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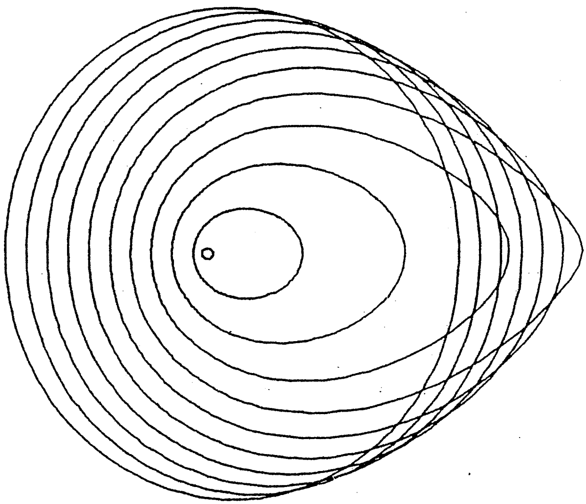

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Function is a mathematics magazine addressed principally to students in the upper forms of schools, and published by Monash University.

It is a "special interest" journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. *Function* is a counterpart of these.

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(Continued on back cover)

CONTENTS

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THE FRONT COVER

Ovals are defined as egg-shaped curves - indeed the word derives from the Latin *ovum*, meaning egg. The best known such curve is the ellipse, which we encounter as the shape of a circle seen in oblique perspective. This is one special case and indeed the circle is a very special case.

The mathematical properties that an oval shape might adopt have never been completely agreed by mathematicians, who tend to regard the word as rather imprecise (like "average", which you will have been warned about at school). Certainly an oval is to be a closed curve, that is to say we should be able to trace around it from any point on it and arrive back where we started. Probably most people would also insist that it have a smooth boundary without corners or sharp points and perhaps that it be convex - i.e. "bulge outward" everywhere.

In practice, it is convenient to forget this last requirement at times : there are curves, known as ovals, that disobey it.

Many ovals are expressed by equations in *bipolar coordinates*. Take two points, O_1, O_2 a distance $2c$ units apart. These will be our two *origins*. Now consider a point P. This will be distant r_1 from O_1 and r_2 from O_2. See Figure 1. These are called the *bipolar coordinates* of P. Note that P always has the same bipolar coordinates as another point P' lying on the other side of the line joining O_1 and O_2.

The ellipse has a very simple equation in bipolar coordinates. It is

$$r_1 + r_2 = 2a. \qquad (1)$$

Refer to Figure 1. If a string of length $2a$ were fixed by its ends (e.g. pinned down) at O_1, O_2 and stretched out by a pencil at P, the pencil could be swung round and would trace out an ellipse. This is, in fact, one of the standard ways to draw an ellipse. You may already know it.

The ratio c/a is called the eccentricity and defines the shape of the ellipse. Clearly $c \leq a$, as otherwise the string would not reach from O_1 to O_2, and also $c/a \geq 0$. Thus $0 \leq c/a \leq 1$.

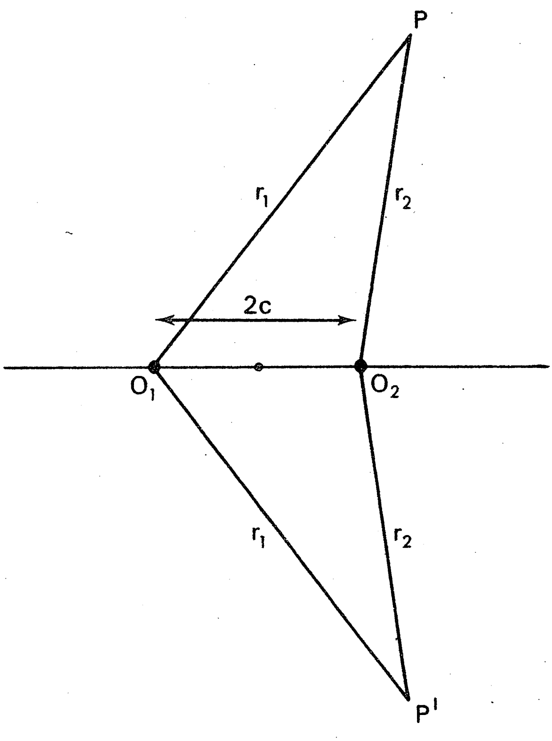
When the eccentricity is zero, we have a circle of radius a, and as c/a increases, the corresponding ellipses become more flattened, as well as becoming shorter. In the other extreme case, $c/a = 1$, the ellipse is flattened into the line segment O_1O_2.

A generalisation of Equation (1) is

$$mr_1 + nr_2 = 2a, \qquad (2)$$

where m,n are constants. This equation produces many curves, known collectively as the *Ovals of Descartes*. All are symmetric about the line O_1O_2. Indeed any curve whose equation is given in bipolar coordinates must possess this property because of the symmetry between P,P' already noted.

Our curves, computed by Geoff Bryan of Monash University, give $m = 3$, $n = 2$, a = 10, and c going from 0.5 to 5 in steps of 0.5. The corresponding curves for $n = -2$ are shown on the back cover.



CHINESE GAMBLING GAMES IN NSW IN 1891†

Frank Hansford-Miller,
Murdoch University

1. Introduction

The Census of 1891 showed that the Chinese population of New South Wales in Australia was 14,156. This total was sub-divided into Chinese of Full Blood or of Mixed Blood, and into Male and Female. There were 13,289 Chinese of Full Blood, of whom as many as 13,133 were Males (98.83%) with only 156 Females (1.17%). The Chinese of Mixed Blood were of a much smaller group and were almost equally divided into 422 Males (48.67%) and 445 Females (51.33%) in a total of 867 persons. Taking the Chinese of Full and Mixed Blood together, the grand total becomes 14,156 persons, with 13,555 Male (95.75%) and 601 Female (4.25%) (Census, 1891).

In such a situation Chinese males resorted in large numbers to gambling, so much so that a Royal Commission was set up in Sydney on "Alleged Chinese Gambling and Immorality and Charges of Bribery against the Police Force", and this Royal Commission published its Report in 1891. The Chinese at this time were not confined to Sydney although the report reveals that the Chinese population in Sydney and suburbs was some 3500. Of these, however, no less than 700 were said to be practically subsisting on the proceeds of gambling houses, which were principally centred in George Street North, Goulburn Street and its neighbourhood, and at Alexandria (Report, 1891-92).

The situation was similar in Victoria, as shown by the "Minutes of Evidence of the Select Committee of the Legislative Council on Chinese Immigration" in the Report of 1856-1857 (Minutes, 1856-57), and the "Report on the Condition of the Chinese Populations in Victoria" by the Rev. W. Young of 1868 (Young, 1868). Many of the Chinese market gardeners and cabinet makers on their Sundays off went along at this time to the gambling dens of Little Bourke Street in Melbourne, just as they did to Campbell Street in Sydney. The Minutes of Evidence also show that gambling was rife among the Chinese immigrants on the Victorian Goldfields.

2. Fan-tan

A very popular Chinese gambling game of this period was Fan-tan. The name is derived from the Chinese "fan t'an", meaning "repeated divisions", which is an apt description of the game (O.E.D., 1933). Writing in 1878 in her "Voyage of the

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Sunbeam", Lady Brassey, traveller, refers to "natives playing at fan-tan" (Brassey, 1878) and among the early Chinese goldfield workers in Australia the only known gambling game was in fact Fan-tan (Report, 1891-02).

