A Second Attempt

To save the situation, Kalippus, a pupil of Eudoxus, had a go at improving the system particularly for the motion of Mars. He came up with a flowery variety of the lemniscate shown in Figure 3. It was able to make a planet retrograde in a few more cases.

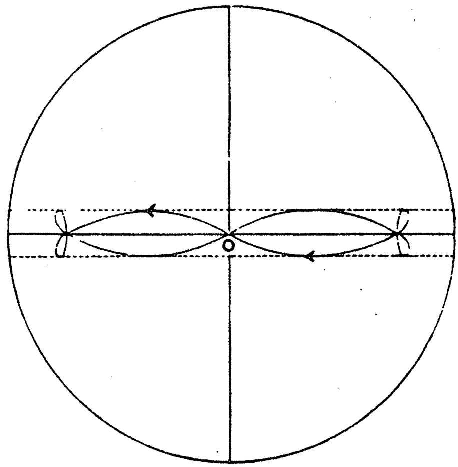


Figure 3

... and according to Kalippus

The work of Eudoxus and Kalippus is the first scientific astronomy — trying to tie theory and observation. They were the first to go beyond mere philosophical thought.

The Epicycles – The Third Attempt

By 230 B.C. astronomers were more practical, determining the positions of stars and planets by graduated instruments at the Museum of Alexandria. This showed marked errors in the old theories to the extent that it was hopeless to retain them. There was also the practical problem of regularising the calendar that provided an impetus to astronomy.

The new theory was the epicycle theory (Figure 4), where the path of the planet was one of the epicycloids. This theory developed gradually with the major contributors being Apollonius (B.C. 230), Hipparchus (B.C. 130) and Ptolemy (A.D. 140). Strangely it was Apollonius who separately studied the ellipse — a fact that was not to find relevance for almost 2000 years.

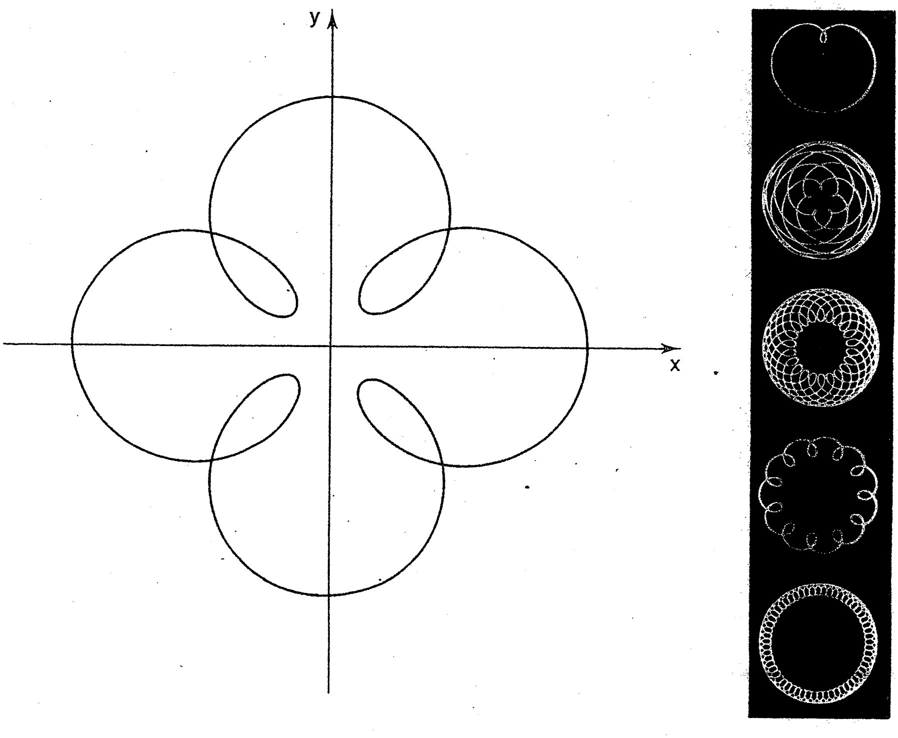
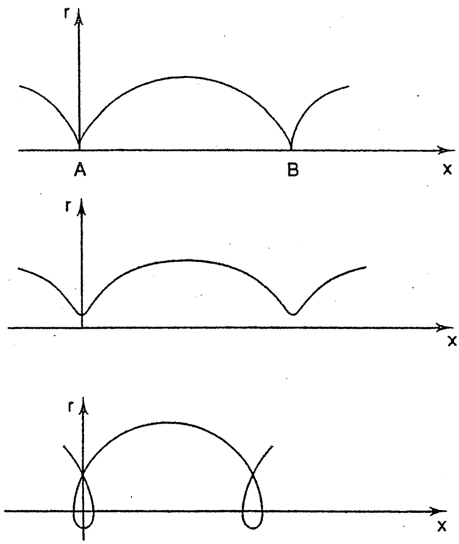
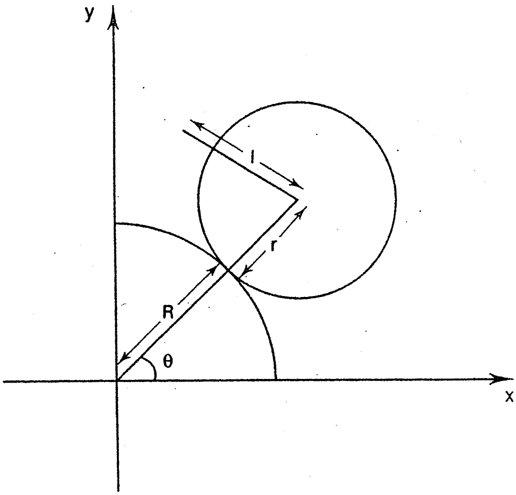


Figure 4

... according to Apollonius, Hipparchus, and Ptolemy, showing the paths of Mercury, Venus, Mars, Jupiter and Saturn

The epicycloid is the same as the cycloid, the course traced out by a point attached to a "rolling hoop" (Figure 5), except that the rolling hoop rolls around the circumference of a large circle, not along a straight line (Figure 6).





Figures 5 and 6

The cycloid

For the epicycloid the hoop rolls around a circle rather than along a straight line. The epicycloid ends up wrapped around the circle.

In the notation of Figure 6, equations of the epicycloid are

$$x = (R+r) \cos \theta - \ell \cos \left\{ \left[\frac{R+r}{r} \right] \theta \right\}$$

$$y = (R+r) \sin \theta - \ell \sin \left\{ \left[\frac{R+r}{r} \right] \theta \right\}.$$

The number of loops can be adjusted almost at will to fit the paths observed for the planets from Earth. And so the theory became more complex, as more combinations of circles were used as a means of computing the position of each planet at each moment.

