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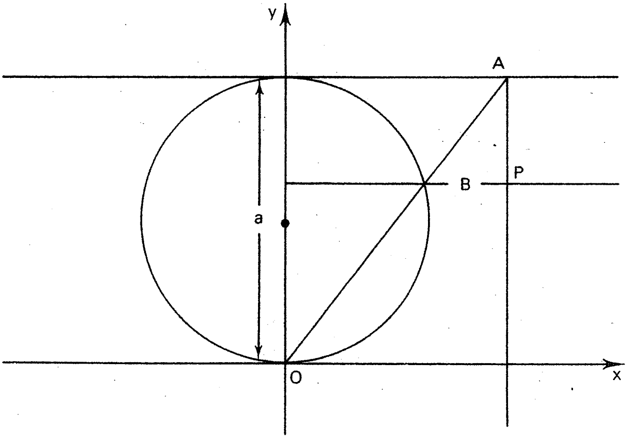

A SCHOOL MATHEMATICS MAGAZINE

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Function is a mathematics magazine addressed principally to students in the upper forms of schools, and published by Monash University.

It is a "special interest" journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. *Function* is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

Function does *not* aim directly to help its readers pass examinations. It does guarantee to enrich and inform their mathematical lives, to provide high quality articles and stimulating reading for the mathematically inclined.

It is hoped that the readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you. Address them to:

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(continued on back cover)

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THE FRONT COVER

Take a circle of diameter a with centre at $(0,a/2)$, resting, if we might put it so, on the origin. The line $y = a$ is tangent to this circle and is parallel to the x-axis, which is also a tangent. Through 0, draw the line $y = mx$, and suppose this intersects the circle in the point B and the line $t = a$ in the point A.

Through B draw a horizontal line and through A a vertical line. Call their point of intersection P. The position of P depends upon the slope (m) of the line OA. As m varies, OA may be thought of as rotating about the pivot at the origin. As it does so, P traces out a curve (shown opposite), called the *Witch of Agnesi*. Its equation is

$$y = a^3/(x^2 + a^2).$$

It is not difficult to verify this fact and readers are invited to do so as an exercise.

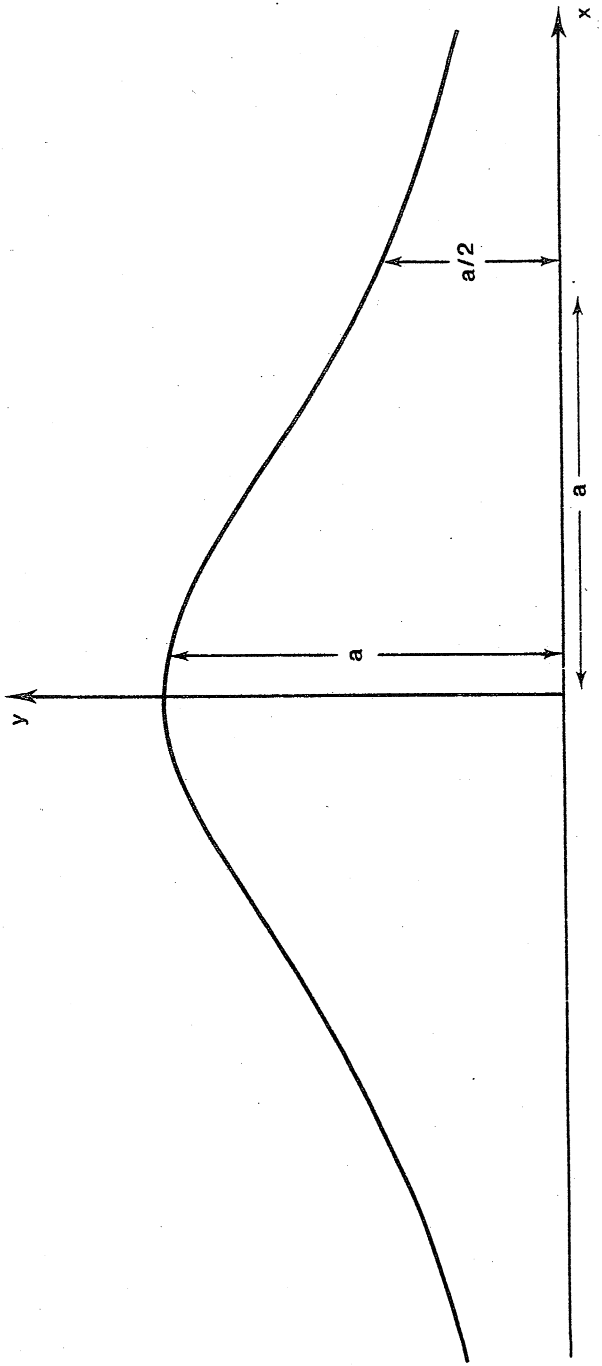
What is less clear is why this, relatively simple, curve should attract so bizarre a name. The story is interesting, and, because it is often mistold, we take pains to recount it correctly here.

Maria Gaetana Agnesi (1718-1799) was, as her biographer Edna Kramer states, "the first woman in the Western world who can accurately be called a Mathematician." She was the eldest child of Pietro Agnesi, a wealthy man and professor of mathematics at the University of Bologna. The University of Bologna is the oldest European university, and held that prestigious status then, as now.

Pietro Agnesi married three times and fathered a total of 21 children, so perhaps it was well that he was wealthy.

Maria showed not only talent, but genius, from a very early age. By the time she was eleven, she spoke, besides her native Italian, French, Latin, Greek, German, Spanish and Hebrew. Before her 21st birthday, she had published two books. Corespondence exists to show that at the age of seventeen she was already a very accomplished mathematician.

Her father, justifiably proud of her achievements, took every opportunity to display these to an adulatory public - a situation the young Maria found increasingly distasteful. In 1738, she thought to escape this by attempting to enter a convent. Her father dissuaded her from doing this, but some accommodation seems to have been reached, for she retired from social life and worked in some seclusion on mathematics.


x
y
a
a
a/2

This bore fruit in 1748 with the publication of her major work *Istituzioni analitiche ad úso della gioventu italiana* (*Lessons in calculus for young Italians*). The title sounds perhaps all too trite to modern ears, but one must remember the date of its appearance.

Newton, one of the founders of the calculus, was still alive when Maria was born, so her work presented an account of what was then difficult and indeed controversial front-line research. Perhaps the most remarkable tribute paid to her text was its translation into English (1801) by Professor John Colson of Cambridge "that the British Youth might have the benefit of it as well as the Youth of Italy."

But there were other, more immediate, tributes. In 1749, Pope Benedict XIV sent her a gold medal and an ornate jewelled wreath in recognition of her achievements and the next year, and perhaps more to the point, offered her a professorship of mathematics and physics at the University of Bologna.

She thus became the first woman ever to be made a professor of mathematics. She held the post for two years (from 1750-1752), but without, it would seem, either teaching or drawing any pay. When her father died, in 1752, she began to withdraw from mathematics and devoted herself more and more to religion, social work and the care of her numerous younger brothers and sisters.

By 1762, she was so far removed from mathematical work that she declined to examine a major paper by the young *Giuseppe Lagrangia* (now known as *J.L.Lagrange*, a mathematical superstar).

That then is the woman. What of her curve - and why is it called *the Witch*?

Well, one irony is that this remarkable mathematician is remembered best for a solitary example from her major book, and a none too important example at that. Another irony is that *Fermat*, a French mathematician, had discussed this curve before her and so, if justice were to be done, should have his name affixed to the curve. However, justice is rarely done in such matters. If this were not enough, Guido Grandi† had also discussed the curve and given it the name *versiera*.

Go back to the line *OA* on the cover diagram. Let m vary - this corresponds to the line *OA* turning on its pivot at *O*. The Latin word *vertere* means "to turn", and from this Grandi, and later Maria Agnesi, derived the word *versiera* - "the result of the turning".

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† See *Function*, *Vol.8*, *Part 2*.

The mainstream of the way the Italian language evolved, however, was different. *Versiera* is not a common word in contemporary Italian. You need to go to quite a large dictionary to find it. But find it you will.