Fan-tan remained popular in New South Wales in 1891 and is described in the Report of the Royal Commission, as follows:

"Fan-tan is played on a table with the aid of square sheet of metal, a cup, and a few dozen brass coins. The sides of the square are numbered 1, 2, 3, 4, and the players select the particular side upon which they will place their stakes. The banker then takes a handful of counters and throws them in a heap on the table, covering a portion of them with the cup, and after sweeping the remainder away, lifts the cover and counts the coins that were beneath it in sets of four, and the player whose money lies on the side of the square corresponding to the number of coins left after the last four have been subtracted trebles his stakes."

If only four counters are left on the sheet, then these are not gathered up into the cup, but are left to indicate that side 4 has won.

Owners of gambling houses are not usually in the game for amusement so it is reasonable to assume that, contrary to to-day's racing practice, the punter's original stake was not returned in addition to the winning odds of 3 to 1. Without this assumption the banker would be working on a nil profit basis.

Fan-tan is thus seen to give the gambler a probability of 1/4 of winning each time he plays, and when he wins he will receive three units of his stakes. Theoretically, therefore, on a long series of games the gambler will receive back only 75% of his stakes. On the other hand the gambling house banker will be getting the very comfortable return of 25% on all stake money.

3. Pak-ah-pu

The other highly popular activity of the Chinese in their gambling houses in New South Wales in 1891, according to the Royal Commission Report, was Pak-ah-pu (Report, 1891-92). This is the spelling of the Report but other versions are pakapoo, pakapu, pak-a-peu and paka pu, all deriving from the Chinese. This is a gambling game resembling lottery, with entries made on sheets of paper which to the uninitiated are said to be indecipherable. This incomprehensibility has produced the Australian slang phrase "like a pakapoo ticket", with the Oxford Dictionary supplement definition of "untidy, disorder": (Burchfield, 1982).

To develop this Australian phrase further, "Packapoo ticket" appears in "The Macquarie Dictionary of Australian Colloquialisms - Aussie Talk" as a noun, with the meaning of "something that looks confusing or incomprehensible", as in the usage "marked like a pakapoo ticket" (Delbridge, 1984). The Partridge "Dictionary of Slang" relates the phrase not just to Australia but especially to Sydney (Partridge, 1970). It would be false, however, to consider the game itself localised to Sydney and New South Wales. The Chinese took it with them wherever they went. The "Daily Mail" of Fleet Street in England reported in 1923:

"Five Chinese pleaded guilty at Liverpool Assizes to charges of running a gaming house ... For the defence it was argued that Fak-a-peu (or Fuck-a-pu) was a game of skill." (Daily Mail, 1923).

Another case of the game being played in England was reported in the London "Daily Express" in 1927:

"A Japanese ship's captain ... appealed against a conviction ... for employing two other Japanese to sell chances in an unlawful lottery known as 'Fuka pu'. It is a favourite game with the Japanese and Chinese living in Limehouse (in the East End of London),' explained Mr Horace Fenton." (Daily Express, 1927).

So we see that it was also popular with the Japanese as well as the Chinese. New Zealand also had its Pak-ah-pu gambling areas, as reported by the "New Zealand Listener" in 1960:

"Some of the last of the old Chinese dwellings of the opium-smoking and pakapoo-playing generation are being pulled down in Haining Street in Wellington." (New Zealand Listener, 1960).

The Royal Commission Report describes how the game was played in 1891 in New South Wales as follows:

"Pak-ah-pu is a species of lottery. For Sixpence the gambler is entitled to draw a mark through ten out of eighty Chinese characters printed on a ticket, and the game is determined by the number of characters thus obliterated corresponding with mottoes subsequently drawn by the banker from a bowl containing twenty characters selected haphazardly from a total of eighty similar to those on the ticket. If all the tickets were effective, when the bank is drawn the odds would be eight to one against any of the mottoes marked by the client corresponding with any of those drawn by the bank. But as a matter of fact the laws of the game demand a correspondence in five mottoes to entitle the purchaser to the smallest prize : 1s.2d. The prize for six successful marks is 10s.0d.; for seven £4.3s.4d.; for eight £23.6s.8d.; for nine £41.13s.4d.; and for ten £83.6s.8d." (Report, 1891-92).

To examine Pak-ah-pu statistically we need only consider the basic population of the eighty Chinese characters. These can be divided into those with an attribute, namely those that have been marked by the gambler on his ticket, and those without, namely those not so marked. Numerically, with the gambler entitled to mark 10 such characters for his stake of sixpence, this means that 10 characters have the attribute, and 70 have not.

The game thus becomes a problem in sampling without replacement, and the hypergeometric distribution applies. For the Pak-ah-pu game as played in 1891 the probability of a score of x is

$$p(x) = \frac{\binom{10}{x}\binom{70}{20-x}}{\binom{80}{20}} \qquad x = 5,6,7,8,9,10.$$

The probabilities of success, p, for the winning scores of 5 to 10 correct mottoes are shown in the second column of the Table. They range from 0.0514276 for 5 correct marks on the ticket to the very low probability of 1.12212×10^{-7} for 10 correct marks.

4. Mathematical Expectation in Pak-ah-pu

The Chinese who frequented the New South Wales gambling salons at this time were addicted gamblers. The average gambler at Pak-ah-pu would obviously not buy just a single ticket for sixpence, but would settle in for a session of play in which his purchases would probably run into hundreds. The Table shows the expected winnings of the gambler and, by subtraction, the resultant expected gain to the banker. The expectations, using the hyper-geometric probabilities, have been displayed for a single bet of 6d. and for a more realistic stake of £100. For his £100 stake, the gambler can, in the long run, expect to win back only £75.5075. The banker's long run gain is thus 24.4925%.

TABLE[†]

Number of Successful Markings	Hypergeometric Probability	Prize Money per 6d. Ticket	Expected return to Gambler (a) Per Single bet of 6d.	Expected return to Gambler (b) Per £100 Stake (=4000 tickets@ 6d.)
5	0.0514277	1s.2d.	0.72d.	£11.9998
6	0.0114794	10s.0d.	1.38d.	£22.9588
7	1.61114×10^{-3}	£4. 3s.4d.	1.61d.	£26.8523
8	1.35419×10^{-4}	£23. 6s.8d.	0.76d.	£12.6391
9	6.12065×10^{-6}	£41.13s.4d.	0.06d.	£1.0201
10	1.12212×10^{-7}	£3. 6s.8d.	0.002d.	£0.0374
	Gambler's expected winnings		4.532d.	£75.5075
	Banker's expected gain		1.468d.	£24.4925

[†] The money predates decimal currency. In the old system, 12 pennies (d) made one shilling(s), and 20 shillings made one pound (£). Thus £23.6s.8d. is 23 pounds, 6 shillings and 8 pence, or 23 $^1/3$ pounds. At the time of conversion, one pound became two dollars and so we would now represent this as $46.67. Of course, the value of money has also changed greatly over the years. [Eds.]

The psychology of the game no doubt contributed to its widespread appeal. It will be seen from the table that the first four categories of prizes provide a reasonable overall return. In every £75 winnings nearly £27 comes from winning £4.3s.4d. prizes for 7 correct marks. For 6 correct marks he receives back nearly £23 in Ten Shillings prizes, whilst the very nice win of £23.6s.8d. for 8 correct provided £12.6 of the winnings in every £100. No doubt, too, the gambler was helped towards his addiction to the game by the fairly continuous 1s.2d. returns for 5 correct, amounting to a total of £12 in £100. At the same time the gambler would obviously keep in his sights the possibility of winning one of the two top big prizes of £41.13s.4d. and £83.6s.8d., for 9 and 10 correct marks, respectively. Yet the Table shows that his mathematical expectation of return for these was very small, the hypergeometric chances against winning tickets being astronomically high. Ten correct marks will occur, theoretically, only once in 8,911,703 games. Perhaps this was another source of attraction of the game, leading to addiction - seeking the almost unattainable.

