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FUNCTION

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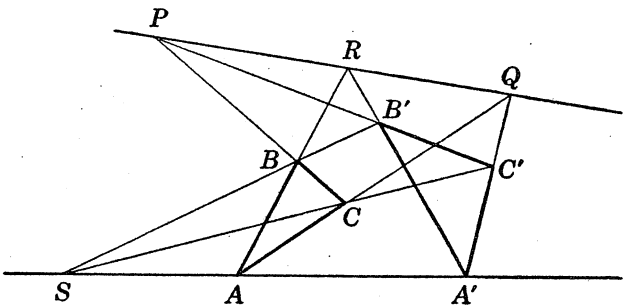

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools, and published by Monash University.

It is a "special interest" journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. *Function* is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

Function does *not* aim directly to help its readers pass examinations. It does guarantee to enrich and inform their mathematical lives, to provide high quality articles and stimulating reading for the mathematically inclined.

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(continued on back cover)

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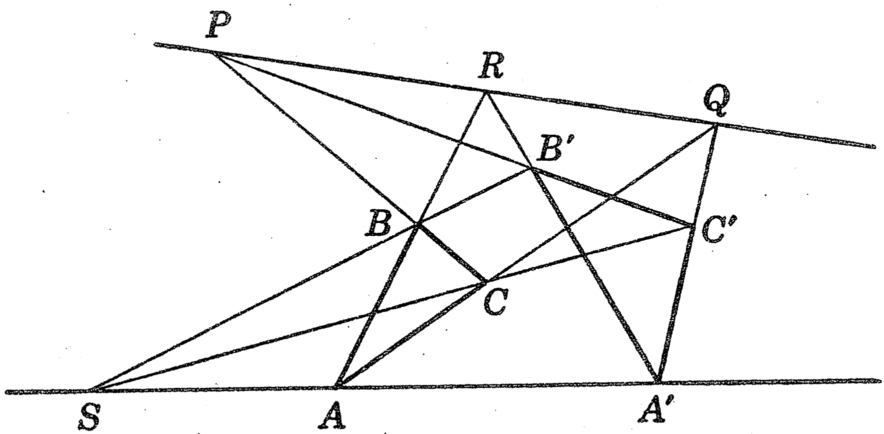
THE FRONT COVER

M.A.B. Deakin, Monash University

If the lines joining the corresponding vertices of two triangles are concurrent (i.e. all pass through a common point), then the three pairs of corresponding lines intersect in collinear points.

This is Desargues' Theorem, named after Girard DESARGUES (1591-1661), a brilliant but much misunderstood and vilified geometer whose work was not accorded its due recognition until the 19th century.

In the cover diagram, here reproduced, the two



triangles are *ABC*, *A'B'C'*.

The corresponding vertices lie on the lines *AA'*, *BB'*, *CC'* which all meet in the *point of concurrency* *S*, so the conditions of the theorem (the words following the 'if') are satisfied.

The corresponding sides *AB*, *A'B'* meet in *R*; *BC*, *B'C'* in *P* and *CA*, *C'A'* in *Q*. *R*,*P*,*Q* are collinear – that is, they lie on one straight line.

A surprising fact is that the theorem, true in both two and three dimensions, is easier to prove in three. Let *SA'*, *SB'*, *SC'* form a tripod resting on a flat base *A'B'C'*. Let *A*,*B*,*C* be points on the 'legs' of this tripod.

Then *A*,*B*,*C* lie in a plane which (typically) intersects the base plane *A'B'C'*. This intersection will be a straight line.

Furthermore *BC* and *B'C'* both lie in the plane *SB'C'* and so (unless they are parallel) intersect. The point *P* where they intersect must lie in the plane *ABC*, as this contains the line *BC*. It must also lie in the plane *A'B'C'*, as this contains the line *B'C'*.

So *P* lies in the line where the two planes *ABC*, *A'B'C'* intersect. So, similarly, do *Q*, *R*.

If *BC*, *B'C'* are parallel, the argument requires a small modification. *P* is then said to be at infinity and we *deem* infinity to be collinear with any two points *Q*, *R*. In particular, if all these points lie at infinity, they are said to form a line at infinity. I.e. if the planes *ABC*, *A'B'C'* are parallel, they meet in the line at infinity.