And it means "witch" - one, feminine (it ends in -a), who turns the right order of things upside down.

So a concatenation of linguistic and historical happenstances preserves the name of a remarkable mathematician in an inadequate context.

To the detriment, indeed, of her memory. One strident article, at the ill-informed end of the feminist spectrum, even claimed that Maria Agnesi had been denounced as a witch. With friends like that do feminists need enemies?

But truth will always outdo such slick fictions. And truth leaves us much to ponder on.

Mathematicians and feminists alike will mourn the loss to the world of intellect of a major talent. Why she withdrew, we may never know. What effect that withdrawal had on the subsequent history of mathematics, we can, at best, speculate.

We do know what she did when she abandoned mathematics. She led a life of piety, devotion and service. In other words, those roles society then and now assigns to women. Why did she choose these - and how free was her choice?

Hard to know, and, even if we did, hard to enter a moral judgement. Her decision may well have been wrong, but it was, by all accounts, hers. And we know one thing about her with glowing certainty - she wasn't stupid.

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INTERNATIONAL MATHEMATICAL OLYMPIAD RESULTS

Australia placed 15th overall.

Bronze medals awarded to:

David Hogan
Ross Jones
Catherine Playoust
Ben Robinson
Terence Tao.

Terence Tao is the youngest competitor ever to have taken part in the international olympiad.

Catherine Playoust is the first girl to obtain a medal for Australia.

∞ ∞ ∞ ∞ ∞

A MATHEMATICAL MODEL OF AN ARMS RACE

Peter Kloeden, Murdoch University

During the four years (1910-1914) preceding the outbreak of the First World War, there was a massive growth in the armaments held by the future protagonists, which became known as an *arms race*. When hostilities finally broke out many people believed that they were an unavoidable consequence of the arms race. So too thought Lewis Fry Richardson (1881-1953) a young English Quaker who had turned his back on a brilliant career as a meteorological scientist[†] because of the way in which meteorology was being used for aerial and gas warfare. Instead he served as volunteer ambulance driver with the French Army. In the lulls between battles he thought a lot about the causes of war, and indeed devoted the rest of his life to its study. He particularly believed that mathematics could be applied here with the same success as for the physical sciences. Thus he constructed a simplistic mathematical model of an arms race, which he hoped would elucidate the mechanisms and consequences in a context free from emotional and political prejudice.

Richardson's model consisted of a pair of *differential equations* describing the evolution in time of the armament levels $x(t)$ and $y(t)$ of two countries X and Y. These equations differ from familiar algebraic equations in that they involve the rates of change of the armament levels as well as the actual armament levels themselves. The theory of differential equations is based on differential calculus, yet a detailed knowledge of calculus is not essential for an understanding of what differential equations are or for a rudimentary analysis of how their solutions behave.

Consider a known function $x = x(t)$ of time t with a nice smooth graph as illustrated in Figure 1. The slope $m(t_0)$ of

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[†] See my earlier article "L.F.Richardson's Weather Forecast Factory" which appeared in *Function Vol.10, Part 3* (1986).

the tangent to the graph at the point $(t_o, x(t_o))$ represents the instantaneous rate of change of the function $x = x(t)$ at $t = t_o$. Generally this rate will vary as t_o varies, so the t_o is included in its value $m(t_o)$. This tangent is in fact the limit of the straight line segments, or chords, joining the point $(t_o, x(t_o))$ to the nearby points $(t_o + \Delta t, x(t_o + \Delta t))$ as Δt is made arbitrarily close to zero, i.e. as Δt converges to 0. The slope of such a chord is

$$\frac{x(t_o + \Delta t) - x(t_o)}{(t_o + \Delta t) - t_o} = \frac{\Delta x(t_o)}{\Delta t} ,$$

where $\Delta x(t_o)$ is written for $x(t_o + \Delta t) - x(t_o)$, and this converges to the slope $m(t_o)$ of the tangent to the graph at $(t_o, x(t_o))$ as Δt converges to zero, i.e.

$$m(t_o) = \lim_{\Delta t \to 0} \frac{\Delta x(t_o)}{\Delta t} \qquad (1)$$

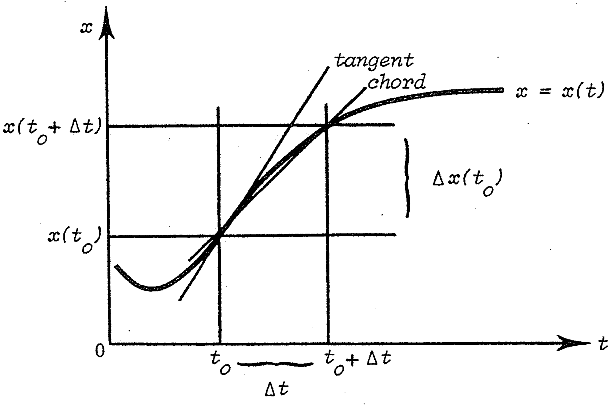

Figure 1: Chords and Tangent at $(t_o, x(t_o))$.

It is now conventional to write the limiting value in (1) as

$$\frac{dx}{dt}(t_o) ,$$

or simply as $\frac{dx}{dt}$ where the dependence on t_o is not explicitly stated. This is called the *derivative* of $x = x(t)$ with respect to t at $t = t_o$. It is the slope of the tangent to the graph of $x(t)$ at the point $x = x(t_o)$ and represents the *instantaneous rate of change* of $x(t)$ at $t = t_o$. Clearly it is negative if $x(t)$ is decreasing, zero if $x(t)$ is instantaneously not changing and positive if $x(t)$ is increasing at $t = t_o$. See Figure 2.

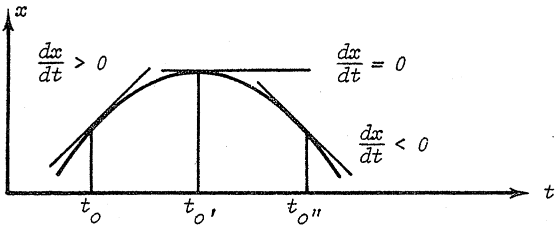

Figure 2: The sign of $\frac{dx}{dt}$.

A simple example of a differential equation is

$$\frac{dx}{dt} = ax \qquad (2)$$

where a is a constant. Here the rate of change $\frac{dx}{dt}$ is proportional to the value of x with the same constant of proportionality a for each instant t . A *solution* of the differential equation (2) is a function $x = x(t)$ which satisfies (2) for each instant t. In this case all of the solutions have the form

$$x(t) = x(0)e^{at}$$

where $x(0)$ is the *initial value* that the solution takes at the initial instant $t=0$. There are thus infinitely many different solutions, each one corresponding to a different initial value. Note also that $x(t) = 0$ for all $t \geq 0$ when $x(0) = 0$. This is called an *equilibrium* or *steady state* solution, as it has zero rate of change $\frac{dx}{dt} = 0$ for all $t \geq 0$. (The differential equation (2) models exponential population growth when $a > 0$ and exponential or radioactive decay when $a < 0$).

Just how one goes about finding solutions for differential equations is beyond the scope of this article. In fact solutions are known only for fairly simple types of equations and in general only approximate solutions can be found by approximating the differential equation

$$\frac{dx}{dt} = f(x)$$

by an algebraic equation

$$x(t + \Delta t) = x(t) + f(x(t))\ \Delta t\ . \qquad (3)$$

Here the derivative $\frac{dx}{dt}$ has been replaced by the quotient $(x(t+\Delta t) - x(t))/\Delta t$, which will be fairly accurate provided Δt is sufficiently small. The algebraic equation (3) is then solved for discrete instants of time Δt, $2\Delta t$, $3\Delta t$, ... starting from a given initial value $x(0)$. These calculations can be easily carried out on a computer. In fact Richardson used a very similar method to find approximate solutions for the extremely complicated differential equations used to model the dynamics of the atmosphere.