5. Relationship with the Present Time

The Royal Commission Report on Alleged Chinese Gambling and Immorality of 1891 was concerned with the harm and privation done not only to the Chinese, but also to the European children, sailors, wharf-labourers and coal humpers. It recommended that the police authorities should take tougher measures in the suppression of gambling (Report, 1891-92). In 1904 the Secretary of the Shop Assistants Union advocated the segregation of Chinese so as to reduce their "demoralizing influences" (Sydney Morning Herald, 1904). In the same year the Anti-Chinese and Anti-Asiatic League of Sydney was founded and this also emphasised the evils of Chinese opium taking and gambling (Sydney Morning Herald, 1904). In Victoria such sentiments had surfaced earlier, for it was in 1868 that the Rev. William Young published his "Condition of the Chinese Population in Victoria" (Young, 1868) which asked for legislation to be provided "to save the Chinese from ruining themselves and the society around them." (Young, 1977).

Today the story is very different. Seven legal casinos already operate in Australia, and others are under construction or planned. It is estimated that as many as 15 casinos could be operating within 15 years (The Weekend Australian, 1985).

We have seen that the house percentage in the 1891 Chinese gambling saloons was 25% in Fan-tan and 24.49% in Pak-ah-pu. These percentages are much higher than those for Roulette in casinos today. The house percentage in Roulette in United States casinos is from 5.26% to 7.89%, whilst in European casinos it is 1.35% to 2.70%. In the dice game of Craps, however, American casinos return from 0.6% to 2.7% less than the correct odds, depending on the type of bet made (Encyclopaedia Britannica, 1976).

Pak-ah-pu is, of course, a form of lottery, and these go back a very long way in the history of mankind. In the Old Testament we can read that the Lord instructed Moses to take a census of the people of Israel and then to divide the land among them by lot (Old Testament, Numbers). In France there were lotteries in 1520 and 1539, but the first public lottery to have paid money as prizes is believed to be La Lotto de Firenze, in Florence, in 1530. It was a great success and spread to other Italian cities.

In their love of gambling the Australian and the Chinese can be seen to have much in common. If the Chinese can be called "the world's most enthusiastic gamblers", together with other South-east Asians (Encyclopaedia Britannica, 1976), then as counterweight we find "Australia has been called the real home of State lottery" (Encyclopaedia Britannica, 1976). New South Wales had lotteries as early as 1849 and the glorious Sydney Opera House is one of its results. Today, however, instead of marking off our ten from eighty Chinese characters in an opium-filled Chinese gambling saloon we go along in millions to our local Lotto agent and fill in our choice of six numbers from 45, ensuring, I am sure, that our coupon is neat and tidy and with little resemblance to its Chinese "incomprehensible" antecedent - a Pak-ah-pu ticket.

Acknowledgement

The author wishes to thank Brenon Clarke for useful discussions, and for checking the computations.

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[Continued on p.30.]

THE PROBLEM OF FACTORING NUMBERS

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Ever since Eratosthenes discovered his "sieve" method to determine whether a number is prime or composite, mathematicians have been trying to find a formula which produces only prime numbers. Many attempts have involved algebraic expressions for generating primes : for example $n^2 - n + 41$ which yields prime numbers for all values of n up to and including 40, or the famous suggestion by Fermat : $2^n + 1$, where n is a power of 2, but this fails for $n = 32$.

As early as 300 BC, Euclid proved that there is no largest prime number. He did so by multiplying a given list of primes $2 \times 3 \times 5 \times \ldots \times p$, where p is the largest prime on the list, and then adding 1 to this product. This new number, if it is composite and not itself prime, is divisible by at least two primes not on the original list, so no such list can ever be complete. The reason for this is that no prime on the list can divide the number exactly - rather the result will always be a remainder of 1.

Consider these cases:

If $p = 7$, $\quad 2 \times 3 \times 5 \times 7 + 1 = 211$, which is prime
If $p = 11$, $\quad 2 \times 3 \times 5 \times 7 \times 11 + 1 = 2311$ which is prime
If $p = 13$, $\quad 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30031 = 59 \times 509$.

In this last example, the factors (59 and 509) were both larger than 13.

We could, instead of adding 1, have subtracted 1. This produces similar results. Consider these cases:

If $p = 7$, $\quad 2 \times 3 \times 5 \times 7-1 = 209 = 11 \times 19$
If $p = 11$, $\quad 2 \times 3 \times 5 \times 7 \times 11-1 = 2309$, which is prime
If $p = 13$, $\quad 2 \times 3 \times 5 \times 7 \times 11 \times 13-1 = 30029$, which is prime
If $p = 17$, $\quad 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17-1 = 510509 = 61 \times 8369$.

Notice that $11 > 7$, $19 > 7$, $8369 > 61 > 17$.

It is interesting to notice the prime factors that appear when the number is composite. 209, for example, is composite and, since $\sqrt{209} \approx 14.45$ and 11 divides 209, the co-factor, 19, must also be prime because $19 < 11^2 = 121$. Thus 209 is the product of exactly two prime factors.

In general, let P be the smallest prime factor of a number N and let P^* be its co-factor. If P^* is not itself prime, it must be the product of two or more primes, each being greater than or equal to P, the smallest such factor. So $P^* \geq P^2$. Conversely if $P^* < P^2$, it cannot have two such factors and must therefore be prime.

So if all the factors of a number N are required it is useful to know the cube root as well as the square root, and to test N for divisibility by all primes up to $\sqrt{N}$. Consider the case $N = 30031$. Once we have found that 59 is a factor, there is no need to test for the remaining primes up to 173, as $\sqrt[3]{30031} \simeq 31.09$. As $59 > 31$, we know that the co-factor (509) is prime as it must be smaller than $31^2 (= 961)$. However, in the case of 510509, whose cube root is approximately 79.92, we discover that 61 divides 510509. But because $61 < 79$, its cofactor, 8369, could be either prime or composite. $\sqrt{8369} \simeq 91.48$, so it must be tested for divisibility by 61 (again), 67, 71, 73, 79 and 89.

We leave to the reader the task of factorizing 510511.

Fortunately there is an even better method to test if a number is prime or composite. It was discovered by a French judge, Pierre de Fermat (1601-1665), when he was investigating numbers of the form

$$\frac{a^n - 1}{a - 1} \; (= a^{n-1} + a^{n-2} + \ldots + a + 1).$$

At present, numbers of this form have been investigated for large values of n in the cases $a = 10$, 2 and for smaller values in the cases $a = 3$, 5. In the case $a = 10$, the numbers have the form $(10^n - 1)/9$ and are called "repunits", which is short for *repeated units*, as they are made up entirely of ones (units). When $a = 2$, the term 'repunits' is also used (especially if they are written in base 2), but more frequently they are called Mersenne numbers.