So the theorem is regarded as holding true in these cases as well.

But what about two dimensions? In this case, think of the diagram as a perspective drawing of the 3-dimensional case, and so reach the result!

A similar device provided the proof of another geometric result in Problem 3.1.3.

Desargues' Theorem has been stated above in the logical form:

If *A*, then *B*,

where *A*,*B* are geometric statements. Its *converse* is

If *B*, then *A*.

Sometimes a true theorem has a true converse, sometimes not. A common fallacy is to assume converses to be equivalent to the theorems from which they are derived. But

If *P* is prime, then *P* = 2 or *P* is odd

is true, but its converse is not.

In the case of Desargues' theorem, however, the converse does hold. See if you can state and prove it.

COMPUTER ANIMATION

Binh Pham, Monash University

Computer animation is the art of creating movement using computer graphics. Both film and television rely on speed to create the illusion of movement. As films move at 24 frames per second and television at 25-30 cycles per second, our vision cannot separate still images. In fact, the illusion of movement is achieved whenever we see a progressive sequence of more than 9 images per second.

In computer animation, many still images are created; the computer is then used to calculate the displacement of objects from one frame to the next. There are 3 types of computer animation: film and television graphics, cartoon animation and realistic simulations.

1. Film and television graphics

Film and television graphics - often called 'animatics' - describe movement in an abstract way. Their main use is for communicating information. An animatic consists of just a few frames, continuously cycled, e.g. 1-2-3, 1-2-3, etc. and these images are held on the screen for a few seconds. Animatics are often inserted into a television advertisement or a news bulletin. Figure 1 shows two stills from a graphics sequence to illustrate the working of a Nikkon camera.

2. Cartoon animation

Cartoon animation represents ways in which figures and objects move by exaggeration. It is designed to capture all the expressive qualities in the tradition of Walt Disney cartoons, and hence it can be much more complicated than the animatics.

An animator begins by sketching the *key frames* which show the positions of the characters at the beginnings and ends of movements. The computer is then used to produce the *in-betweens* (i.e. the intermediate frames).

There are two methods for producing in-betweens automatically: skeleton animation and moving point constraints. In *skeleton animation*, a match stick figure is manipulated by the animator who fixes the positions of the character's arms, legs, etc., in each key frame. These skeletons which can be drawn very quickly are then covered with curved lines produced by the computer.

In the technique of *moving point constraint*, the animator specifies the key frames and the curves which describe the path and the speed of points within a key frame. The computer will be used to calculate the shape of the intermediate frames with the information provided.

If there is a mathematical relationship between two images, the computer can be effectively programmed to transform one line drawing into another. As a matter of interest, in 1917 the biologist D'Arcy Wentworth Thompson published his theory of transformation, *On Growth and Form*. Figure 2 shows how the shape of one species can be derived from the shape of another related species. These observations of natural forms are very useful in computer animation.

3. Realistic simulations

Instead of working with outline drawings, the simulator uses full 3-D images. A realistic simulation requires 4 stages of production: preliminary modelling, motion direction, full modelling and rendering.

A preliminary model is constructed using a few polygons to represent the outline. This model is used in the computer to plot the movement of an object in the intended sequence. 3-D coordinates and perspective projection are used. Techniques which are similar to those of a live action director are used to carry out the motion tests. The simulator controls the point of view of the 'camera', the positioning of the 'lights' and the movement of the 'actors'.

Once the motion tests are completed, the model will be refined to show all details. Rendering - or displaying - each frame is done automatically. Programs will be written to determine the hidden surfaces and the shading of the objects. The latter involves the calculation of the colour and intensity for each pixel (picture element) in every frame.

High cost and slow rates of production have limited the use of realistic simulation in motion pictures. Prior to *Star Wars*, only one major film, Disney's *Tron*, had contained more than a few minutes of computer-generated effects which were done by subcontracting to 4 specialist computer animation companies. *Tron*, unlike *Star Wars*, was wholly generated by computer techniques. Apart from Disney, Lucas film is the only other major studio which has invested seriously in computer animation. However, things might change in the near future since a new generation of more powerful computers will be able to process realistic simulations in minutes rather than days.