Now for Richardson's model of an arms race. Richardson supposed that two countries X and Y wanted peace, but were apprehensive of the other's intentions and were prepared to fight if attacked. He let $x(t)$ represent the armaments level of country X and time t and $y(t)$ that of country Y. He assumed that neither country had an incentive to have weapons if the other country had none and that the rate of acquiring arms for one country would be directly proportional to the armament level of the other country. These two assumptions yield a coupled pair of differential equations

$$\frac{dx}{dt} = ay \quad \text{and} \quad \frac{dy}{dt} = bx \quad , \qquad (4)$$

where a and b are two positive constants of proportionality, the specific values of which need not concern us just now.

The model described by the differential equations (4) is far too simplistic because it disregards any limiting factors to growth, such as the financial burden of large levels of armaments, which one would expect to depress the rate of increase. Richardson thus subtracted a positive multiple mx of x from the $\frac{dy}{dt}$ equation in (4) to account for such limiting factors. This lead to the pair of differential equations

$$\frac{dx}{dt} = ay - mx \qquad (5X)$$

$$\frac{dy}{dt} = bx - ny\ . \qquad (5Y)$$

He also considered effects that were not due to mutual stimulation (i.e. the x or y values) but to permanent underlying attitudes or grievances. For this he added a constant g to (5X) and a constant h to (5Y) to obtain

$$\frac{dx}{dt} = ay - mx + g \qquad (6X)$$

$$\frac{dy}{dt} = bx - ny + h \qquad (6Y)$$

where $g,h > 0$ correspond to grievances and $g,h < 0$ to feelings of goodwill between the countries X and Y.

The pair of differential equations (6X) and (6Y) is *Richardson's model of an arms race*. He analysied them using specific coefficients a,b,m,n,g and h relevant to the arms race which preceeded the First World War. His method of analysis does not require finding solutions for (6X) and (6Y) (which is possible) but merely examining where the rates of change are negative, zero or positive. In fact from equation (6X) $\frac{dx}{dt}$ is zero for

$$0 = ay - mx + g \qquad (\text{line } L_1)$$

and from (6Y) $\frac{dy}{dt}$ is zero for

$$0 = ax - ny + h \qquad (\text{line } L_2) ,$$

which represent the equations of straight lines. Moreover on one side of L_1, $\frac{dx}{dt}$ will be always negative and on the other side always positive. A similar situation holds for $\frac{dy}{dt}$ and the line L_2 (but just which side what holds depends on the particular values of the coefficients a,b,m,n,g and h). In addition the point $(\bar{x}, \bar{y})$ of intersection of the two lines, assuming they do intersect, is a steady state or equilibrium solution $x(t) \equiv \bar{x}$, $y(t) \equiv \bar{y}$ for all $t \geq 0$ of the pair of differential equations.

Let us consider solutions starting at some point $(x(0),y(0))$ away from the equilibrium solution, which Richardson called the *balance of power*. Just what happens to the solution $(x(t),y(t))$ as $t \to \infty$ depends on the particular values of the coefficients. We will consider two cases with the lines and derivative signs oriented as in Figures 3 and 4, respectively. Remembering that a negative derivative means the function is decreasing, whereas a positive derivative means it is increasing, we can roughly sketch what will happen to $(x(t),y(t))$ as t increases. (Strictly speaking we have to take into account the magnitudes of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as well as their signs). In Figure 3 the two solutions which are representative of all other solutions, tend towards the balance of power equilibrium, which is thus called a stable equilibrium. In contrast in Figure 4 for sufficiently small initial values the solutions get even smaller, that is the countries appear to be disarming. However, if the initial values are too large, the goodwill between the countries and the limits to growth factors will not be sufficient to reduce or hold in check the arms levels and there will be a runaway arms race with armament levels becoming arbitrarily large.

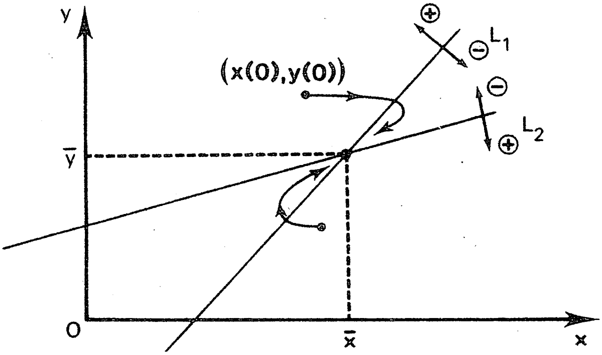

Figure 3: Stable balance of power.

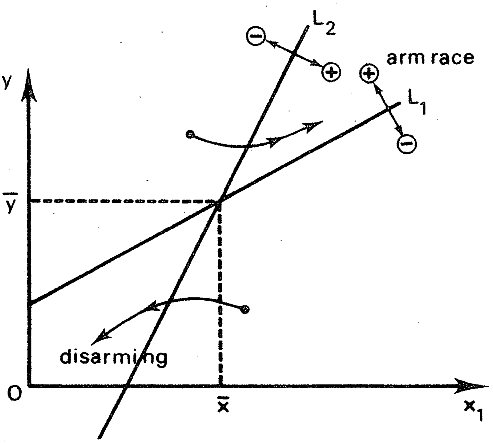

Figure 4: Unstable balance of power with initial values determining if disarmament or an arms race occurs.

Without going into details here, it can be shown that there are four typical cases:

(I) If $mn > ab$, $g > 0$, $h > 0$ there is a stable balance of power (Figure 3);

(II) If $mn > ab$, $g < 0$, $h < 0$ there is total disarmament;

(III) If $mn < ab$, $g > 0$, $h > 0$ there is a runaway arms race;

(IV) If $mn < ab$, $g < 0$, $h < 0$ the situation is ambiguous as in Figure 4 with a runaway arms race or disarmament depending on the initial levels of armaments.

Other cases can occur too with g and h taking opposite signs or being zero. These are left to the reader to analyse. Note that when $mn = ab$ the straight lines L_1 and L_2 are parallel and do not intersect, unless they coincide everywhere.

Richardson was well aware that his model was contrived and artificial, yet it does bear some resemblance to what can be abstracted from the dynamics of actual arms races. The results seem to coincide with what common sense tells us what should happen, so have we really gained anything by having such a model? Yes we have, because the model shows that certain mechanisms and relationships lead to certain results, independently of any particular moral, emotional or political point of view. We must remember that these factors often distort what we may think is common sense. In Richardson's words "The equations are merely a description of what people would do if they did not stop to think". The equations certainly give us something to think about!

A readable and fairly elementary book on the use of mathematics to model conflict and conflict resolution is Anatol Rapaport's "Fights, Games and Rebates" (Univesity of Michigan Press, 1974). The reader could also consult the article by Bruce Taplin and myself on "The Prisoner's Dilemma Game", which appeared in *Function Vol.9, Part* 1 (1985), p.14.

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Everything should be made as simple as possible, but no simpler.

A.Einstein

When a mathematician has no more ideas, he pursues axiomatics.

Felix Klein

Fantasy, energy, self-confidence and self-criticism are the characteristic endowments of the mathematician.

Sophus Lie

The essence of mathematics lies in its freedom.

Georg Cantor

CHEATING, STEALING, PIANO TUNING

Hans Lausch, Monash University

Tunings of the musical scale in which most or all concords are made slightly impure in order that few or none will be left distastefully so, are called musical temperaments. When the medium of performance allows little or no flexibility of intonation – compare the suppleness of the human voice with the inflexibility of the keyboard instruments – music theorists are obliged to contrive specific mathematical schemes.

Equal temperament, in which the octave is divided into 12 uniform semitones, is, with a few exceptions, the standard Western temperament today. As early as 1588, the year of the Armada, the abbot of San Martino in Sicily, Girolamo Roselli, was said to have reached these forward looking conclusions:

"This way of dividing the diapason or octave into 12 equal parts ... could alleviate all the difficulties of singers, players and composers by enabling them generally ... to sing or play ... DO-RE-MI-FA-SOL-LA upon whichever of the 12 notes they wish, touring through all the notes, making a circular music; hence all the instruments will be able to keep their tuning and be in unison, and organs will be neither too high nor too low in pitch."