If n is composite, then $(a^n - 1)/(a - 1)$ is also composite, for if $n = pq$, then

$$a^{pq} - 1 = (a^p - 1)(a^{pq-p} + a^{pq-2p} + \ldots + a^p + 1).$$

If, on the other hand, n is prime, $(a^n - 1)/(a-1)$ may be prime or composite. Take the case $a = 6$ and tabulate $6^n - 1$ in factorised form.

n	$6^n - 1$	New Prime Factors			
		Base 10		Base 6	
1	5		5		5
2	5 x 7		7		11
3	5 x 43		43		111
4	5 x 7 x 37		37		101
5	5 x 5 x 311		311		1235
6	5 x 7 x 31 x 43		31		51
7	5 x 55987		55987		1111111
8	5 x 7 x 37 x 1297		1297		10001
9	5 x 19 x 43 x 2467	19	2467	31	15231
10	5 x 5 x 7 x 11 x 101 x 311	11	101	15	245
11	5 x 23 x 3154757	23	3154757	35	151341205
12	5 x 7 x 13 x 31 x 37 x 43 x 97	13	97	21	241

Notice that all the primes in the last column end in 1 or 5 (all primes greater than 3 are of the form $6k \pm 1$) and that all primes in the previous column are of the form $6kn + 1$, if n is prime. Hence it appears, and Fermat proved, that if n is prime and $a^n - 1$ is composite, then one of its factors will be of the form $2kn + 1$. So in order to write out the 13th line of the table, we need only test

$$(6^{13} - 1)/5 \ (= 2612138803)$$

for divisibility by primes of the form $26k + 1$, up to its square root, which is approximately 51109.09. This is still quite a big task, and the number may in fact be prime (factor tables go only to 10^7). However there are two further methods for reducing the number of primes to test.

The first has to do with the "primitive root". If $1/p$ is written as a repeating "decimal" in base a and if the number of digits in the period is $p-1$, then a is called a primitive root of p. Fermat showed that no primitive root of p could be a factor of $(a^{p-1}-1)/(a-1)$. The primes having 6 as a primitive root are all of the forms $24k + 7$, $24k + 11$, $24k + 13$ or $24k + 17$. So we can test 2612138803 completely by testing for its divisibility by primes of the form $26k + 1$, which are not also of the form $24k + 7$, $24k + 11$, $24k + 13$ or $24\ k + 17$. This means they must be of the form $24k + 1$, $24k + 5$, $24k + 19$ or $24k + 23$, as all other possibilities give numbers that are never prime.

The first prime we need to test is 53 as

$$53 = 26 \times 2 + 1 = 24 \times 2 + 5.$$

We check the primes successively from 53 on. Most do not need to be tested. E.g. $131 = 24 \times 5 + 11$ and so we need not test it. We do need, however, to test 313 (why?) and this is the next one requiring testing, after 53.

The second method I call "pinching the prime". Suppose we want to find factors of $(6^{13} - 1)/5$. We need only test primes of the form $62k + 1$ which are also of the form $24k + 1$, $24k + 5$, $24k + 19$, $24k + 23$. For example,

$$311 = 62 \times 5 + 1 = 24 \times 12 + 23$$

so we should test it. But notice that 311 already appears in the table as a factor of $(6^5 - 1)/5$ and it cannot also divide $(6^{31} - 1)/5$; this is because it divides exactly into 11111 (base 6) and this is the smallest number made up of ones into which it will divide, so it cannot possibly divide into a number made up of 31 ones (base 6), as 31 is not a multiple of 5. We say that the prime 311 is "pinched" from row 31 of the table to row 5, and need not be tried in row 31.

For any prime of the form $2nk + 1$, where $k > n$, the prime is pinched from row n.

I tested $(6^{13} - 1)/5$ on my scientific calculator, making use of the above simplification, and found that 2612138803 is divisible by 3433. The cofactor is 760891 which is also prime.

I have found that $(6^n - 1)/5$ is prime if n is 2, 3, 7 or 29 and I have complete factorisations for n = 5, 11, 13 and 17 (counting only prime n). I have incomplete results for n = 19, 23, 31, 37, 41, 43, 47, 53, 59 and 61.

Other workers have looked at $(a^n - 1)/(a - 1)$ and its factorisations for different values of a going from a = 2 to a = 13. In fact all the easy work has been done, but much remains which is not known.

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WHAT INDEED?

What science can there be more noble, more excellent, more useful for men, more admirably high and demonstrative, than this of the mathematics.

Benjamin Franklin

HOLY MATHEMATICIANS!

Michael A.B. Deakin,
Monash University

A recent article in *American Mathematical Monthly* (April 1986, p.324) drew the mathematical world's attention to a 15-year old development. In 1971, the mathematician Francesco Faà di Bruno (1825-1888) took a major step towards sainthood.

First, let's look at the process involved in this. Then we'll come back to the man.

When a man or woman of the Catholic Faith dies, it may be that he or she is regarded as having led a particularly pious life. A local cult may then grow up in which the faithful see in this person's life an example worthy of being admired and followed. In the belief that the person is now with God, they may even address prayers to him or her, in the hope of his or her intercession with God, Who, in recognition of the virtuous life led on earth, will be favourably disposed to such requests.

At this stage, such cults have no official status, but in some cases local bishops take up the cause and ask the Vatican to endorse them. The Vatican may then open a file on the individual in question, who acquires the title "Servant of God". If, as a result of these enquiries, they do grant an initial limited endorsement of the cult, the person becomes known as "the Venerable" (Francesco Faà di Bruno, in this case).

There are two further and less limited endorsements. If the next stage, known as "Beatification", is reached, the person is referred to as "Blessed ....", and if the final stage, "Canonisation" is achieved, as "Saint ....". These latter two stages are the culmination of legal processes probing the status of miracles (usually cures lying beyond the power or scope of orthodox medicine) attributed to the venerable person's intercession with God.

Now to the mathematician. Francesco Faà di Bruno (Faà di Bruno was his surname) was born on March 29, 1825. He was a sickly infant, in danger of death at his birth; however, he not only survived but came to thrive and grow to manhood. His family was wealthy and perhaps he owed some of his new-found robustness to their ability to provide adequate care. Indeed, he made his career in the army and would probably have continued to do so but for his being provoked into a duel.

As a firm Catholic, he could not, in conscience, go through with this, and so, according to the code of honour then in force, found it necessary to resign his commission. He travelled to Paris to attend classes in Mathematics and received his doctorate in 1856, so founding a new career for himself. He then returned to Turin, his home town, and took up a professorship of Mathematics, a post he held for the rest of his life.

During the years 1856-1886, he published over 30 technical papers in Mathematics, four treatises on Mathematics or Astronomy, and a two-volume text-book on Physics; he was also responsible for a number of inventions, most notably a typewriter for use by the blind. Mathematicians today remember him best for "Faà di Bruno's formula" - the generalisation of the chain rule, for finding the nth derivative of $f(g(x))$. This formula is very complicated and I do not give it here. Interested readers will find it on p.50 of D. Knuth's *The Art of Computer Programming, Vol*.1.

As well as working consistently at Mathematics, he was active in church affairs and good works, administering funds for the poor, and indeed supplementing these from his own resources. He founded a school for the education of girls and a hospice for the rehabilitation of "fallen women", as the Vatican put it. A church and a religious order were founded by his initiative and his financial help. At the age of 51, later in life than is usual, he was ordained to the priesthood. He lived till 1888, so his centenary is imminent.

It is undoubtedly this aspect of his life, rather than his Mathematics, which has led to the campaign by the diocese of Turin for the canonisation of one of their faithful. He did in some of his writings, draw together the two aspects of his life, in that he sought a higher unity, we are told, between the factual truths known by reason and the moral truths known by faith.

R.P.Boas, drawing the mathematical community's attention to these developments, wonders if any other mathematician has proceeded so far towards sainthood. I don't know either, but I'll offer a few comments.

There are at least two full saints with tenuous claims to being mathematicians. *Albertus Magnus (Albert the Great)* (c.1200-1280) was a pioneer scientist, who wrote two volumes, unfortunately lost, on Mathematics. He seems not to have grasped, however, what later thinkers did, that much of Science is best expressed in mathematical form. *Robert Bellarmine* (1542-1621), a controversial figure, was finally canonised in 1930. He had some mathematical skill, though he can hardly be called a mathematician. Regrettably, he is now most readily remembered for the part (it is a matter of dispute how extensive this was) he played in the persecution of a much greater mathematician, Galileo.