Simplicity – Niklas Koppernigk

The first great simplification in finding the paths of the planets had to wait for Niklas Koppernigk (1473-1543), a canon of the church at Frauenberg in Poland. He simply put the sun at the centre and the planets in circular orbits around the sun.

This was a shock, since man naturally assumed himself to be at the centre of the universe. Yet at first it caused little trouble — it was not until Galileo publicised the new theory that the fun started. In fact, a repeat of Anaxagoras's problem.

The reason was twofold. Firstly Niklas was a mild, contemplative man, seeking the truth but not stirring, as with Anaxagoras or Galileo. For example, he wrote under the Latin name Copernicus, by which he has been known ever since. Secondly, Copernicus' book was considered an important contribution to the reform of astronomy on which the calendar and the accurate determination of the date of Easter depended. In other words, the system was a practical improvement.

Kepler – The Third Attempt

It was Johannes Kepler who cracked the problem in 1609. And it was by checking another assumption — that of circular motion — which led to the final simplification. The path of the planets is an ellipse. Simple. An ellipse with the sun as one focus (Figure 7), with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

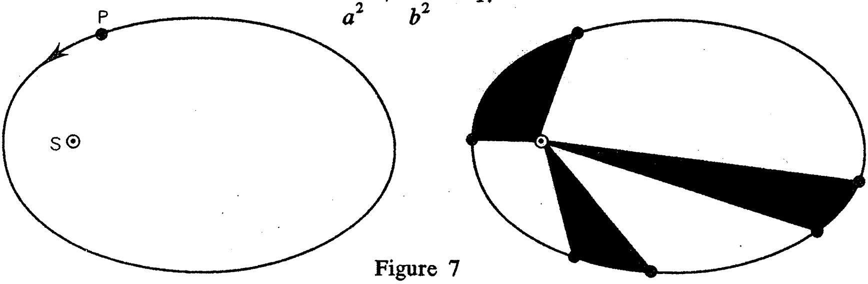


Figure 7

... and finally according to Johannes Kepler

The path of the planets had started out as "the wandering stars" with no apparent logic, part of the mystery of the universe. Then the lemniscate or figure eight curve was tried. Then a further complication. Then a whole range of epicycloids took to the heavens.

Yet, all along, it was a matter of getting the perspective or assumptions right. That the planets went around the sun, and that they went around in ellipses, the curve studied by Apollonius (Figure 8) in the first place.

That solved the problem.

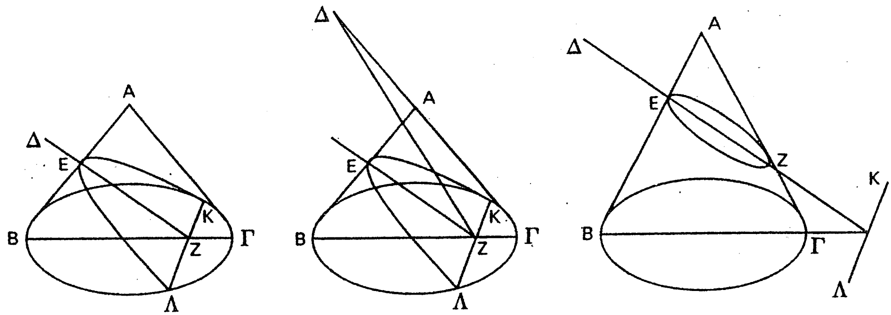


Figure 8

The ellipse is one of the conic sections from Apollonius's book "Conics" here shown in a 1710 edition

Cassini – Yet Another Attempt

But Giovanni Domenico Cassini (1625–1712) did not think so. It's a historical curiosity that he believed one of the curves bearing his name — the Cassini Curves — was a better representation of the earth's orbit than the ellipse. Although this was nonsense — Kepler had already tried an egg-shaped curve — the curve he discovered became the subject of numerous investigations.

(Nevertheless, we should not be too hard on Cassini. He was a celebrated astronomer with many planetary discoveries to his credit, including the fact that the rings about Saturn are broken into two sets of rings. This break in the rings is now called the Cassini Division. A plan to send the space probe Pioneer II through the Cassini Division was abandoned in 1979 since the division does contain ring particles which could damage the craft even though the division is so clear that stars can be seen through it.)

The Cassini Curves are given by the equation

$$(x^2+y^2)^2 - 2x^2(x^2-y^2) = a^4 - c^4.$$

They are often called Cassini Ovals, yet are not always ovals, as shown in Figure 9. (An oval is a closed curve such that a straight line cannot cut it in more than two points.)

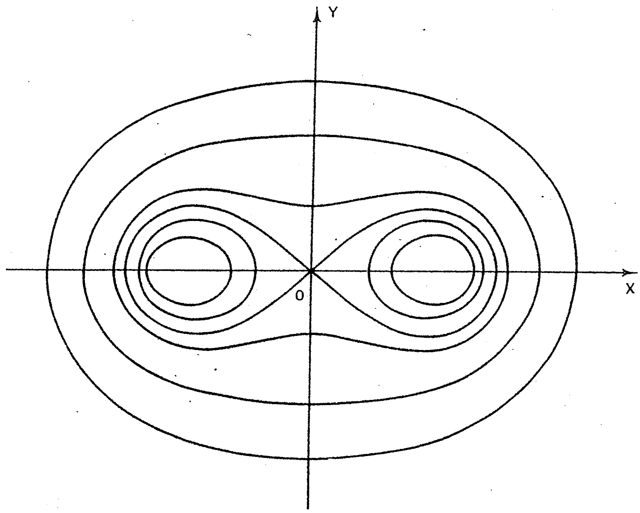


Figure 9

... although Cassini did have one last attempt with his Cassini Curves

For $a < c$ the Cassini Curves consist of pairs of separate ovals — shown for $a = 0.8c$ and $a = 0.9c$.

For $a > c$ the Cassini Curves are closed, the outer two shown for $a = \sqrt{2}c$ and $a = \sqrt{3}c$ are ovals; but the curve for $a = 1.1c$ is not oval.

For $a = c$ the Cassini Curve is the Lemniscate.

And that takes us right back to Eudoxus.

The Solution

And what is the solution to the path of Mars shown in Figure 1? It is shown in Figure 10. As both the Earth and Mars follow their elliptical orbits about the Sun, what is the path of Mars as observed from the Earth? This is what Kepler worked out. Figure 10 shows the path of Mars as observed from Earth against the background of the stars. Here, finally, is the explanation of Figure 1.