About 50 years later, we are told, an old man in rags, who had spent most of his life in Sicily and Calabria and knew nothing except how to play the harpsichord, retired to Rome and triggered excitement by advocating equal temperament on the harpsichord and even inducing the influential composer Frescobaldi, with the aid of frequent and gratuitous beverages, to recommend it for the organ in Bernini's new apse at San Lorenzo in Damaso. The mathematician Father Marin Mersenne recommended the use of equal temperament about the same time.

In the late 17th century and early 18th a circle of German theorists became very interested in equal temperament, including Werckmeister, Neidhardt and Mattheson. In England the organ builder Renatus Harris, wishing to discredit a competitor, brought the mathematician John Wallis to write in the Philosophical Transactions of 1698 a letter to Samuel Pepys Esquire, relating to some supposed imperfections in an organ.

Wallis asserted that equal temperament had been found necessary on organs. In his *Generation harmonique* (1737) the French composer Rameau endorsed equal temperament.

Whether Johann Sebastian Bach, who used the term "well-tempered clavier" in the title of his first book (1722) of 24 preludes and fugues to signify some kind of tuning suitable for all 24 keys, was an advocate of equal temperament, is debated by musicologists. His son, C.P.E. Bach, however, is the best candidate if the music of any leading 18th-century composer ought to be performed in equal temperament.

It was in 1761 when the Berlin music theorist F.W.Marpurg published the article "Attempt to find a perfectly equal temperament by construction". Marpurg, in his introduction to this treatise, makes the following comments:

"Mr Kirnberger, one of our best local musicians, ... who wished to see an equal temperament on the monochord which would please both the ear and the eye, came to read what Neidhardt ... wrote of the geometric construction in view of the temperament. He took the opportunity to talk about it with an acute Berlin mathematician, whose name to mention I have no permission, and to ask him: Whether one could not investigate in more detail, and perhaps more satisfactorily than through arithmetical approximation, what Neidhardt had touched only superficially. Mr Kirnberger's learned friend undertook the investigation and after a brief effort took pleasure in solving the riddle and filling the wide gap left by Mr Neidhardt. Here is his essay on this subject which gives so much honour to his excellent insights by not only pleasing every authority on musical temperament but certainly even the mathematicians."

We shall turn to this mystery writer and his essay in a moment, but first a few words about Mr Kirnberger. Johan Philipp Kirnberger (1721-1783) was a well-known music theoretist, composer, and music teacher who was tutor of Princess Anna Amalia of Prussia.

He belonged to the Berlin group of theorists, which included Quantz, C.P.E.Bach and Marpurg, and is commonly described by his contemporaries as emotional and ill-tempered, inflexible, conservative, tactless and pedantic, but his detractors acknowledged his devotion to students and friends and his dedication to the highest musical standards. In 1764 he edited a second edition of the essay on equal temperament in which he omitted Marpurg's introduction, and in the preface to one of his collections 'Piano Exercises' of 1766 he progressed to purporting to have written the essay himself.

In 1776, at a time when his relations with Kirnberger had soured, Marpurg published the most articulate treatise of the late 18th century on the subject, 'Attempt on the musical temperament'. And here we learn the identity of our writer: as Marpurg put it, it was "the famous Mr Mendelssohn".

Just to keep your mind in the right century, let me assure you that in spite of the musical context, Marpurg does not refer to Felix Mendelssohn Bartholdy (1809-1847), but to his grandfather Moses Mendelssohn who was born in 1729. This year the world remembers the 200th anniversary of his death.

To give you only a sketchy story of his life, would lead far beyond the limitations of a magazine article. The best biolgraphical account is the one by Alexander Altmann, *Moses Mendelssohn - A Biographical Study*, London 1973.

Moses Mendelssohn founded a great dynasty of artists, bankers and scientists. Of his mathematical descendants, the most famous representative is the number theorist Kurt Hensel (1861-1941), and should you play the mathematical strategy game NIM or a related game, then think of Roland Sprague (1894-1967), one of the pioneers of the modern theory of NIM-like games.

Professor Walter Hayman, who was instrumental in founding the British Mathematical Olympiad and is a well-known expert in the theory of complex functions, also descends directly from Moses Mendelssohn. Three mathematicians, whose results have been in the tool kits of succeeding generations, married women of the house of Mendelssohn: P.G.Lejeune-Dirichlet, who accepted Gauss' position in Göttingen, married Rebecka Mendelssohn Bartholdy, the composer's younger sister, E.E.Kummer married Ottilie Mendelssohn, another of Moses' granddaughters, and Hermann Schwarz became the Kummer's son-in-law.

Mendelssohn's mind had been occupied with probability theory ever since his first paper in German 'On Chance Happenings' (1753). In 1779, he also contributed to a reader for the best pupils of a recently founded school, and one of his last great works called 'Morning Hours', which refers to the part of the day during which he gave lessons to a number of young people, contains a section on probability.

In Berlin he became member of the "Learned Coffeehouse", a closed society of about hundred people, Members of the Royal Academy and other intellectual leaders of the Kingdom of Prussia. There he met the mathematician Johann Albrecht Euler, whose prolific father Leonhard was then in Berlin, and Mr Kirnberger, who gave Moses piano lessons so that, in the end, he managed to play a minuet.

It was found about this time when Mendelssohn wrote the logical commentary 'Bi'ur milot hahigayon' on the famous work "Terms of Logic" by the philosopher Maimonides (1135-12-4), to whom also a *Function* article was devoted (*The Rambam*, *Function*, *Vol*.9, *Part* 5, October 1985). Also in this case, Mendelssohn fell victim to an apparently not quite honest publisher, who presented himself as the author of the treatise: in those days, pirates were identified as far upstream as Frankfurt on the Oder.

Mendelssohn's treatise on equal temperament begins with translating the musical requirements for equal temperament into mathematical language. Given two strings of equal thickness and equal tension, but the one being only half as long as the other, the shorter one will produce a pitch which is exactly an octave higher than that of the longer.

The task is to cut out another 11 strings of lengths appropriate to produce all the semitones within the octave. mathematically this amounts to the following problem: if the longer string has length 2 and the shorter has length 1, and if $r = \sqrt[12]{2}$, then one has to cut out strings of length r, r^2, r^3, ..., r^{11}, respectively.

Note that if two line segments of lengths p and q are given, then it is easy to construct a line segment of length h such that $p:h = h:q$. There are numerous ways of getting h: e.g. draw a line segment AH of length p, extend it beyond H as far as B such that HB has length q; let M be the midpoint of AB and draw a semicircle c with centre M having AB as its diameter; draw a line l through H perpendicular to AB and let C be an intersection of l with c; then HC has length h, which is an assertion I ask you to prove.

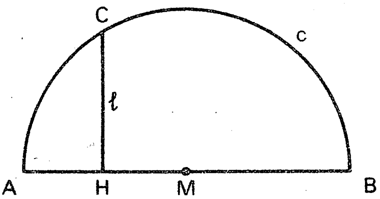

Figure 1

Since $1:r^6 = r^6:2$, r^6 can be easily constructed. Likewise, since $1:r^3 = r^3 : r^6$ and $r^6 : r^9 = r^9:2$, we have also simple constructions for the string lengths r^3 and r^9.

Since $1:r = r:r^2 = r^2:r^3$, we will be able to construct the strings of length r and r^2 (and all the other remaining ones) as soon as we can solve the following construction problem: given two line segments of lengths p and q, find line segments of lengths h and k such that $p:h = h:k = k:q$. Note that especially when p=1 and q=2, then $h = \sqrt[3]{2}$.

Mendelssohn remarks:

"It thus depends simply on the well-known Delian problem which, in antiquity, made so much stir. Plato, Hero of Alexandria, Philo, Apollonius, Diocles, Pappus, Sporus and Erathostenes, at various times provided solutions. ... These great people found only mechanical solutions; it looks as though the construction might be impossible without the help of curves."

What did Mendelssohn mean by these remarks? In antiquity, one school of thought in geometrical constructions was to limit oneself to only two tools, namely compass and ruler, and use these within the constraints of the following rules: suppose you have already obtained a number of points and let us call them the "old points", then, new points can be constructed only as points of intersections of two straight lines, of a straight line and a circle, or of two circles; each of these straight lines must pass through at least two old points, and each of these circles must have an old point as its centre and pass through at least one old point.