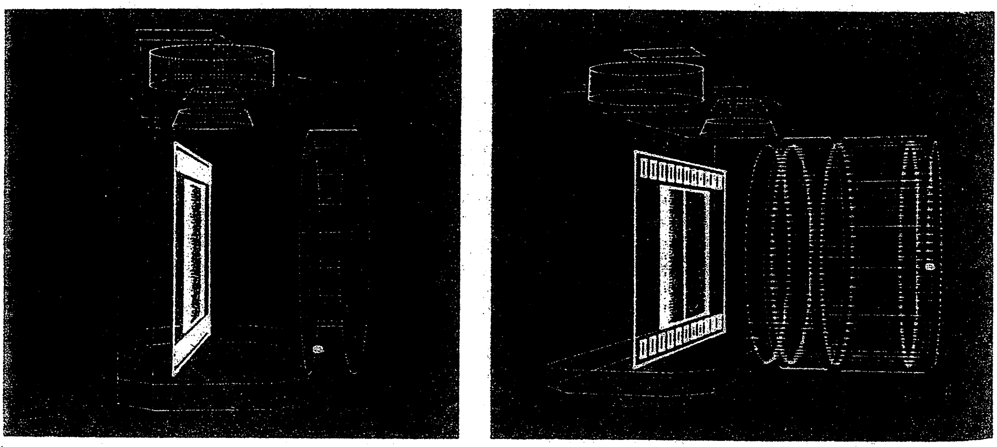


Figure 1

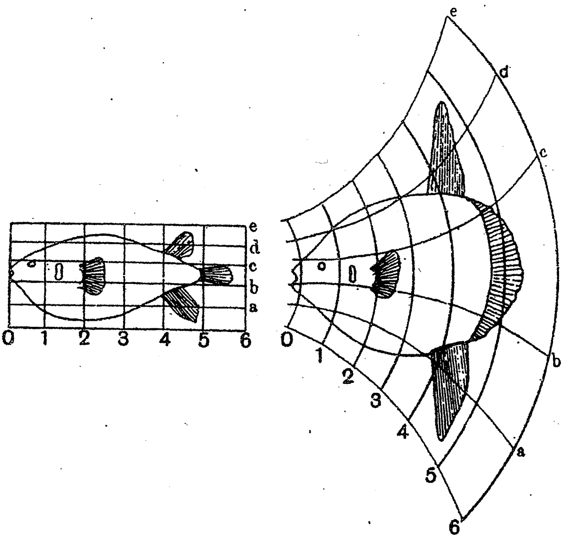


Figure 2

L.F. RICHARDSON'S WEATHER FORECAST FACTORY

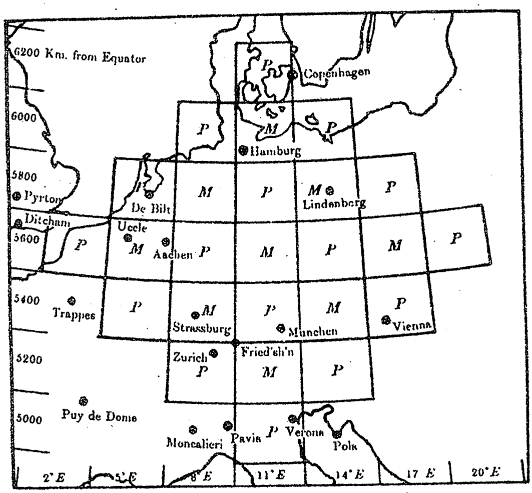
Peter Kloeden, Murdoch University

You probably have not heard of Lewis Fry Richardson, an Englishman who was born in 1881 and died in 1953, but you certainly enjoy the benefits of his scientific genius every time you hear or read a weather forecast from the Bureau of Meteorology. While a young man Richardson made many valuable scientific contributions in Physics and Meteorology, particularly concerning the nature of turbulence. Yet it was one of his least successful, as it seemed at that time, ideas that has proved to be most beneficial. This was Richardson's attempt to solve numerically the equations of atmospheric dynamics in order to obtain an accurate weather forecast.