Three members of the Jesuit order, to which Bellarmine also belonged, achieved fame as scientists and mathematicians: *Christoph Clavius* (1537-1612), *Athanasius Kircher* (1602?-1680) and *Rudjer Bosković* (1711-1787). The first named is the best-known, and his most immediate claim to fame is his part in the setting up of the Gregorian calendar, which replaced the previous, and less accurate, Julian one. (See *Function, Vol*.1, *Part* 1, p.19.) Of these three, however, Boscović is probably the greatest mathematician. He worked on the theory of optics, the paths of comets, and so also the theory of conic sections (see *Function, Vol*.10, *Part* 2), meteorology and geophysics. He promoted international cooperation on geodesy (determining the detailed shape of the earth) and worked on the theory of gravity.

Of course, membership of a religious order is no guarantee of heroic sanctity, although it is said to help in this direction. Presumably it also helps in the promotion of the prospective saint's cause with Rome. I don't know how far, if at all, these three have advanced along the path to sainthood.

Similar remarks apply to *Marin Mersenne* (1588-1648), not a Jesuit, but a Franciscan, and a much better-known mathematician. He is the Mersenne after whom Mersenne Primes (i.e. primes of the form 2^n-1) are named. He also contributed to the foundation of probability theory and to the theory of equal temperament. (See *Function*, *Vol*.10, *Part* 4.) It was he who discovered that the period of a pendulum varies as the square root of its length. Several of his biographers make reference to his outstanding piety.

Another possible candidate is *Blaise Pascal* (1623-1662), after whom Pascal's Triangle is named. His mathematical achievements were considerable, involving projective geometry, mechanical computation, theory of fluids, probability theory, and what we now recognise as an early form of the integral calculus. He also produced an extensive body of religious writing and has achieved a reputation as a mystic. The Catholic Church, of which he was at all times a loyal member, has not always looked with favour on his writings, regarding them at times with suspicion, or even banning them. Other Catholic writers, however, see them as possessing great spiritual significance.

Among the things he is remembered for is "Pascal's Wager", nowadays viewed as a misuse of probability theory. If one lives one's life on the basis of a belief in God, the argument goes, one has an infinite expectation if He exists and a finite (possibly negative) one if He doesn't. The total expectation is thus infinite. If, by contrast, one lives one's life on the basis of a belief in there being no God, one's expectation, depending on how one calculates, is either finite, or minus infinity! This sort of moral book-keeping does not today strike all who read it as being commendable.

Yet another possible mathematical saint of the future is *Maria Agnesi* (1718-1799), whose story was told in the last issue of *Function*. It is interesting to contrast her life with that of Francesco Faà di Bruno.

Whereas Agnesi abandoned Mathematics for good works, Faà di Bruno combined the two. Vatican II, the last council of the Catholic Church, is widely seen as having endorsed the view that a life of piety lived in the world of work and day-to-day activity is preferable to such a life lived in withdrawal from that world. So, if one were a gambler, perhaps Faà di Bruno would be the one to back. There might then one day be prayers to Blessed (or even Saint) Francesco Faà di Bruno.

LETTERS TO THE EDITOR

POWER'S POSTULATE AND THOMPSON'S THEOREM

Begin with a definition.

A multiply-perfect number is one for which the sum of its factors, including one and the number itself, is a multiple of the given number.

Power's postulate states that:

This multiple is equal to the number of distinct prime factors of the given number.

This problem was prompted by Question 30 in the senior division of this year's Australian Mathematics Competition. Our bursar, Mr Frank Power, advanced his postulate in the course of discussing this question.

The following BASIC computer program will print multiply-perfect numbers and the sum of all their factors.

```
10 N=2                               start search at 2
20 S=0                               initialize factor sum to zero
25 R=SQR(N)                          limit of first factor
30 FOR D=1 TO R                      `crude' loop for finding factors
40 IF N/D=INT(N/D)THEN S=S+D+N/D     add factor pair to factor sum
50 NEXT D                            increase divisor by one
60 IF R=INT(R) THEN S=S-4            remove twice counted factor in
                                     square number
70 IF S/N=INT(S/N)THEN PRINT N;S     check of multiply-perfect
80 N=N+1                             try next number
90 GOTO 20
```

Using this program, I constructed the following table, going on until I found a counterexample to Power's postulate; the number 32760 is multiply-perfect but the multiple is 4, whereas there are 5 distinct prime numbers. (It is easy to overlook this case as 32760 = 91 x 5 x 3^2 x 2^3 and many may (wrongly) assume 91 to be prime.)

MULTIPLY-PERFECT NUMBER	SUM OF FACTORS	MULTIPLE	PRIME FACTORS	NUMBER OF DISTINCT PRIME FACTORS
6	12	2	3x2	2
28	56	2	7x2^2	2
120	360	3	5x3x2^3	3
496	992	2	31x2^4	2
8128	16256	2	127x2^6	2
30240	120960	4	839x5x3^2x2^3	4
32760	131040	4	13x7x5x3^2x2^3	5

I gave up checking odd numbers for being multiply-perfect as I conjecture (*Thompson's Theorem*) that none are.

Leigh Thompson
Bairnsdale High School.

[The so-called perfect numbers are a special case of Mr. Thompson's multiply-perfect numbers: that for which the multiple is two. Thus there are more multiply-perfect numbers than there are perfect numbers. (E.g. 120 is multiply-perfect but not perfect.) It is not known whether there are odd perfect numbers (if there are, they are greater than 10^{50}), so Thompson's Theorem, which would imply as a corollary that there are not, is open. It would seem to be very difficult to prove or to disprove it. Eds.]

MORE ON PYTHAGORAS

The June, 1986, edition of *Function* presented a pictorial proof of Pythagoras' Theorem using six diagrams. Here is a proof using only two diagrams.

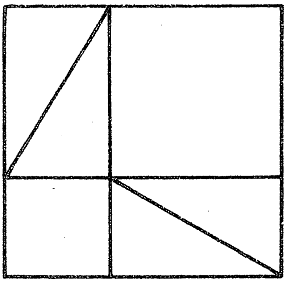


Figure 1

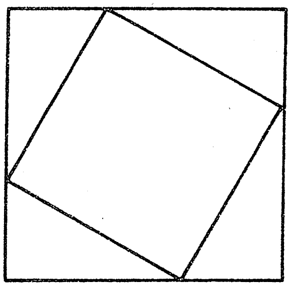


Figure 2

Both diagrams represent the square on the sum of the legs of the right triangle. In Fig.1, this square is decomposed into four copies of the triangle together with the squares on the legs. In Fig.2, this square is decomposed into four copies of the triangle together with the square on the hypotenuse.

J.G.Kupka
Monash University.

1986

In regard to R.D.Coote's letter on p.25 of the June, 1986, issue of *Function*, describing the formation of expressions for numbers 1 - 100 using the digits 1986 in order, I found

$$97 = 1 + ((\sqrt{9})!)! \div 8 + 6.$$

Anthony Roylance,
Geelong West T.S.