The Delian problem consisted of constructing a line segment of length $\sqrt[3]{2}$ from one of unit length, or, as it is often put, to construct a cube of volume 2 from one of volume 1. The ancient Greeks found it impossible to perform this construction when constrained by the rules laid down above and resorted to all kinds of "mechanical" solutions: the drawing of various curves (e.g. spirals) or cheating by surreptitiously carving marks into the ruler which turned out to be of some help.

Mendelssohn's suspicion, that such a construction might be impossible without infringing the rules, was prophetic, indeed: it was only in the 19th century that it could be proved by means of agebraic methods that no such construction was possible. No less a mathematician than Isaac Newton, in his 'Arithmetica universalis' was one of those transgressors.

Mendelssohn explains:

"Newton .... divides the line segment *AB*, the first of the two given line segments into two equal parts at *E* [Fig.2]. He then draws a circle with centre *A* through *E* and fits in the second given line segment *EC* such that the point *C* is on the circle. Next he extends the line segments *EC* and *BC*. While keeping the ruler placed at *A*, he moves it between the two lines just drawn until *GF* becomes as long as *AE* or *EB* and draws the line *FGA*. After this, he says, *CF* and *AG* will be the desired line segments ... Constructio nota est, adds Newton. I may be permitted to prove what Newton assumes as known. Great geniuses reach their aim in one step where common minds must be led by a long sequence of conclusions. The theorem was ...

$$AB : CF = CF : GA = GA : CE."$$

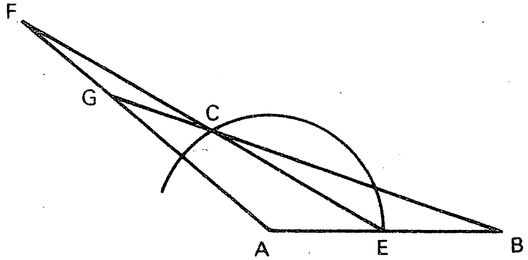

Figure 2

Before letting Mendelssohn proceed to the proof, we need a result from geometry. Please send your proofs of it to the Editors.

"Let c be a circle, P a point outside the circle, T a point on c, PT a tangent of c, and R and S two distinct points on c such that the extension of RS contains P. Then

$$PR \times PS = PT^2."$$

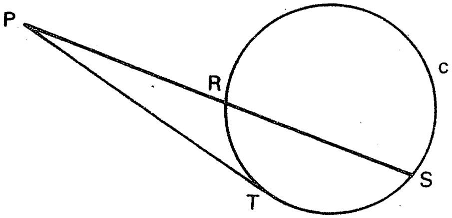

Figure 3

Mendelssohn now continues:

"Proof . ... Extend FA as far as H (Fig.4) and draw the line segment AK parallel to EC. Since AK is parallel to EC, we have

$$BA : BE = AK : EC.$$

Now, $BE = \frac{1}{2} AB$, hence also $EC = \frac{1}{2} AK$. Further since the triangles FGC and KGA have the same angles [i.e. are similar] (as FC is, by construction, parallel to KA), we see that

$$CF : FG = KA : GA.$$

Consequently, $CF : 2\ FG = \frac{1}{2} KA : GA$. But $2\ FG = AB$ (by assumption), $KA = 2\ CE$ (as demonstrated), hence $CF : AB = CE : GA$ or, this turned around,

$$AB : CF = GA : CE.$$

Likewise,

$$(AB + GA) : (CF + CE) = AB : CF = GA : CE.$$

Now,

$$AB + GA = FH$$

because

$$AH + FG = AB,$$

and also

$$CF + CE = FE;$$

therefore

$$FH : FE = AB : CF = GA : CE.$$

Further [by the result from geometry as stated above],

$$FH : FE = FC : FL.$$

Since

$$FL = AG$$

(as $AL = AG$, by assumption), we see that

$$FH : FE = FC : AG.$$

Consequently, from the equation (already shown)

$$FH : FE = AB : CF = GA : CE,$$

we obtain

$$CF : AG = AB : CF = GA : CE.,$$

and finally,

$$AB : CF = CF : AG = AG : CE,$$

which was the theorem to be proved.

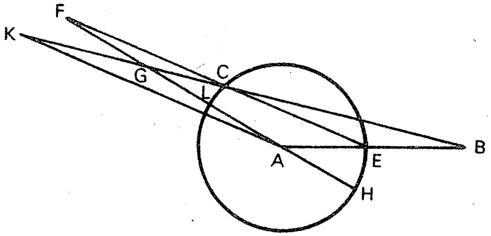

Figure 4

... The mechanical artist can accept this on trust if he does not wish to concern himself with mathematical reasoning. But he has to apply all possible care to execute what has been prescribed to him."

Mendelssohn did understand not only the concerns of the "mechanical artist" but also the day-to-day problems of many a mathematician: One day, three of his friends, all of them mathematicians, had a game of cards in which one can score twenty-one. They could not reach agreement on their individual points, and so asked Mendelssohn, who was standing nearby, to act as their umpire. Mendelssohn helped out, not without exclaiming: "Lo and behold, here are three mathematicians and they cannot count twenty-one!"

****

"Pure Mathematics is the mathematician's real wand."

"One may be a mathematician of the first rank without being able to compute. It is possible to be a great computer without having the slightest idea of mathematics.

[*Novalis* was the pen-name of the German epigrammist von Hardenburg, who died in 1802. Eds.]

INFINITE LADDERS

M.A.B. Deakin, Monash University

A ladder is an expression of the form a^{b^c} or $a^{b^{c^d}}$, etc., where a number a is raised to a power, which is itself a number raised to a power, etc. It is a little easier to write (and to think about) ladders if we use a different notation. In BASIC, exponentiation (the raising of numbers to powers) is indicated by a vertical arrow , and in this notation our examples become

$$a\uparrow(b\uparrow c) \quad \text{and} \quad a\uparrow(b\uparrow(c\uparrow d)) \ .$$

Note that the brackets are important here:

$$2\uparrow(3\uparrow4) = 2\uparrow81 = 2.41785 \ldots \times 10^{24} \ ,$$

a very big number indeed, compared with the more modest

$$(2\uparrow3)\uparrow4 = 2\uparrow12 = 4096 \ .$$

One question that began to interest me was what sense I could give to an infinite ladder made up of identical numbers:

$$a\uparrow(a\uparrow(a\uparrow(a\uparrow(\ \ldots \ . \qquad (1)$$

Could any meaning be assigned to this?

My first move was to restrict the investigation to the case $a > 0$ and to look only at those cases where real (as opposed to complex) arithmetic was involved. These restrictions are not entirely necessary, but the whole flavour of the investigation alters if they are removed.

It next seemed that two approaches to the infinite ladder (1) were possible. Both proceed from the assumption that a new expression like (1) has no intrinsic meaning at all. The problem is to *define* a meaning for it.

A similar case arises in secondary school mathematics: a^{-1} cannot be defined in the way that a^2, a^3, a^4, etc. are. However, where all the exponents are positive, we have $a^p.a^q = a^{p+q}$ and if we define a^{-1} as $1/a$, this property, along with others, remains. Similarly, we define $a^{1/2}$ as $\sqrt{a}$ to preserve properties such as $(a^{1/2})^2 = a$, and the like.

So in trying to define the expression (1), we look for approaches which provide satisfying and natural ways to regard this expression.

One obvious way is to consider the sequence

$$a,\ a\uparrow a,\ a\uparrow(a\uparrow a),\ a\uparrow(a\uparrow(a\uparrow a)),\ \ldots.$$

and see what happened as the number of a's tended to infinity. I'll come back to this, but I postponed it in my exploration, because I saw what seemed like a more promising approach. It went like this.