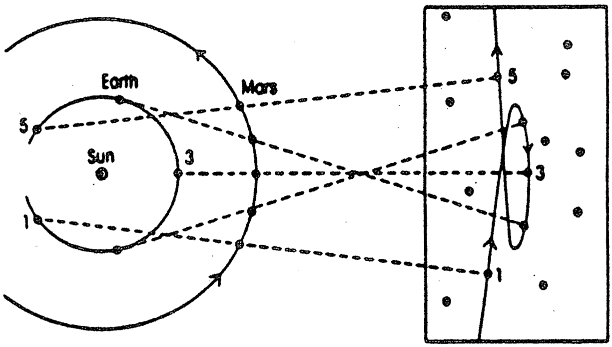


Figure 10

The solution to the path of Mars in Figure 1.

References

Dreyer, J.L.E. *A History of Astronomy from Thales to Kepler.* Dover.

Heath, T. *A History of Greek Mathematics.* Dover.

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"Those who have an excessive faith in their ideas are not well fitted to make discoveries."

Claude Bernard, as quoted with approval by Jacques Hadamard in "An essay on the psychology of invention in the mathematical field", Princeton University Press, 1945.

TOURNAMENT OF THE TOWNS

Peter Taylor, University of Canberra

Background

The Tournament of the Towns is an inter-city mathematical problem-solving competition for high school students which has been organised in the Soviet Union since 1979.

Each town which enters arranges for its students to sit for the paper on a given day. The scoring system is discussed below, but essentially the town's score is adjusted according to its population, enabling each town to enter the competition on an approximately equal footing.

The competition has spread into other countries of the Eastern Bloc, particularly Bulgaria, which has several towns competing. The first I heard of the competition was while attending ICME-6, in Budapest. At this conference, two Bulgarian mathematicians, Petar Kenderov and Jordan Tabov, approached me to see if we could enter an Australian city in the competition.

They pointed out that the International Mathematical Olympiad (IMO) had also started as a competition between countries of the Eastern Bloc. However, it was relatively expensive to run, particularly because of transport and accommodation costs, and was limited to those students fortunate to gain selection in their national team.

The Tournament of the Towns, on the other hand, was accessible to students on a much wider basis, with virtually no operating costs, as students attempt the problems in their home towns. It was envisaged that this competition could become the second major international event.

In inviting an Australian city to enter the competition, Professors Kenderov and Tabov said that they were aware of the success of the Australian Mathematics Competition (AMC), organised at the Canberra CAE (now the University of Canberra), and the successful hosting, in 1988, by the same institution (under the chairmanship of my colleague, Peter O'Halloran), of the International Mathematical Olympiad (IMO), and that we were consequently likely to have the infrastructure to organise such an entry.

The Competition Structure

The Tournament of the Towns is held twice a year, in Autumn and Spring. There are two papers, a Senior Paper for students in Years 11 and 12, and a Junior Paper for students in years 8, 9 and 10. Furthermore, each paper has two versions, a training version and a main version. A student may attempt either version or both (they are scheduled on separate days). A student who attempts both versions is awarded the higher score of the two. The main paper is distinctly the more difficult, but more points are obtainable in this paper. The main paper normally has 6 questions, all worth varying point values ranging upwards from 3 points for the first question. However, students only score for their best three solutions, so to get a high mark it may be necessary to select the harder questions.

It is strongly recommended that all students attempt the training version of the paper, and that they only attempt the main paper understanding that the paper is difficult, and that it is an achievement to get even one complete solution.

Those students in the lower age groups receive an upwards adjustment of their score to enable them to compete on an equal footing with the older students. For example, Year 8 students have their mark increased by a factor of 3/2.

There has been some debate about the standard of the papers, and how they compare, say, with the Australian Mathematical Olympiad. There is no doubt that the training paper is quite accessible to most students of high ability. However, the main paper is very difficult. Whereas comparisons can be misleading, it is thought that this paper may be less technical than the Australian Mathematical Olympiad, it still requires high innate ability, and may be of about the same standard.

The local organisers do some preliminary assessment and send the papers likely to form part of the city score to the central jury in Moscow for central assessment. The papers are formally marked there by University students.

If the population of the city is N00 000, the average of the best N papers forms the city's score. However, if the population is less than 500 000, as is the case in Canberra, 5 papers are still required. In these cases the city has its score adjusted by a further compensatory factor.

Canberra's Entry

On my return to Canberra late in 1988 I discussed the proposal with two people. First there was my colleague, Peter O'Halloran, referred to above, who is also Director of the AMC, and is Deputy Chairman of the Australian Mathematical Olympiad Committee (AMOC). Peter was very encouraging about entering a local team, particularly as he saw this event as a focus for training and motivating our potential IMO team members.

Secondly I spoke to Mike Newman, of the ANU. Mike, together with his colleague Laci Kovács and others, have been running Friday night enrichment classes for Canberra's secondary students for a period of about 25 years. Mike had already closed down his 1988 program when I spoke to him but he showed a strong interest in the competition. We arranged to invite a number of students to attend a training session one Friday afternoon in November 1988 prior to entering the competition on the following Friday afternoon.

Virtually all invited students attended and the results were very encouraging to us. Of 41 cities which entered the competition (Canberra being the only western city and the only city using the English language), Canberra's score placed it in 13th position, a situation described as highly commendable by the Moscow organisers, especially at its first attempt. Furthermore, one student, Simon Wilson, was awarded full marks for three questions (this being the maximum number of questions which could be attempted, under the scoring system), including the question rated the most difficult in the paper. He was, in due course, presented a Diploma issued by the USSR Academy of Sciences, a unique honour for an Australian student.

During 1989, the Friday evening sessions were restructured. Students of all secondary year levels were invited to attend these sessions. There are now two classes: one is open to all, and meets only during terms 1 to 3. The other is at a higher level, and by invitation only, and runs throughout the school year. It is essentially this group which trains for and enters the Tournament of the Towns (although we do focus on other competitions during the year, such as the IBM competition organised by the University of NSW, and questions from the AMOC program).

Entries By Other Australian Cities

In addition to Canberra, other Australian cities are now taking part. The main vehicle for entry is likely to be through local AMOC representatives, although this does not have to be the case. In November 1989 Lawrence Doolan, of Melbourne Church of England Grammar School, with assistance of Hans Lausch, of Monash University, organised a Melbourne entry with over 30 students. Scores are not yet to hand, but the Melbourne students attempted only the main version of the paper, which will limit their ability to obtain a high score. Several other cities are likely to enter the next round, in April 1990, such as Hobart, Newcastle, Wollongong and Armidale.

Any reader of *FUNCTION* who is interested in entering a city team or just seeks further information is welcome to write to me. All Australian entries need to be coordinated through the national committee, which I chair. This means that I can supply the question papers (when available) and return scripts to the central organisers, but local organisers need to organise their own students and perform their own preliminary assessments. There is no entry fee.

Furthermore, the Australian committee awards certificates to Australian students, based on local assessments. These vary from a Participation certificate upwards, with a complete solution normally required for a Credit certificate. The highest level is a High Distinction, which normally requires three complete solutions.