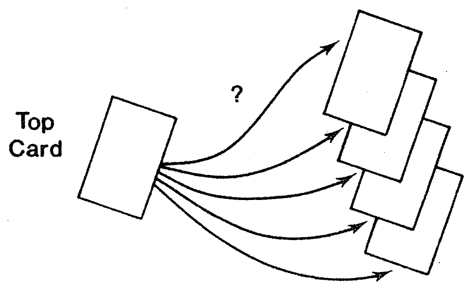
CARD SHUFFLING

G.A. Watterson, Monash University

In the Monash Mathematics tea room, a lot of Bridge is played at lunch times. But this not why I decided to write this article. One morning recently, I went to morning tea only to find that there was nobody there. So I sat down and read the first article in the May issue of the *American Mathematical Monthly*, which happened to be on the table. The authors of the article (David Aldous and Fersi Diaconis) solved the problem of how long you have to shuffle a pack of cards to get it into random order. And their results really surprised me, although I must admit that I didn't understand all that they had written. About 7 (or so) "riffle shuffles" are needed, for randomness for a pack of 52 cards. We will describe a riffle shuffle later. For a pack of n cards, perhaps about $(3/2)\log_2(n)$ riffle shuffles are needed. We shall look at a simpler shuffle first.

ONE-CARD-AT-A-TIME SHUFFLE

Aldous and Diaconis first describe a very simple shuffling method, different from the riffle shuffle. They then use a very simple argument to show that about $n\log_e(n)$ shuffles are needed for randomness of n cards. Define a shuffle to consist of taking the first card off the top of the pack, and then placing it in any one of n positions: above the second card (i.e. back where the first card was, originally), or between the second and third cards, or between the third and fourth card, or .... , or between the second bottom and bottom card, or underneath the bottom card. You continue taking top cards off and slotting them in somewhere. Each time, we assume that it is equally likely just which of the n positions is chosen to slot the card back in.



Now, how do we argue that approximately $n\log_e(n)$ shuffles are needed? Concentrate on the card which was originally on the bottom; suppose it was the Ace of Spades. As shuffles are carried out, gradually more and more cards will be slotted in underneath the Ace of Spades, and these cards will be in perfectly random order amongst themselves, because they were slotted in at random. By the time the Ace of Spades comes up to the top position, *all* the other cards will be in a random order. And then it is the Ace of Spades' turn to be taken from the top, and slotted in somewhere, at random. At exactly that time, all the cards are in random order.

We let T_i be the number of shuffles required to increase the number of cards underneath the Ace of Spades from $i - 1$ to i. In particular, T_1 is the number of shuffles required to get one card slotted in beneath the Ace of Spaces, while $T_{(n-1)}$, is the number of shuffles required to bring the Ace of Spades from being second, to being at the top, of the pack. Consider T_i; while there are $i - 1$ cards underneath the Ace of Spades, there are i positions below the Ace of Spaces, and $n - i$ positions above it, where the top card may be slotted in. At any shuffle, the probability is $p = i/n$ that the top card if slotted in somewhere *below* the Ace of Spades, and the period T_i ends. The average number of shuffles required for this to happen when it has probability i/n of happening on any one shuffle, is n/i. That is,

$$\text{mean of } T_i = E(T_i) = n/i. \qquad (1).$$

This is easy to understand. For instance, if you toss a six-sided die until a six turns up, it will take six tosses, on average, because the probability of it happening in any one toss is 1/6.

The total time for the Ace of Spades to rise from the bottom to the top, and then to be slotted in at random, is

$$S = T_i + T_2 + T_3 + \ldots + T_{(n-1)} + 1, \text{ shuffles} \qquad (2)$$

so that the expected (mean, or average) time for all the cards to be shuffled into random order is

$$E(S) = n/1 + n/2 + n/3 + \ldots + n/(n-1) + 1$$

$$= n[1/1 + 1/2 + 1/3 + \ldots + 1/(n-1) + 1/n] \text{ shuffles.} \quad (3)$$

But it is known that, for fairly large n,

$$1/1 + 1/2 + 1/3 + \ldots + 1/n = \log_e(n) \text{ approx.}$$

(Actually, you might like to check that an even better approximation is $\log_e(n) + 0.57721$. Get your computer to calculate these quantities for various values of n.) So, a good approximation to $E(S)$ is

$$E(S) = n[\log_e(n) + 0.57721]. \qquad (4)$$

For instance, we can use (3) to get exact answers and (4) to get approximate answers, as illustrated in Table 1.

Table 1

n	2	6	10	30	52	100
$E(S)$ in (3)	3	14.7	29.3	119.8	236.0	518.7
$E(S)$ in (4)	2.5	14.2	28.8	119.4	235.5	518.2

Remember that here, a single "shuffle" consists of moving only one card, so that each card is moved rather few times (for instance, once!) in order to achieve a random order.

The above discussion highlights only the *expected* number of shuffles needed. But sometimes S will be smaller than expected and sometimes it will be larger. S is a random variable, and its variance is given by

$$\sigma^2 = \mathrm{Var}(S) = \mathrm{Var}(T_1) + \mathrm{Var}(T_2) + \ldots + \mathrm{Var}(T_{(n-1)}),$$

$$= n(n-1)/1^2 + n(n-2)/2^2 + \ldots + n.1/(n-1)^2,$$

$$= n[(n-1)/1^2 + (n-2)/2^2 + (n-3)/3^2 + \ldots + 1/(n-1)^2]. \quad (5)$$

The standard deviation of S is, of course, the square root of its variance:

$$\sigma = s.d.(S) = [\mathrm{Var}(s)]^{1/2}.$$

To see why (5) is correct, you might like to check out that T_i has a *geometric* probability distribution:

$$Pr(T_i = t) = q^{(t-1)}p, \text{ for } t = 1, 2, 3, 4, \ldots,$$

where

$$p = i/n \quad \text{and} \quad q = 1 - i/n$$

and this geometric distribution has mean and variance given by

$$E(T_i) = 1/p \quad \text{and} \quad \mathrm{Var}(T_i) = q/p^2 .$$

As usual, it is very likely that a random variable, such as S, will take values within two standard deviations of its mean value. So we are able to put bounds on how many shuffles will probably be enough to achieve randomness, namely

$$E(S) - 2\sigma < S < E(S) + 2\sigma \text{ , very probably.} \quad (6)$$

The bounds in (6) are illustrated in Table 2.

Table 2

n	2	6	10	30	52	100
$E(S) - 2\sigma$	0.2	2.2	6.9	46.9	107.0	267.1
$E(S) + 2\sigma$	5.8	27.2	51.7	192.8	365.0	770.4

Notice that the bounds are somewhat far apart. Take the upper bound, to be safe!

RIFFLE SHUFFLES

Let us now get back to riffle shuffles. A riffle shuffle is when you divide the pack of cards into roughly equal halves, and then interleave (or "riffle") the two halves together. I won't discuss the theory that Aldous and Diaconis used to draw their conclusions about riffle shuffling, as it is too complicated. But one comment in their article interested me very much. Before Diaconis became a mathematical statistician, he was a professional magician and gambler. He says that such professionals can do a riffle shuffle which is not random at all. They can divide the pack into two exactly equal halves (26 cards each). Suppose that originally the cards were numbered 1, 2, 3, ... , 52 from the *bottom*. The perfect division would have cards 1, 2, ... , 26 in one pack and cards 27, 28, ... , 52 in the other. Then, in riffling, card 1 would be dropped first, then card 27, then card 2, then card 28, and so on. The perfect riffle shuffle would produce a pack, from the bottom, in the order:

1, 27, 2, 28, 3, 29, 4, 30, ..., 52, 51, 26, 52.