Put

$$x = a\uparrow(a\uparrow(a\uparrow(a\uparrow(\ \ldots \quad . \qquad (2)$$

Then

$$a\uparrow x = a\uparrow(a\uparrow(a\uparrow(a\uparrow \ \ldots = x\ , \qquad (3)$$

or in more usual notation

$$a^{x} = x. \qquad (4).$$

Graphically, we solve this by finding the intersection of the graphs of $y = a^{x}$ and $y = x$.

Now the behaviour of the graph of a^{x} depends on the value of a. If $0 < a < 1$, a^{x} decreases as x increases and we get (see Figure 1) a single intersection with $y = x$. If $a = 1$, the graph is a straight line which intersects the line $y = x$ when $x = y = 1$. So we have our first result

$$1\uparrow(1\uparrow(1\uparrow(1\uparrow(\ \ldots \ = \ 1 \qquad (5)$$

as $1^{x} = x$ has a single solution : $x = 1$.

When $a > 1$, matters are more complicated. For some values of a, there will be two intersections, but beyond a certain critical value, the curve $y = a^{x}$ rises too steeply to reach the graph of $y = x$ and there are no intersections. In between, at the critical value of a, the two graphs would just graze tangentially, for a unique solution.

See Figure 1 again. The top graph has $a = 2$ and clearly 2 exceeds the critical value. Thus we do not expect any meaning to be available for the expression

$$2\uparrow(2\uparrow(2\uparrow(2\uparrow(\ \ldots.. \quad .$$

On the other hand, when $a = \sqrt{2}$ (the second case shown), we get two intersections, at $x = 2$, $x = 4$, since

$$\sqrt{2}^{2} = 2 \qquad \sqrt{2}^{4} = 4 \qquad \text{[redacted]}, \qquad (6)$$

So

$$\sqrt{2}\uparrow(\sqrt{2}\uparrow(\sqrt{2}\uparrow(\sqrt{2}\uparrow(\;\ldots\ldots \quad = 2 \text{ or } 4. \qquad (7)$$

This last example is also instructive in another way. For we can write $\sqrt{2} = 2^{1/2}$ and the first of Equations (6) now becomes

$$(2^{1/2})^{2} = 2 \;,$$

an equation which easily generalises to

$$(x^{1/x})^{x} = x \quad , \qquad (8)$$

so that we have a form equivalent to Equation (4):

$$a = x^{1/x} \; . \qquad (9)$$

Figure 2 shows the graph of $y = x^{1/x}$. (For an account of the related function $y = x^{x}$, which also considers negative x and the negative values introduced by (e.g.) $(\frac{1}{2})^{1/2} = \pm\sqrt{\frac{1}{2}}$, see *Function, Vol. 6 Part 2.*) The expression $x^{1/x}$ is known to tend to zero as x gets very small and to one as x gets very big. It is also known to rise steadily to a maximum height and to fall slowly once this is attained.

It is also known at what value of x that maximum is attained. The value in question is e . e is the base of the natural logarithms, also known as Euler's constant, after the very great mathematician who discovered many of its properties. Perhaps the simplest way to define e is to say that it is that value of a for which the graph of $y = a^{x}$ passes through the point (0,1) with a slope of 1. There are, however, many other ways of looking at e, and those of you studying Year 12 mathematics might have met it in other ways.

It is known that the graph of $y = x^{1/x}$ achieves its maximum when $x = e$, and that maximum value will of course be $e^{1/e}$. So if a lies between 1 and $e^{1/e}$, Equation (9) will define two values for x and these correspond to the two values that may be assigned to Expression (1).

To view it another way, we could include more members of the set of functions $\{(x,y)\colon\ y = a^{x}\}$ in Figure 1. One of these would just graze the line with equation $y = x$ and for this one we

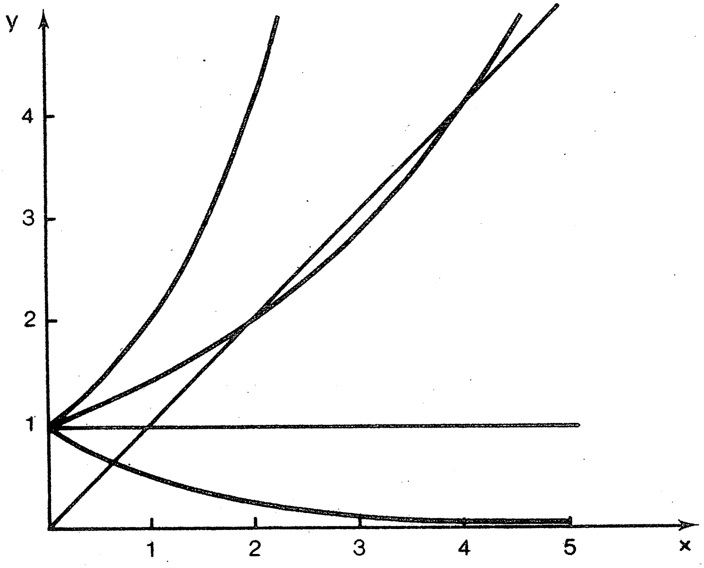

Figure 1: Graphs of $y = 2^x$, $\sqrt{2}^x$, 1^x, $(1/2)^x$ and $y = x$

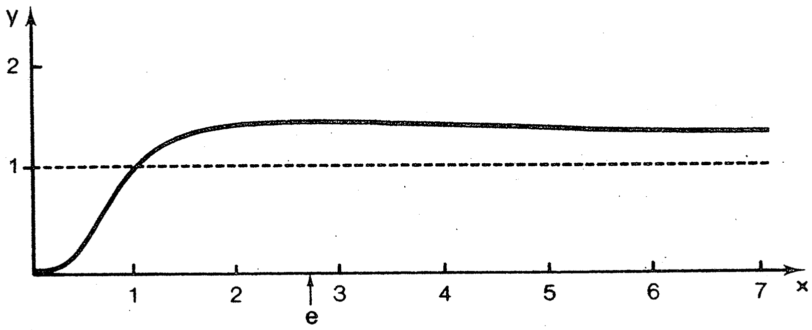

Figure 2: Graph of $y = x^{1/x}$

would have a = take place just at the point $x = y = e$.

Now let us come back to our other approach. Put

$$f(1,a) = a\ ,\ f(2,a) = a\uparrow a,\ f(3,a) = a\uparrow(a\uparrow a)\ ,$$

and so on. We then have the relation

$$f(n,a) = a\uparrow f(n-1,a)\ . \qquad (10)$$

This provides a convenient way to compute $f(n,a)$.

For example, consider $a = \sqrt{2}$. I put $\sqrt{2}$ into the store of my HP-35 and also entered it into the register. I next pressed RCL x^y to find 1.632526919, a good approximation to $\sqrt{2}\uparrow\sqrt{2}$. To apply Equation (10) again press ENTER RCL x^y to get 1.760839555, a good approximation to $f(3, \sqrt{2})$. Continuing in this way, I generated successive values of $f(n,\sqrt{2})$, till I called it quits at $f(50,\sqrt{2}) = 1.999999989$, convergence to 2 being quite evident.

Indeed it was possible to prove that the limit was exactly 2. To do this, we need to prove first that if $f(n,a) < 2$, then

$$f(n,a) < f(n + 1,\ a) < 2.$$

So $f(n,a)$ increases as n increases but remains always less than 2. This implies that $f(n,a)$ tends to a limit ℓ ($\ell \leq 2$) as n gets larger and larger. It is fairly easy then to show that ℓ cannot, in this case, be less than 2.

So it seemed that the "limit" way of looking at the question gave only one of the two possible values of Equation (7). What, I wondered, had happened to the other? Where, in the case $a = \sqrt{2}$, was the solution 4? Well, it took a while to find it and it turned up in what you might think of as an unusual way.

Equation (10) follows a line which I left unnumbered, but which included the statement

$$f(1,a) = a\ , \qquad (11)$$

a natural enough thing to require. But, as I reflected on it, it seemed that it wasn't forced on us.

Why not try

$$f(1,a) = b \text{ (say)?} \qquad (12)$$

Then $f(2,a) = a\uparrow b$, $f(3,a) = a\uparrow(a\uparrow b)$, and so on; the recalcitrant b, as n gets larger, disappears off to infinity and, when Equation (1) finally gets re-established, does it matter if, way out on the right, where no-one will ever see it, is a b and not an a? Intuition would say 'no'.