International Organisation

Whereas the competition is organised in Moscow, it is recognised that if the competition is to become truly international it will need an international committee and more formal sets of rules. Mathematicians from the USSR, Bulgaria and Australia are currently working on the procedures and regulations and are hoping to agree on such a structure at the first conference of the World Federation of National Mathematics Competitions to be held in Waterloo, Canada in 1990. The competition is expected to be known as the International Mathematics Tournament of the Towns, to more accurately reflect the nature of the event.

Some Past Questions

A selection of some past questions is given below. A more detailed selection of past questions, together with strategy essays and solutions, prepared by mathematicians in Australia, has been prepared and is available from the AMC office, University of Canberra, PO Box 1, Belconnen, ACT 2616, at a cost of $8.

JUNIOR, Years 8, 9, 10: November 1988

1. One of the numbers 1 or -1 is assigned to each vertex of a cube. To each face of the cube is assigned the integer which is the product of the four integers at the vertices of the face. Is it possible that the sum of the 14 assigned integers is 0 ?

(3 points)

2. A point M is chosen inside the square $ABCD$ in such a way that $\angle MAC = \angle MCD = x$. Find $\angle ABM$.

(3 points)

3. (a) Two identical cogwheels with 14 teeth each are given. One is laid horizontally on top of the other in such a way that their teeth coincide (thus the projections of the teeth on the horizontal plane are identical). Four pairs of coinciding teeth are cut off. Is it always possible to rotate the two cogwheels with respect to each other so that their common projection looks like that of an entire cogwheel? (The cogwheels may be rotated about their common axis, but not turned over.)

   (b) Answer the same question, but with two 13-tooth cogwheels and four pairs of cut-off teeth.

(3 points)

4. A convex n-vertex polygon is partitioned into triangles by non-intersecting diagonals. The following operation, called *perestroyka* (= reconstruction), is allowed: two triangles ABD and BCD with a common side may be replaced by the triangles ABC and ACD. By $P(n)$ denote the smallest number of perestroykas needed to transform any partitioning into any other one. Prove that

   (a) $P(n) \geq n - 3$; (2 points)

   (b) $P(n) \leq 2n - 7$; (2 points)

   (c) $P(n) \leq 2n - 10$ if $n \geq 13$. (3 points)

5. Does there exist a natural number which is not a divisor of any natural number whose decimal expression consists of zeros and ones, with no more than 1988 ones?

(8 points)

JUNIOR, Years 8, 9, 10: March 1989

1. The convex quadrilaterals $ABCD$ and $PQRS$ are made respectively from paper and cardboard. We say that they suit each other if the following two conditions are met:

   (1) It is possible to put the cardboard quadrilateral on the paper one so that the vertices of the first lie on the sides of the second, one vertex per side, and

   (2) If, after this, we can fold the four non-covered triangles of the paper quadrilateral on to the cardboard one, covering it exactly.

(a) Prove that if the quadrilaterals suit each other, then the paper one has either a pair of opposite sides parallel or (a pair of) perpendicular diagonals. (2 points)

(b) Prove that if $ABCD$ is a parallelogram, then one can always make a cardboard quadrilateral to suit it. (3 points)

2. Prove that if k is an even positive integer then it is possible to write the integers from 1 to $k - 1$ in such an order that the sum of no set of successive numbers is divisible by k.

(5 points)

3. (a) Prove that if $3n$ stars are placed in $3n$ cells of a $2n \times 2n$ array, then it is possible to remove n rows and n columns in such a way that all stars will be removed. (4 points)

(b) Prove that it is possible to place $3n + 1$ stars in the cells of a $2n \times 2n$ array in such a way that after removing any n rows and n columns at least one star remains. (4 points)

SENIOR, Years 11, 12: November 1988

1. What is the smallest number of squares of a chess board that can be marked in such a manner that

(a) no two marked squares may have a common side or a common vertex, and
(b) any unmarked square has a common side or a common vertex with at least one marked square?

Indicate a specific configuration of marked squares satisfying (a) and (b) and show that a lesser number of marked squares will not suffice.

(3 points)

2. Prove that $a^2pq + b^2qr + c^2rp \leq 0$, whenever a, b and c are the lengths of the sides of a triangle and $p + q + r = 0$.

(3 points)

SENIOR, Years 11, 12: March 1989

1. Find a pair of 2 six-digit numbers such that, if they are written down side by side to form a twelve-digit number, this number is divisible by the product of the two original numbers. Find all such pairs of six-digit numbers.

(3 points)

2. The point M, inside ΔABC, satisfies the conditions that $\angle BMC = 90^\circ + \frac{1}{2}\angle BAC$ and that the line AM contains the centre of the circumscribed circle of ΔBMC. Prove that M is the centre of the inscribed circle of ΔABC.

(4 points)

3. A club of 11 people has a committee. At every meeting of the committee a new committee is formed which differs by 1 person from its predecessor (either one new member is included or one member is removed). The committee must always have at least three members and, according to the club rules, the committee membership at any stage must differ from its membership at every previous stage. Is it possible that after some time all possible compositions of the committee will have already occurred?

(6 points)

4. We are given N lines $(n > 1)$ in a plane, no two of which are parallel and no three of which have a point in common. Prove that it is possible to assign, to each region of the plane determined by these lines, a non-zero integer of absolute value not exceeding N, such that the sum of the integers on either side of any of the given lines is equal to 0.

(7 points)

5. We are given 101 rectangles with sides of integer lengths not exceeding 100. Prove that among these 101 rectangles there are 3 rectanglues, say A, B and C such that A will fit inside B and B inside C.

(7 points)

* * * * *

MATHS AND SCIENCE :

GOOD FOR GIRLS, GOOD FOR VICTORIA !

Robyn Arianrhod, Monash University

Most of you will have seen the recent *Maths Multiplies Your Choices* TV commercials, and perhaps you will have wondered what all the fuss was about? The commercials were, in fact, part of a Victorian government campaign which can best be summed up by the title of one of the government's own information brochures: MATHS AND SCIENCE: GOOD FOR GIRLS, GOOD FOR VICTORIA.

The brochure (the *Minister for Labour News*) describes the campaign as follows: "By encouraging girls to keep studying maths and science, the State government plans to boost the economy and get more girls into rewarding, technical careers. At present, more than 50% of girls decide to drop maths before Year 12. Without maths at Years 11 and 12, these girls are unable to enter most scientific and technical careers. Unless more girls take up technical careers, Victoria could face critical shortages of technically skilled labour ... . Victoria's international competitiveness depends on the growth of technologically advanced manufacturing and services. Working at the leading edge of technology, export industries and import-competing industries are being developed ... . With a bigger pool of skilled young people, Victoria's growing technology-based industries will be set to make the most of new opportunities."