Suppose that we continue riffle shuffling the pack, in the same perfect way, for several shuffles. It surprised me to learn that, in only *eight* shuffles, the pack returns back to exactly its initial order 1,2,3, ..., 51,52 (from the bottom). I wondered if that happened for packs with *any* even number of cards. So I wrote the following computer program to find out, for each even number n (up to 100), how many perfect riffle shuffles would be needed to return the pack of n cards back to their original position. The program is for running on an APPLE II computer; your computer might need a slightly different program.

```
 10 REM RIFFLE SHUFFLE PROBLEM
 20 NMAX = 100
 30 REM N DENOTES THE SIZE OF PACK< NMAX THE LARGEST N
 40 DIM X (NMAX), Y(NMAX)
 50 REM ********************
 60 REM *WE START THE N LOOP*
 70 REM *********************
 80 REM X(I) DENOTES THE NUMBER OF THE I-TH CARD FROM THE BOTTOM
 85 REM AND SO DOES Y(I), TEMPORARILY DURING SHUFFLING
 90 REM WE START WITH X9I0 = 1
100 REM:
110 FOR I = 1 TO NMAX
120 X(I) = I
130 NEXT I
140 COUNT = 1 : N2=N/2 :REM THESE ARE USED OFTEN LATER
150 REM *************************
160 REM *NOW WE SHUFFLE THE PACK*
170 REM *************************
180 FOR 1 = 1 TO N STEP 2
185 J = (I+1)/2
190 Y(I) = X(J) : REM BOTTOM HALF CARD DROPS
200 Y(I+1) = X(N2+J) : REM TOP HALF CARD DROPS
310 FOR I = 1 TO N
320 X(I) = Y(I) : REM THIS NUMBERS THE NEW PACK ORDER
330 NEXT I
340 REM ******************************************
350 REM *NOW WE CHECK FOR RETURN TO ORIGINAL ORDER*
360 REM ******************************************
370 FLAG = 1 : REM THIS INDICATES ORIGINAL ORDER
380 FOR I = 1 TO N
390 IF X(I)     I THEN FLAT = 0:REM NOT ORIGINAL ORDER
400 NEXT
410 IF FLAG = 0 THEN COUNT = COUNT + 1 : GOTO 160
420 REM COUNT COUNTS THE NUMBER OF SHUFFLES
430 PRINT "SIZE OF PACK = "N" NUMBER OF SHUFFLES = "COUNT
440 NEXT N
450 END
```

I found the results given in Table 3.

Table 3

n	shuffles	n	shuffles	n	shuffles	n	shuffles	n	shuffle
2	1	22	6	42	20	62	60	82	54
4	2	24	11	44	14	64	6	84	82
6	4	26	20	46	12	66	12	86	8
8	3	28	18	48	23	68	66	88	28
10	6	30	28	50	21	70	22	90	11
12	10	32	5	52	8	72	35	92	12
14	12	34	10	54	52	74	9	94	10
16	4	36	12	56	20	76	20	96	36
18	8	38	36	58	18	78	30	98	48
20	18	40	12	60	58	80	39	100	30

There are some very interesting results in Table 3. For instance there are never more shuffles needed than $n-2$ (except when $n = 2$). You might have expected that for some sizes n, the pack would *never* return to its original order. But $n - 2$ shuffles *are* quite often needed; on other occasions, very many fewer shuffles are needed. For instance in the important case with $n = 52$ cards, only 8 shuffles are needed, as Diaconis said, to return the pack to its original order.

A couple of problems suggest themselves. Can you work out a formula which tells you how many perfect shuffles are needed to restore a pack of n cards? And what happens when there is an *odd* number of cards?

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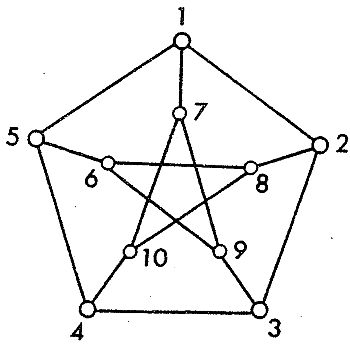
PROBLEM SECTION

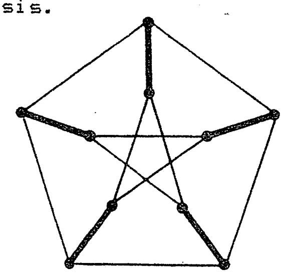
We have quite a large number of outstanding problems and so here we "clear the decks".

SOLUTION TO PROBLEM 10.2.1

The problem was to describe the possible m-circuits of the Petersen graph connecting 10 points by 15 lines as shown. An m-circuit is a path through the graph that takes one of the lines from marked point to marked point, followed by another to a new marked point and so visits on points in turn each exactly once and returns to the initial point. In particular we asked for a proof that no 10-circuit can exist.



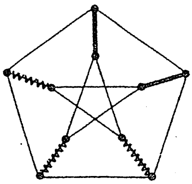
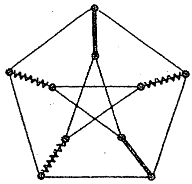
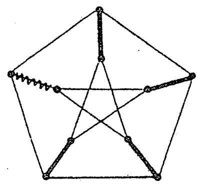
Mark Short, a student at Monash, supplied this analysis.



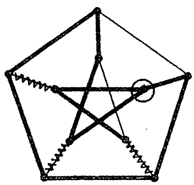
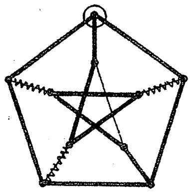
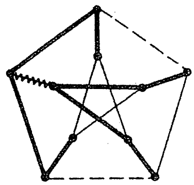
Any 10-circuit must use some of the thick edges - otherwise we would have two 5-circuits instead.

In fact, it must use an even number of them : for each edge on the circuit going inwards there must be another going outwards, since the circuit is a closed path.

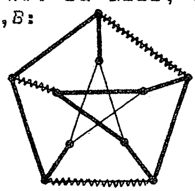
This means either 2 or 4 of the thick edges are used and symmetry reduces these possibilities to 3 cases:

1.  2.  3.  *

Now, if one edge at a vertex is *not* used in the 10-cycle, the other two must be used. This implies that the following edges (also drawn thick) are necessarily present in the 10-circuit in the 3 cases:

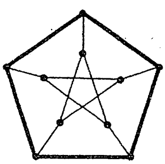
1.  2.  3. 

Cases 1 and 2 now violate the circuit condition at the circled vertices. As for Case 3, the dashed edges can *not* be used, so we're forced to use *AB* to get a circuit through *A*,*B*:



But now we have closed a 5-circuit, so we cannot extend it to a 10-circuit.

Examples of 5-, 6-, 8-, 9-circuits: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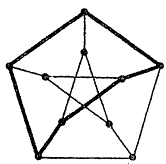
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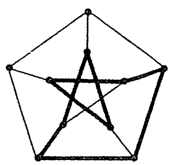
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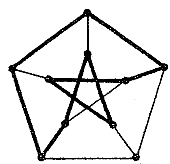
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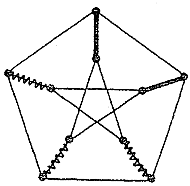
I think it is fair to claim there are no 2-, 3- and 4-circuits "by inspection", though this may be stretching it a bit for the 4-circuits.

* denotes prohibited edge

As for the nonexistence of 7-circuits, one can argue again that any 7-circuit must use an even number of the edges connecting the outer pentagon to the inner star, and the number has to be 2, because if there were 4 their ends on either the outside or the inside could not be joined up by the 3 remaining edges in the 7-circuit.

So we are reduced to the cases:

4.



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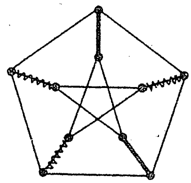
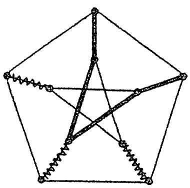
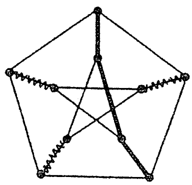


The inner ends of the thick edges can only be connected like so.