Well, intuition is partly right and partly wrong. And the place where intuition fails gives the answer to the question as to where the missing value went.

Figure 3 (on p. 28) shows graphs of $y = f(n,\sqrt{2})$ for $b = f(1,\sqrt{2}) = 1$, $\sqrt{2}$, 2, 3, 3.9, 4, 4.1. (The graphs are the sets of dots joined together by lines not themselves part of the graphs.) The first five of these graphs converge to the value $y = 2$ when n is large, the convergence being most obvious in the special case $b = 2$.

For $b < 2$, $f(n,\sqrt{2}) > f(n-1, \sqrt{2})$. This is easily proved from Equation (10) and so the graph tends upwards, ultimately toward the value 2.

For $2 < b < 4$, $f(n,\sqrt{2}) < f(n-1, \sqrt{2})$, which is also readily proved, and so the graph tends downwards, ultimately toward the value 2.

But if $b > 4$, $f(n, \sqrt{2}) > f(n-1, \sqrt{2})$ and the graph continues to rise, more and more steeply.

The special value $b = 4$ allows $f(n,\sqrt{2})$ to merely duplicate $f(n-1,\sqrt{2})$, but any value of b differing from 4, no matter how slightly, will mean that $f(n,\sqrt{2})$ tends to 2 or increases forever.

We speak of 2 as being a stable value and 4 an unstable one.

So we see that the two approaches to the ladder (1) give consistent answers, but with some differences of emphasis.

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"The real mathematician is an enthusiast *per se*. Without enthusiasm no mathematics."

Novalis

"Insofar as the theorems of mathematics relate to relaity, they are not certain, and, insofar as they are certain, they do not relate to reality."

Albert Einstein

PRODUCTS OF CONSECUTIVE INTEGERS

John Mack, University of Sydney

We begin with a simple example. Can the product of two consecutive positive integers be the square of a positive integer? That is, are there positive integers m and n such that

$$n(n+1) = m^2?$$

It is easy to construct a proof that this cannot happen. Two consecutive positive integers always have greatest common factor 1, so the given equality implies that both n and $n+1$ are squares of integers. But for any integer $k \geq 1$,

$$(k + 1)^2 - k^2 = 2k + 1 \geq 3$$

and hence consecutive integers cannot be squares.

The same argument shows that

$$n(n + 1) = m^\ell$$

(where $\ell = 2, 3, \ldots$) is also impossible.

What about the product of three consecutive positive integers? Is there a solution in positive integers to

$$n(n + 1)(n + 2) = m^2$$

or more generally, to

$$n(n+1)(n+2) = m^\ell \quad , \ell = 2, 3, \ldots ?$$

One can construct a direct proof that this is also not possible. A recent Australian Mathematical Olympiad test problem asked if the product of five consecutive positive integers is ever a square. This was, understandably, found rather difficult by those who tried it.

In fact, it was conjectured long ago that the product of consecutive integers is never a power, that is, the equation

$$n(n+1)\ldots(n+k) = m^\ell$$

has no solutions in positive integers $n \geq 1$, $k \geq 1$ and $\ell \geq 2$.

How might one try to attack such a problem?

Clearly, as we increase the number of terms in the product, we increase the likelihood of common factors — in every four consecutive terms, for example, two are even and one is at least a multiple of 4, while at least one is a multiple of 3 — so the chances of grouping terms into relatively prime blocks and arguing in that way don't seem to be very good. Thus we might expect that the larger the value of k, the more difficult the problem becomes for that value of k. Fortunately, a simple idea provides a different attack when k is large compared to n.

Suppose we could guarantee that one of the integers n, $n+1$, ..., $n+k$ was a prime p. Since the multiples of p are p apart and since $p \geq n$, it follows that there can be no other multiple of p among the integers n, ..., $n+k$ if $n+k < 2n$, that is, if $k < n$. In this case, the product $n(n+1) \ldots (n+k)$ contains p to the first power only and cannot be an ℓth power. So, what do we know about the occurrence of primes in a set of consecutive integers?

The simplest and best known result is "Bertrand's Postulate" (proved in the nineteenth century) that, for any integer n, there is a prime p satisfying $n < p < 2n$. This prime p could happen to be the integer $2n-1$, so to apply this result to our problem we would need to have the entire product $n(n+1) \ldots (2n-1)$ on the left-hand side. Thus we obtain the result:

$$n(n+1)\ldots(2n-1) = m^{\ell}$$

has no solution in integers $n \geq 1$, $m \geq 1$, $\ell \geq 2$.

Can we not obtain any more from Bertrand's Postulate? We can, by remembering that if p is a prime, then there are no multiples of p less than p. So if we work with $n+k$ instead of n, we see that there is a prime p satisfying

$$n+k > p > (n+k)/2.$$

[If $n+k$ is even, this is obvious. If $n+k$ is odd, apply Bertrand's postulate to the set $\{r, \ldots, 2r\}$, where r is the integer just below $(n+k)/2$.]

From this, we see that if $(n+k)/2 \geq n$, then there is exactly one power of p in the product $n(n+1)\ldots(n+k)$ and so this product cannot be an ℓth power. Thus the problem has no solution if $k > n$. Combining the two results we have so far, we see there is no solution if $k \geq n - 1$.

Thus, the "problem zone" is identified as that for relatively small values of k. By use of a different result on prime factors, which states that if $n > k+1$, then among the numbers n, $n+1$, ..., $n+k$, at least one is divisible by a prime p greater than $k+1$, we can deduce that

$$n(n+1)\ldots(n+k) = m^{\ell}$$

has no solutions if $n \leq (k+1)^{\ell}$.

No easy method of proof is known for the case $n > (k+1)^{\ell}$. When $\ell = 2$, a proof was given by Paul Erdos in 1939 and he and John Selfridge gave a proof for $\ell \geq 2$ in 1975. This latter proof depends on showing the existence of a prime factor of the product which occurs to a power which is not a multiple of ℓ.

Paul Erdös visits Australia regularly and will celebrate his 75th birthday next year. Maybe someone will present him with a simpler proof of the result that the product of consecutive integers is never a power.

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Continued from p.25.

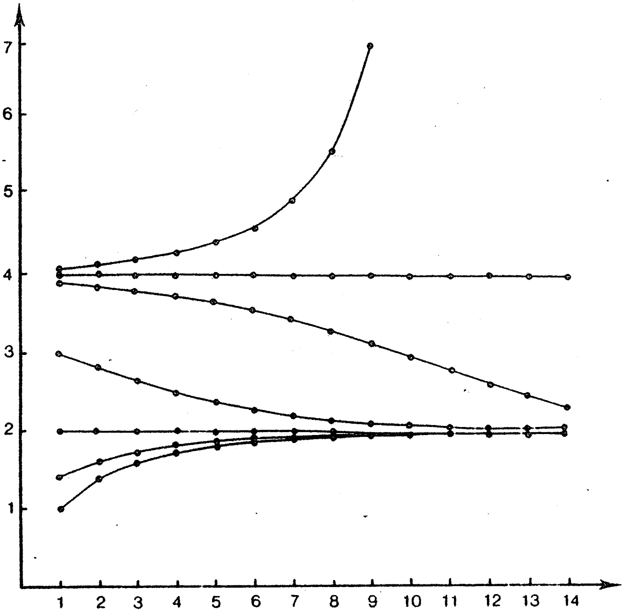

Figure 3. $f(n,\sqrt{2})$, for $f(1,\sqrt{2}) = 1, \sqrt{2}, 2, 3, 3.9, 4, 4.1$.

PROBLEM SECTION

MORE ON PROBLEM 10.1.1

This problem, submitted by D.R. Kaprekar, read as follows.

A man had 115 dollars. He spent 40 of them and 75 were left. He went out again and spent 46, leaving 29. A third time he went out and spent 19 leaving 10. Finally he went out and spent the 10, leaving nothing. Here is a table.