That's the planned benefit for Victoria of having more girls doing science and maths to Year 12, and to help the process along, the government is working with companies in the establishment of work-based childcare units for employees. Politically, it's an exciting time to be a girl in the process of planning her career! But what are the personal benefits for girls of taking maths and science to Year 12? And why the emphasis on girls anyway?

In 1966, about one-third† of women aged 25–34 were working. Now the figure is closer to two-thirds. Although the stereotype in past generations — perhaps your parents'! — has been that women work only up until they have children, this has often been far from the reality for many women; nevertheless, the stereotype has prevented girls from actually *preparing* for a career, and so they have tended to be concentrated in the less interesting, less well-paid jobs. Currently, over half of all women employed work in either clerical or sales jobs.

As our world becomes more technology-oriented, jobs which perhaps in your mother's day did not require a maths or science background, jobs such as nursing or physical education teaching, nowadays do require such a background. Thus, even traditionally "female" jobs can be denied to girls who drop maths, physics and chemistry too early, for the fact is that boys are much better prepared by the end of Year 12 for the widest range of careers than are girls.

In Victoria, approximately the same proportions of male and female students study at least one maths and science subject at Year 10; in Year 11, slight gender differences appear in these numbers, with fewer girls studying maths and science, while by Year 12, substantial gender differences appear, with 70.1% of boys but only 50.7% of girls doing at least one maths subject, and 20% of boys compared with 6% of girls taking at least two maths and at least two science subjects. In Year 12, 49.0% of girls have no maths, compared with 30.0% of boys.

Having maths at Year 12 prepares you for a much greater variety of careers than would otherwise be possible, and as can be inferred from the figures in the previous paragraph, many of these jobs are male-dominated. This creates a vicious circle in that many girls avoid such jobs — and the school subjects they require — because they feel uncomfortable in a classroom or work environment in which they are in the minority. So, boys, please take note of the Victorian Minister for Labour (Neil Pope):

"Both employers and employees will need to make an effort to make female trainees and workers 'at home' in the workplace. This means looking at how new employees are welcomed. It means making sure that girls aren't expected to do the lunchroom dishes."

And it means giving girls a fair go in the classroom!

Girls, to find information to help you choose subjects and plan your careers, encourage your parents to come with you to see your school Careers teacher, or to ring the Vocational Orientation Centre, 663-5800, or the Career Reference Centre, 665-8466.

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† All figures in this article are from Department of Labour publications.

LETTERS TO THE EDITOR

A Theorem on Trigonometric Functions

I have discovered a theorem which may be of interest to readers of *Function*. It goes as follows:

For any three points A, B, C in the Cartesian co-ordinate plane, with rational co-ordinates, $\sin(2k\angle ABC)$ is rational for any integer k and so is $\cos(2k\angle ABC)$.

Proof: Dilate the plane so that A, B and C are lattice points. By Pick's theorem[†], the area of ABC, $(= S)$, is rational. Let $BA = a$ and $BC = b$. Then we know that $2S = ab\sin(\angle ABC)$. So $ab\sin(\angle ABC)$ is rational. $\overline{BA}$ can be written as $e\underset{\sim}{i} + f\underset{\sim}{j}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are the unit-vectors in the x and y direction. $\overline{BC}$ can be written as $c\underset{\sim}{i} + d\underset{\sim}{j}$. Since e, f, c, d are integers, then $\overline{BA}\cdot\overline{BC} = ab\cos(\angle ABC) = ec + fd$, which is an integer. By Pythagoras, a^2 and b^2 are integers. So $a^2b^2\cos^2(\angle ABC)$ is the square of an integer. Hence it is an integer. Hence $\cos^2(\angle ABC)$ is rational. Hence $\cos(2\angle ABC) = 2\cos^2(\angle ABC) - 1$ is rational. Multiplying $2ab\sin(\angle ABC)$ by $ab\cos(\angle ABC)$ gives a rational number, since both factors are rational. Hence $2a^2b^2\sin(\angle ABC)$ is rational. Dividing this by the integer a^2b^2 gives a rational number. Hence $2\sin(\angle ABC)\cos(\angle ABC)$ is rational, and this is $\sin(2\angle ABC)$.

We have proved the theorem for $k = 1$. For $k = 0$ the theorem can be seen to hold. We now proceed by induction. If the theorem holds for some integer, say n, then

$$\sin[2(n+1)\angle ABC] = \sin(2n\angle ABC)\cos(2\angle ABC) + \cos(2n\angle ABC)\sin(2\angle ABC)$$

$$\cos[2(n+1)\angle ABC] = \cos(2n\angle ABC)\cos(2\angle ABC) - \sin(2n\angle ABC)\sin(2\angle ABC).$$

By the inductive assumption, all terms on the right-hand sides of these equations are rational. Hence the left-hand sides are also rational. Hence if the statement holds for some positive integer, n, it holds for all positive integers k, such that $k > n$. As it holds for $n = 1$ it holds for all positive integers k.

If $k < 0$ use the fact that $\cos(-\theta) = \cos(\theta)$
and $\sin(-\theta) = -\sin(\theta)$ to see that if the statement holds for all positive integers, then it holds for all negative integers too and hence for all integers.

Brian Weatherson,
Glen Waverley.

[†] See *Function, Vol.* 8, *Part* 1.

More on Calculating Ages

Recently I was sent a copy of *Function Vol.* 13, *Part* 3, by my friend John Burns. This was because of John's article on "Bowling averages" which I was partly involved in. The article by Garnet Greenbury on "Calculating age by formula" caught my attention. In case you have had only a small response to this article, I add the following comments.

Let A be any age.
If N_1, N_2, N_3 are integers then

$$A = 3N_1 + x$$

$$A = 5N_2 + y$$

$$A = 7N_3 + z.$$

Substitution into

$$A = 70x + 21y + 15z - 105n$$

yields

$$n = A - 2N_1 - N_2 - N_3$$

where $N_1 = [A/3]$, $N_2 = [A/5]$, $N_3 = [A/7]$.

A simple computer program using integer values [ ] can be written to list n for all integers A from 1 to 130 (or other appropriate upper bound). The output shows that from 1 to 104 the values for n are either 0, 1 or 2 without exception. At $A = 105$ the value of n is -1 and for $105 \leq A \leq 209$ the values for n are either -1, 0, 1 (that is, $n = 2$ no longer occurs). For $210 \leq A < 314$ the values are $n = -2, -1$ or 0 and so on. Within each group of 105 integers there doesn't seem to be any discernible pattern.

Neville de Mestre
Bond University

* * * * *

"Mathematicians are like Frenchmen; whatever you say to them, they translate into their own language, and forthwith it is something entirely new."