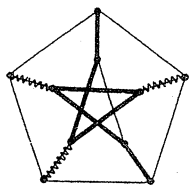


in which case it is impossible to connect the outer ends by the 3 remaining edges

The inner ends of the thick edges can be connected in two ways:



in this case the outer ends cannot be connected by the 4 remaining edges



in this case the outer ends cannot be connected by the 1 remaining edge.

SOLUTION TO PROBLEM 10.3.1

Take a, b, c to be any three digits, where $a > c$. From these form the three digit number

$$100a + 10b + c$$

and its reverse

$$100c + 10b + a.$$

Subtract to find $99(a - c)$.

Now
$$\begin{aligned} 99(a - c) &= 100(a - c) - (a - c) \\ &= 100(a - c - 1) + 100 - a + c \\ &= 100(a - c - 1) + 90 + (10 - a + c). \end{aligned}$$

The reverse of this is

$$100(10 - a + c) + 90 + (a - c - 1).$$

Add these two together to get

$$\begin{aligned} &101(a - c - 1) + 180 + 101(10 - a + c) \\ = &909 + 180 = 1089. \end{aligned}$$

The problem was to show that 1089 is always the result of this operation. This solution is due to S. Bigelow of Eltham, who also provided the next.

SOLUTION TO PROBLEM 10.3.2

We asked for the value of

$$\sqrt{(1 + \sqrt{(1 + \sqrt{(1 + \ldots}}} = x \text{ (say)}.$$

Square both sides and find

$$1 + x = x^2.$$

This gives

$$x = (1 \pm \sqrt{5})/2,$$

and, as the answer must be positive, we choose the + sign.

SOLUTION TO PROBLEM 10.3.3

To prove that the product of n consecutive numbers is divisible by $n!$, let $(k + 1)$ be the first. The product is then

$$(k + 1)(k + 2)(k + 3) \ldots (k + n),$$

which is $(k + n)!/k!$.

Form

$$\frac{(k + n)!}{k!} \cdot \frac{1}{n!}$$

$$= \frac{(k + n)!}{k!\,n!}$$

$$= \frac{(k + n)!}{k!(k + n - k)!}$$

$$= \binom{k + n}{k},$$

a binomial coefficient, equal to the number of ways in which k objects may be chosen from a set of $k + n$ objects, and clearly integral.

This solution is by Tim Arnold, Year 12, Scotch College.

SOLUTION TO PROBLEM 10.4.1

If x, y, z may be any integers and $\Box$ is an operation which satisfies

$$x\Box(y + z) = y\Box x + z\Box x \ ,$$

show that $u\Box v = v\Box u$ for all integers u,v.

David Shaw of Geelong West Technical School and John Barton of North Carlton both sent solutions. This is David Shaw's.

1. Put $x = y = z = 0$.

   $0 \Box 0 = 2(0 \Box 0)$.

   So $0 \Box 0 = 0$.

2. Put $x = y = 0$.

   $0 \Box z = 0 \Box 0 + z \Box 0$.

   So $0 \Box z = z \Box 0$.

3. Put $y = z = 0$.

   $x \Box 0 = 2(0 \Box x)$.

   So $0 \Box x = 2(0 \Box x)$.

   and $0 \Box x = 0$.

4. Put $y = 0$.

   $$x \Box z = 0 \Box x + z \Box x$$

   $$= z \Box x \ ,$$

and the result follows as x,z may be any integers.
John Barton's solution was similar.

SOLUTION TO PROBLEM 10.4.2.

Sprague's series u_1, u_2, u_3, ... satisfies $u_1 + u_2 + ... = u_n = s_n$, where $s_n = 1$ and $s_{n+1} = s_n + {}^1/s_n$. We asked for a simple approximate formula for s_n.

David Shaw writes:

$$s_n^2 = (s_{n-1} + \frac{1}{s_{n-1}})^2$$

$$= s_{n-1}^2 + 2 + s_{n-1}^{-2}$$

$$\simeq s_{n-1}^2 + 2$$

$$= (s_{n-2} + \frac{1}{s_{n-2}})^2 + 2$$

$$\simeq s_{n-2}^2 + 4$$

$$\simeq s_{n-3}^2 + 6\ , \text{etc.}$$

In general,

$$s_n^2 = s_{n-r}^2 + 2r \qquad (n \geq r + 2).$$

Now put $n = r + 2$ $(r \geq 0)$.

$$s_{r+2}^2 = s_2^2 + 2r$$

$$\simeq 4 + 2r$$

$$= 2(r + 2).$$

I.e.

$$s_n^2 = 2n$$

and

$$s_n \simeq \sqrt{2n}.$$

John Barton solved the problem differently. He wrote the condition

$$s_{n+1} = s_n + {}^1/s_n$$

as

$$\Delta s_n = 1/s_n$$

and approximated this *difference* equation by a *differential* equation which he then solved to find

$$s_n = \sqrt{2n + a}\ ,$$

where a is an arbitrary constant.

He notes that

$$\sqrt{2(n+1) + a} \simeq \sqrt{2n + a} + 1/(\sqrt{2n+a}).$$

the error, if n is large, being approximately $1/(4\sqrt{2}n\sqrt{n})$. If we choose $a = 0$, we get an exact value for $n = 2$, other choices of a give better approximations for other n.

We conclude with some new problems.

PROBLEM 10.5.1

It is easy enough with a calculator to show that e^{π} ($\simeq$ 23.14) $>$ π^{e} ($\simeq$ 22.46), but can you show this without computation.

PROBLEM 10.5.2

If $x \geq 0$, $y \geq 0$ and

$$\begin{aligned} -2x + y &\leq 50 \\ 3x + 2y &\leq 300 \\ x + y &\leq 50 \\ \text{and} \qquad x &\leq 90 , \end{aligned}$$

what is the maximum value of $x + y$?

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[Continued from p.9.]

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"The originality of mathematics consists in the fact that in mathematical science connections between things are exhibited which, apart from the agency of human reason, are extremely unobvious. Thus the ideas, now in the minds of contemporary mathematicians, lie very remote from any notions which can be immediately derived by perception through the senses; unless indeed it be perception stimulated and guided by antecedent mathematical knowledge."

A.N.Whitehead

PERDIX

Here are the questions set in the 1986 International Mathematical Olympiad in Warsaw. The first three questions were set on the first day, July 9, the next three on the second day, July 10. For each paper a time of $4^1/2$ hours was allowed. Each question was worth 7 marks.

Send me your solutions.

1. Let d be any positive integer not equal to 2, 5 or 13. Show that one can find distinct a,b in the set $\{2,5,13,d\}$ such that $ab - 1$ is not a perfect square.

2. A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s = A_{s-3}$ for all $s \geq 4$. We construct a sequence of points P_1, P_2, P_3, ... such that P_{k+1} is the image of P_k under rotation with centre A_{k+1} through angle 120° clockwise (for $k = 0,1,2,...$). Prove that if $P_{1986} = P_0$ then the triangle $A_1A_2A_3$ is equilateral.

3. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all the five numbers is positive. If three consecutive vertices are assigned the numbers x,y,z, respectively and $y < 0$ then the following operation is allowed: the numbers x,y,z are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

\- - - - -

4. Let A,B be adjacent vertices of a regular n-gon ($n \geq 5$) in the plane having centre at O. A triangle XYZ, which is congruent to and initially coincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X.

5. Find all functions f, defined on the non-negative real numbers and taking non-negative real values, such that:

   (i) $f[x\,f(y)]\,f(y) = f(x + y)$ for all $x,y \geq 0$,
   (ii) $f(2) = 0$,
   (iii) $f(x) \neq 0$ for $0 \leq x < 2$.

6. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to colour some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white points and red points on L is not greater than 1? Justify your answer.

INDEX, VOLUMES 8, 9 AND 10