	Spent	Left
	40	75
	46	29
	19	10
	10	0
Totals	115	114

The total at right is 114, not 115. Where is the missing dollar?

In our last issue, we printed a comment by Garnet J. Greenbury and after going to press we received two more letters on the subject.

David Dyte (year 12, Scotch College) writes:

Let us suppose the man had x dollars which he spent in four lots, a,b,c and d dollars each ($a+b+c+d = x$). This readily gives us the "Spent" table:

a	40
b	46
c	19
d	10
x	115

But the "Left" table should be interpreted thus:

$$\begin{array}{lcrr} x-a & = & b+c+d & 75 \\ (x-a)-b & = & c+d & 29 \\ (x-a-b)-c & = & d & 10 \\ (x-a-b-c)-d & = & \underline{0} & \underline{0} \\ & & b+2c+3d & 114 \end{array}$$

Obviously $b+2c+3d = a+b+c+d$. It is interesting to see how the numbers were chosen, though:

$$a+b+c+d-1 = b+2c+3d$$
$$\Longleftrightarrow \quad a-1 = c+2d \ ,$$

which is verified by examining the figures, and shows that the second number does not alter the results.

Simon Kong (year 11, Trinity Grammar) also sent us his analysis.

Let y_0 be the total number of dollars. Then

	Spent	Left
	y_1	x_1
	y_2	x_2
	y_3	x_3
	y_4	0
Totals	A	B

Now $y_4 = x_3$ since he spent the last money left which is x_3.

Also,

$$\begin{aligned} x_1 &= y_0 - y_1 \\ x_2 &= x_1 - y_2 \\ &= (y_0 - y_1) - y_2 \\ &= y_0 - y_1 - y_2 \\ x_3 &= x_2 - y_3 \\ &= (y_0 - y_1 - y_2) - y_3 \\ &= y_0 - y_1 - y_2 - y_3 . \end{aligned}$$

Adding up the number of dollars <u>spent</u>,

$$\begin{aligned} A &= y_1 + y_2 + y_3 + y_4 \\ &= y_1 + y_2 + y_3 + (y_0 - y_1 - y_2 - y_3) \text{ since } y_4 = x_3 \\ &= y_0 . \end{aligned}$$

Adding up the number of dollars <u>left</u>,

$$\begin{aligned} B &= x_1 + x_2 + x_3 \\ &= (y_0 - y_1) + (y_0 - y_1 - y_2) + (y_0 - y_1 - y_2 - y_3) \\ &= 3y_0 - 3y_1 - 2y_2 - y_3 , \end{aligned}$$

since there exist many possible combinations of y_i, $i = 0,1,2,3$, such that B may be greater than A or less than A.

It is irrelevant to compare the total amount spent (A) and the total amount left (B).

For the missing dollar trick, let $B = y_0 - 1$,

i.e. $B = 3y_0 - 3y_1 - 2y_2 - y_3 = y_0 - 1$. (1)

Any combination of y_i, $i = 1,2,3$, that satisfies (1), subject to the conditions established below, will lead to the missing dollar:

(i) Since the number of dollars <u>left</u> must be greater than zero, we have

$$\begin{aligned} & x_1 > 0 \\ \Leftrightarrow\ & y_0 - y_1 > 0 \\ \Leftrightarrow\ & y_0 > y_1 \\ & x_2 > 0 \\ \Leftrightarrow\ & y_0 - y_1 - y_2 > 0 \\ \Leftrightarrow\ & y_0 > y_1 + y_2 \\ & x_3 > 0 \\ \Leftrightarrow\ & y_0 - y_1 - y_2 - y_3 > 0 \\ \Leftrightarrow\ & y_0 > y_1 + y_2 + y_3 . \qquad (2) \end{aligned}$$

(ii) Further, since the number of dollars <u>spent</u> must be greater than zero, we have

$$y_i > 0 \qquad i = 1,2,3. \qquad (3)$$

Note that we do not include y_4 as $y_4 = x_3$ and in the condition stated above, it is seen that $x_3 > 0$.

For the problem given

$$\begin{aligned} B = 3(115) - 3y_1 - 2y_2 - y_3 &= (115) - 1 \\ \Rightarrow\ 3y_1 + 2y_2 + y_3 &= 231. \end{aligned}$$

Subject to (2) $115 > i = 1$

(3) $y_i > 0 \quad i = 1,2,3.$

Check: $3y_1 + 2y_2 + y_3$
$= 3(40) + 2(46) + (19)$ – values of y_1, y_2, y_3 as given
$= 231.$ in the problem

Alternatively, any y_i, $i = 1,2,3$, subject to the inequality constraints (2) and (3) will do the same trick.
E.g. $y_1 = 30$, $y_2 = 60$, $y_3 = 21$

$(y_0 = 115)$	Spent	Left
	30	85
	60	25
	21	4
	4	0
	115	114

SOLUTION TO PROBLEM 10.1.2

David Dyte also sent us his solution to this problem – to show that, given any 17 numbers, it is always possible to choose five in such a way that their sum is divisible by 5. Here is his solution.

If we refer to the numbers in modulo 5 notation, then we need only consider 5 numbers: 0,1,2,3, and 4. Other integers are simply an addition of one of these and a multiple of 5, and so need not be used.

Now there are at least two ways of making 5 of these numbers add to give 0 (mod 5):
Method I: (a set) $0+1+2+3+4 \equiv 0 \pmod 5$
Method II: (5 of a kind) $n+n+n+n+n \equiv \pmod 5$.

There are other ways but these need not be considered. Now, if we try to choose a set of 17 numbers satisfying neither Method I nor Method II:
(i) To avoid a Method I set we must avoid choosing one particular number and only choose from the other four;
(ii) To avoid a Method II set we must choose at most four of each number.

In order to maintain these conditions, having chosen 16 numbers we will have a set of four of the numbers repeated four times. In chosing a 17th number we must choose a fifth of one number (Method II) or complete a set of all the numbers (Method I). So in any set of 17 numbers one of these two methods must be satisfied, and so in any set of 17 numbers 5 can be chosen so that their sum can be divided by 5.
[*We may note that the number 17 is not the smallest number with the stated property. What is? Eds.*]

SOLUTION TO PROBLEM 10.2.2

We asked for the smallest value of α such that $F(x) = 7x^{11} + 11x^7 + 10\alpha x$ is divisible by 77 for all values of x. We had solutions from David Shaw and Devon Cook. Here is Devon Cook's solution. David Shaw's is similar.

For all integral x,
$x^{11} - x \equiv 0 \pmod{11}$ and $x^7 - x \equiv 0 \pmod{7}$, thus
$7x^{11} - 7x \equiv \pmod{77}$ and $11x^7\ 11x \equiv 0 \pmod{77}$.

Adding the last two congruences,

$$7x^{11} + 11x^7 - 18x \equiv 0 \pmod{77}$$

and clearly therefore,

$$7x^{11} + 11x^7 - (18 \pm 77k)x \equiv 0 \pmod{77}$$

The lowest multiple of 10 which equals $-18 \pm 77k$ is 290
Thus $10\alpha = 290$, thus $\alpha = 29$.
In general,
$10\alpha = -18 \pm 77k$.

PROBLEM 10.4.1 (from *Parabola*)

Let $\square$ be an operation that combines two integers to form a third. Given that

$$x \square (y+z) = y \square x + z \square x,$$

prove that

$$u \square v = v \square u,$$

for all u, v.

PROBLEM 10.4.2

The mathematician Roland Sprague invented the following sequence: $u_1 = 1$, $s_n = u_1 + u_2 + \ldots + u_n$, $u_{n+1} = 1/s_n$. Although there is no simple formula for s_n it has an approximate formula which is very simple. Can you find it and say why it works.

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NEWS UPDATE

In *Function, Vol.6, Part 2*, we reported on the ten year old Ruth Lawrence's admission to Oxford University amid pessimistic accounts that she'd "come to nothing". Well four years later she hasn't fulfilled these prophecies. She recently graduated with first class honours in Mathematics and Physics and, at 14, is the youngest graduate ever to emerge from Oxford.

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*$4.00 for *bona fide* secondary or tertiary students.

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