GOETHE

HISTORY OF MATHEMATICS SECTION

Mathematics is not as it is sometimes presented: as a finished product, there to be learned by the all too often unwilling and unconvinced student. In fact, mathematics is an ongoing endeavour, an organic thing that exhibits its own patterns of evolution. This is very clear at the research level, where we mathematicians try to extend the boundaries of mathematics or of its applications. At times it is possible to explain, or at least to describe, these advances to the target audience of *Function* but this is not always the case. Modern mathematics, at the edge of research, is (not surprisingly) difficult; and naturally it calls upon and relies upon concepts and ideas unavailable to the student.

It follows from this that it is not always possible to show the human face of contemporary Mathematics: too many things get in the way. Where possible, *Function* does go in for this, but it's an option not always available.

One way round this dilemma, and a very good one at that, is to look at the history of our subject: to see how real-life men and women grappled with mathematical concepts, to see what drove them, to watch them at work (so to speak), even to document their mistakes.

With this in mind, the editors of *Function* have voted to have a special **History of Mathematics** section in each issue. I have been asked to edit this section, and have agreed to do so. I would welcome your comments and contributions to this section. I feel it has a lot of potential. Over the years, *Function* has carried a good deal of material of this type and it has been well received.

For this inaugural column, I pick up three stories from earlier issues of *Function* and bring them up to date by providing more recent information. It would help to have the earlier stories to hand, but it is not absolutely necessary; enough background will be given to enable new readers to pick up the threads right now.

Michael A.B. Deakin

Li Ma-To

In *Function, Vol. 10, Part 5* (October 1986), I wrote about a number of mathematicians who had been active in the religious life of the Catholic church and who might one day become candidates for sainthood. Later, I recollected two others that I'd forgotten to mention and so wrote a Letter to the Editor (*Vol. 13, Part 1*) giving details of these two.

One of them was the Jesuit priest Matteo Ricci, who spent a great part of his life as a missionary in China. As I stated there, his most lasting contribution to Mathematics is probably his translation (completed by other hands) of Euclid into Chinese.

Ricci is best remembered for what his biographer calls his "complete adaptation to China". Indeed, it is a tribute to the completeness of that adaptation that he is probably better remembered in China than he is in the West. In China, he is known as *Li Ma-To.* This is in fact a rendering of his (Italian) name into Chinese. The *Li* is, modulo the L-R dichotomy that besets transliteration from East Asian languages, the first syllable of the name *Ricci,* and is moreover the closest approximation available from the relatively limited list of Chinese surnames to the name *Ricci. Ma-To* is a quite straightforward transliteration of the Christian name *Matteo.* (Remember that in Chinese the surname precedes the given name.)

We asked Tony Lun, of Monash University's Mathematics Department, to write Ricci's Chinese name for us in Chinese characters, and his calligraphy is reproduced below left. He informs us that in Hong Kong, where he comes from, it is common practice to name schools, particularly those specialising in Mathematics, in honour of Ricci.

Below right is a portrait of Ricci supplied by Dr Paul Rule, head of the Division of Religious Studies at La Trobe University. He writes that the procedures relating to Ricci's possible beatification (a preliminary to canonisation, or declaration of sainthood) seem to be held up by political considerations, although the diocese of Macerata (Ricci's home town) is actively promoting his cause.

Li 利

Ma 瑪

To 竇

The Three Cities Revisited

In *Function, Vol. 11, Part 5,* I wrote about the problem of siting a facility (school, telephone exchange or what have you) to serve three cities. The question at issue is so to place it that we minimise the total distance from the facility, situated at O, to the three cities at (say) A, B and C. I.e., we must so place O that $OA + OB + OC$ is as small as possible. The problem is a quite practical one; indeed, it is an important lead-in to the study of many commercially important areas of Mathematics.

One rather surprising approach is to solve the problem experimentally by making a model of the geography on a table-top. Holes are drilled at the points representing A, B, C through the table. Through these holes are passed strings which are tied together above the table. Weights are hung on the other ends of the strings. In the simplest case, these weights are equal. The equilibrium position of the knot on the surface of the table shows us where to place O.

When our article on this topic was published, it seemed that the first mathematician to consider the matter was the geometer Jakob Steiner, who wrote about it in the years 1835-1837. It was also believed that the experimental approach was first discovered by the Polish mathematician Hugo Steinhaus in the early years of this century.

It has recently been discovered that both Steiner and Steinhaus were preceded in their discoveries. A recent paper by Ole Franksen and Ivor Grattan-Guinness, published in the journal *Mathematics and Computers in Simulation*, points out that the problem is treated in an obscure 1829 memoir by the French mathematicians G. Lamé (1795-1870) and B.P.E. Clapeyron (1799-1864). The paper was, as Franksen and Grattan-Guinness point out, largely ignored at the time and soon "fell into almost complete oblivion".

But it is quite clear that all the discoveries attributed to Steiner and Steinhaus about this problem were known to Lamé and Clapeyron in (or before) 1829. Furthermore, while Steiner in particular saw his analysis as a piece of pure Geometry and did not press the matter of applications, the two French researchers were motivated by practical considerations.

It should, however, be noted that because of the neglect suffered by the early memoir the later accounts by Steiner and Steinhaus were almost certainly independent rediscoveries.

Franksen and Grattan-Guinness also relate the solution to an earlier version of the model, or mechanism, used to find that solution. This is called *Varignon's Frame*, after the mathematician who invented it and used it in his proof of what is now called Lamy's Theorem. Lamy's Theorem is the reason why the Steiner process of drilling holes in the table-top works.

(Lamy, incidentally, is a different person from Lamé, although the two are from time to time and perhaps understandably confused. To add further to the confusion, the theorem attributed to Lamy and named after him was actually discovered by Varignon! Life was not meant to be easy.)

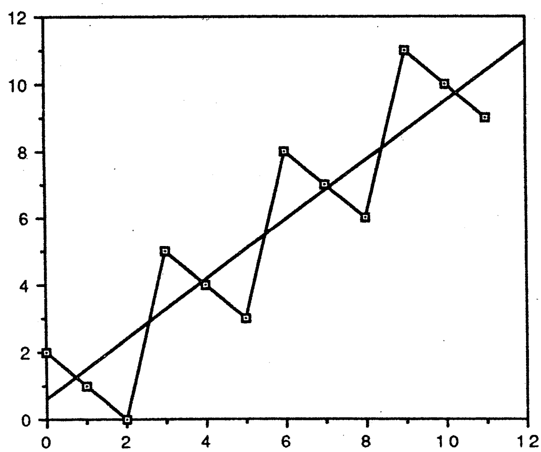
That Varignon's Frame could be used to solve the problem was actually pointed out before Steinhaus by Georg Pick, an Austrian mathematician, in 1909. Pick has another claim to fame. A remarkable and elegant formula for determining areas is known as "Pick's Formula" after its discoverer. For the details of this story see *Function, Vol. 8, Part 1.*

Statistical Paradoxes

A letter forwarded to us from the Monash Law School was published in *Function, Vol. 5, Part 4.* We published it under the title "Blyth's Paradox", which was the one they supplied. At that time neither they nor we knew who Blyth was. However, Blyth is almost certainly Colin R. Blyth, an American statistician. We will not here go into the details of the original example which were rather specialised. However, Blyth's Paradox is related to another known as "Simpson's Paradox".

Quite by coincidence, Simpson's Paradox was discussed in the very same issue of *Function,* by G.A. Watterson under the title "Which School?". Simpson's Paradox points out that if a population is divided into two or more parts, each and every one of them may display characteristics at variance with those of the population as a whole.

The following neat illustration of Simpson's Paradox is based on one due to Blyth. A set of twelve observations display an upward trend, but the first three of them display a downward trend, as do the second three, and so on. In the graph below, the data lie on a zigzag curve, in which the four downward trends are clearly visible. The straight line gives the overall trend, and this trend accounts for nearly 80% of the variance in the original data!



PROBLEM SECTION

In this section, we will be dealing with problems and their solutions. The problems will be categorised into two groups according to difficulty. I would welcome solutions, comments and new problems from readers. To begin with, here are some solutions to problems posed in previous issues of *Function*.

Hans Lausch,
Problems Editor

Solutions

Problem 10.5.1 It is easy enough with the calculator to show that $e^{\pi} = (\approx 23.14) > \pi^{e}(\approx 22.46)$, but can you show this without computation?

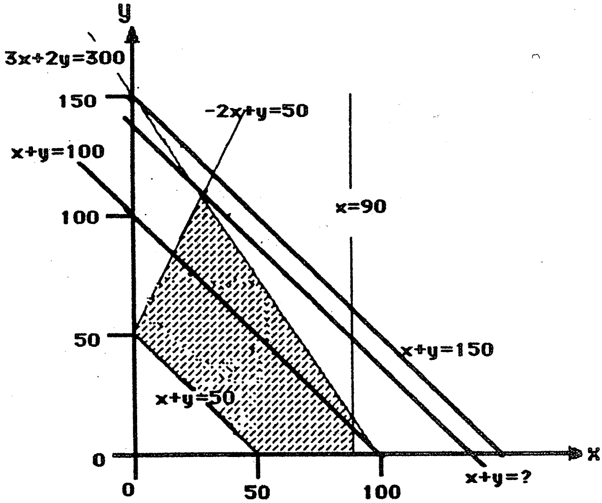
Solution (Garnet J. Greenbury, Brisbane). We note that for all real numbers x we can write $e^x = 1 + x + \frac{x^3}{2!} + \frac{x^3}{3!} + \dots$. Let $x > 0$. Then $e^x > 1 + x$ for all $x > 0$. It follows that $e^{1+x} = e \cdot e^x > e(1+x)$, and hence $e^{e(1+x)} > [e(1+x)]^2$. Put $x = \frac{\pi}{e} - 1$ to obtain the desired inequality.

Problem 10.5.2 If $x \geq 0, y \geq 0$ and

$$-2x + y \leq 50$$
$$3x + 2y \leq 300$$
$$x + y \leq 50$$
$$\text{and} \qquad x \leq 90,$$

what is the maximal value of $x + y$?

Solution. Consider the diagram below. The shaded area (including its boundary) represents the set of all ppoints (x, y) that satisfy simultaneously all the given conditions, i.e. all the inequalities of the problem. From among all the points of this area we have to find one for which $x + y$ is maximal. In order to do this we imagine all lines $x + y = c$, where c is allowed to range over all real numbers. This set is a bundle of parallel lines. In the diagram we have drawn the lines for $c = 50$, 100 and 150. We observe that running through these lines from left to right takes us from lines with smaller c to lines with larger c. Hence the rightmost line $x + y = c$ which contains (at least) one point belonging to the shaded area provides us with a point as required. We note that this is the line $x + y = c$ that passes through the intersection of the lines $-2x + y = 50$ and $3x + 2y = 300$. Consequently the pair (x, y) that is going to provide



us with the answer $x + y$ will be the solution of the system of the two equations

$$\begin{aligned} -2x + y &= 50 \\ 3x + 2y &= 300. \end{aligned}$$

Its solution is $x = \frac{200}{7}$, $y = \frac{750}{7}$. Therefore the answer to the question is $\frac{200}{7} + \frac{750}{7} = \frac{950}{7} = 135\frac{5}{7}$.

Problem 12.2.1 The host at a party turned to a guest and said: "I have three daughters and I will tell you how old they are. The product of their ages is 72. The sum of their ages is my house number. How old is each?".

The guest rushed to the door, looked at the house number, and informed the host that he needed more information.

The host then added: "The oldest one likes strawberry pudding".

The guest then announced the ages of the three girls.

What are the ages of the three dauthers? (All ages are integers.)

Solution. Writing down 72 as a product of three factors (all being integers) tells us all the possibilities for the girls' ages, and if we add these factors for each possibility, we find all candidates for the host's house number. We represent this information in table form:

Daughter no. 1	Daughter no. 2	Daughter no. 3	Host's house number
72	1	1	74
36	2	1	39
24	3	1	28
18	4	1	23
18	2	2	22
12	6	1	19
12	3	2	17
9	8	1	18
9	4	2	15
8	3	3	14
6	6	2	14
6	4	3	13

The only number that appears more than once in the house number column is 14. So 14 must be the host's house number, otherwise the guest would have been able to tell the daughter's age immediately by referring to the row in which the house number occurs. (Had e.g. 15 been the house number, then the guest would have known that the daughters were 9, 4 and 2 years old, respectively.) The extra information that the oldest one likes strawberry pudding eliminates the possibility that two daughters are 6 and one is 2. The guest was thus able to tell his host that one daughter was 8 and the other two were 3.

Problem 12.2.2 A ship is twice as old as its boiler was when the ship was as old as the boiler is now. The sum of their ages is forty-nine years. How old is the ship and how old is the boiler?

Solution. Let the ship's age be S and the boiler's age be B. Then $S + B = 49$, i.e.

$$B = 49 - S. \tag{1}$$

The ship is $S - B$ years older than the boiler; when the ship reached the boiler's present age B, the boiler was thus $B - (S-B) = 2B - S$ years old. Hence $S = 2(2B-S) = 4B - 2S$, i.e.

$$4B = 3S. \tag{2}$$

Substituting (1) into (2) we obtain

$$\begin{aligned} 4(49-S) &= 3S, \\ \text{i.e.} \quad 7S &= 196, \\ \text{i.e.} \quad S &= 28 \end{aligned}$$

is the age of the ship; it follows then from (1) that $B = 21$ is the age of the boiler.

Problem 13.2.1 Prove that, if the sum of any two square numbers is equal to a third square number, then one of the three numbers must be divisible by 5.

Solution. Let a, b and c be integers such that $a^2 + b^2 = c^2$. We note that a square number always leaves remainder 0, 1 or 4 when divided by 5. Suppose that neither a^2 nor b^2 is divisible by 5, then (using modular arithmetic mod 5)

$$\begin{aligned} c^2 - a^2 + b^2 & \\ &= (1 \text{ or } 4) + (1 \text{ or } 4) \bmod 5 \\ &= 2 \text{ or } 3 \text{ or } 0 \bmod 5. \end{aligned}$$

Being impossible remainders modulo 5 for square numbers, 2 and 3 can be discarded. Hence $c^2 = 0 \bmod 5$. In other words, 5 divides c^2 unless a^2 or b^2 is divisible by 5.

New Problems

a. Starter problems

Problem 14.1.1 Let a be an arbitrary real number. Determine all real numbers x with the property that

$$|x| + |x - 1| + |x - 2| = a.$$

Problem 14.1.2 Determine all real numbers x which satisfy the equation

$$\sqrt{x^2 - [x]^2} - [x]^2 = 3 - x,$$

where $[x]$ is the largest integer less than or equal to x.

Problem 14.1.3 Prove that 17 never divides any number of the form $2^{(3^n)} + 1$.

Problem 14.1.4 Prove that the following inequalities hold for every positive integer n :

$$\frac{3}{2} \cdot \left[\sqrt[3]{(n+1)^2} - 1 \right] < 1 + (1/\sqrt[3]{2}) + (1/\sqrt[3]{3}) + \dots + (1/\sqrt[3]{n}) < \frac{3}{2} \cdot \sqrt[3]{n^2}.$$

b. Harder problems

Problem 14.1.5 Let a_0 be an arbitrary integer and suppose that $a_1, a_2, a_3, \dots$ are numbers for which the equation

$$a_n = n + (-1)^n \cdot a_{n-1}$$

holds.

a) Show that the sequence $a_0, a_1, a_2, \dots$ contains at least 1990 numbers that are equal.

b) Determine the least value of N such that the finite sequence $a_0, \dots, a_N$ contains 1990 numbers that are equal.

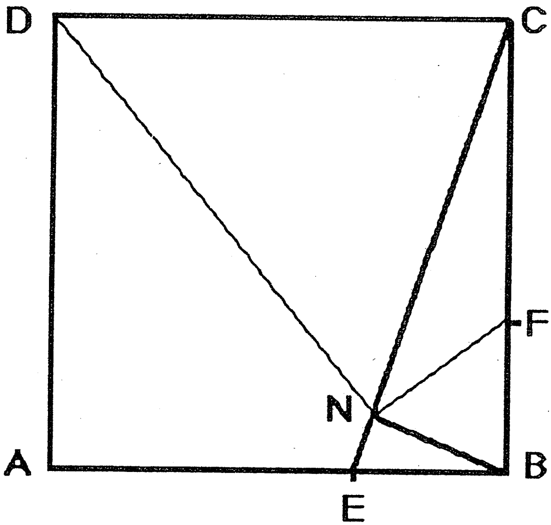
Problem 14.1.6 Let $ABCDE$ be a convex pentagon (i.e. a pentagon with all its interior angles less than $180°$), u its perimeter and p the sum of the lengths of its diagonals.

a) Show that $\frac{p}{u} < 2$.

b) Show that $1 < \frac{p}{u}$.

c) Are the values 2 and 1, respectively, best possible or can, in a), 2 be replaced by a smaller number, or in b), 1 by a larger number?

Problem 14.1.8 Let $ABCD$ be a square. Choose any point E on AB and then let F be the point on BC which is determined by the condition $BE = BF$. Let N be the foot of the altitude of the right-angled triangle EBC.



Show that $\angle DNF$ is a right angle.

Problem 14.1.7 Let n be a positive integer. What is the value of the sum $\sum \frac{(-1)^{[3^k/2]}}{3^k}$, where $[3^k/2]$ is defined as for $[x]$ in Problem 14.1.2?

CONVENTIONS IN MATHEMATICS

Strange how people get hot under the collar about things that don't really matter. Recently there's been a great debate on whether the new decade began on January 1, 1990 or will "really" begin on January 1, 1991. Even stranger, come to think of it, how people see it as a mathematical question. Just imagine the furore there'll be a month or so before January 1, 2000. "The decade" may begin wherever we care to say it begins. And if most people (like me) find it more convenient to let it start on January 1, 1990, so be it. If others want it to begin a year later, then they can be right, too. Just as long as we know what definition we're using. Come to think of it, why should the year begin on January 1? For the tax-man it begins on July 1. It used to begin on March 25. Other cultures count it from the first day of spring.

And when, pray, is that? We tend to say September 1, which would correspond to a Northern Hemisphere date of March 1. But in the USA it's customary to count spring as beginning on March 22 — the vernal equinox. This is why we in Australia sometimes see letters to the papers saying that spring "really" begins on September 22.

Another recent newspaper debate concerned prime numbers. Forget that one correspondent thought that the term "prime number" meant "non-zero single digit integer in base ten". By accepted convention, the word "prime" refers to the divisibility properties of numbers. Once this was established, the debate really heated up. Is 1 a prime? Well, again it depends on your definition. Today most mathematicians would not call 1 prime. An earlier generation would have. Let us see what is involved here.

Clearly, 1 has no factors other than 1 and itself (these being identical in this, and only in this, case). 2, 3 are prime, as are 5, 7, etc. and they also have no factors other than 1 and themselves. 6, by contrast, is composite, because the different numbers 2, 3 divide it; so is 4, as 2 divides 4, and 2 is different from 4.

But now take the number 6. We can express this in terms of its prime factors: $6 = 2 \times 3$. But if we count 1 as prime, we can also write $6 = 1 \times 2 \times 3$ or $6 = 1^2 \times 2 \times 3$, etc. The decomposition is unique if we do not include 1 in our list of primes. Otherwise it is not.

So now we say that positive integers fall into three categories: the primes, the composites and 1 (which is in a class by itself) and so we may say: "Every number may be expressed in a unique way as a product of primes" and forget about putting in a whole lot of extra stuff about the number 1.

Nevertheless, what you gain on the swings, you lose on the roundabouts. Consider, for example, Wilson's Theorem that says that "if and only if p is prime, then $(p\text{-}1)! + 1$ will be a multiple of p". Now if $p = 1$, we have $0! + 1$ and as $0!$ is normally defined as being 1 (and any other definition would do as well), then $(1\text{-}1)! + 1$ is clearly a multiple of 1.

So Wilson's Theorem works for $p = 1$. If we say that 1 is not a prime, we need to amend the statement of Wilson's Theorem to read "if and only if p is prime (or 1), then ...". The same holds true for other theorems as well. You pays your money and you takes your choice!

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