The equations of atmospheric dynamics are a complicated system of partial differential equations involving meteorological variables such as wind velocity, atmospheric pressure and temperature, and their space and time derivatives (which indicate their rate of growth or decay in a particular space direction or in time). As the exact solution of these equations was (and still is) not known, the only way to solve them, and hence to obtain a prediction of the evolution of the weather, was to use an approximate numerical method. For this the area of the forecast, say Europe, is divided into a grid like in a street directory, and then the system of partial differential equations is approximated by a system of algebraic equations involving the values of the unknown meteorological variables at each grid point. Fortunately the values of these variables at one grid point usually depend only on the corresponding values at nearby gridpoints, so the system of algebraic equations is moderately simple. However it involves an enormously large number of equations, and this number grows rapidly as the number of grid points increases. This creates a dilemma because more grid points are needed for a more accurate forecast. Modern supercomputers have now made this task feasible, though there are still shortcomings (as the accuracy of some of the Bureau's forecasts shows!). Such computers were not available in Richardson's day. In fact only slow human operated calculating machines were then in use. To carry out the necessary calculations Richardson envisaged a large factory with an array of desks corresponding to the gridpoints, with a human "computer" at each desk performing (possibly using a calculating machine) all the calculations at that gridpoint and passing them onto the adjacent desks. (See the cartoon on page 10, which gives a Russian view on the matter.)

For a fairly accurate forecast, Richardson estimated that some 64,000 such "computers" would be required. Naturally, the English Government of the day thought this idea rather harebrained and did not take it up. To prove its feasibility Richardson spent many months doing all the calculations himself for a single 24-hour forecast, but his grid was too coarse and the results highly inaccurate (there were also some numerical difficulties). The project seemed to die a natural death, but the ideas made a marked impression on other young scientists, who, 30 or 40 years later when modern electronic computers came into being, were to successfully carry out Richardson's dream. (Interestingly, the most modern supercomputers with parallel or vector array architecture are based on a similar idea to Richardson's weather forecast factory, but with a microprocessor at each desk rather than a human.)

Richardson published his ideas on numerical weather forecasting in a book during the 1920's, though he had done much of the work earlier. In fact during the First World War he made a deliberate decision to do no further work in meteorology, because, as a devout Quaker and Pacifist, he was distressed that this work would be useful for horrible new weapons of war: gas and aeroplanes. Richardson resigned from his job as a government meteorologist and went to France to serve as a volunteer ambulance driver with a medical unit of the French Army (the British Army did not accept civilians in its operational zones). Deeply shocked by the carnage of this war,



he devoted his scientific talents to the study of the causes of war and is now remembered as one of the founding fathers of what is called "Peace and Conflict Studies". I shall describe Richardson's mathematical model of an arms race in the next issue of *Function*. If you are interested in reading more about Richardson, there is a recent biography by O.M.Ashford called "Prophet - or Professor, The Life and Work of Lewis Fry Richardson", which was published by Adam Hilger Ltd. (Bristol and Boston) in 1985.

∞ ∞ ∞ ∞ ∞

REVERSIBILITY

Back in the 1940s and 50s, I and nearly all other Australian schoolboys eagerly followed the exploits of the air ace James Bigglesworth, D.S.O., M.C., D.F.C. (Biggles), the creation of the very prolific author, Captain W.E. Johns.

Recently, in a second-hand shop, I came upon a copy of *Sergeant Bigglesworth, C.I.D.*, and out of mixed curiosity and nostalgia I bought it and read it. It stands up surprisingly well, but one lapse amused me considerably, the more so as it contrasts with a passage in the book where things are worked out right. Begin with the latter.

Biggles has pursued a gang of ex-Nazis to an oasis in the Sahara desert. The place is full of land-mines and Biggles escapes by climbing a palm tree and dropping into the water of the oasis's central pool. But then "he perceived a contingency for which, in his haste to get into the pool, he had made no provision. How was he to get out? The frond from which he had dropped, which had sagged under his weight, was now far out of reach. With a shock he realised that he was, quite definitely, a prisoner in the pool."

He does, of course, get free, although that is not a story I wish to pursue here.

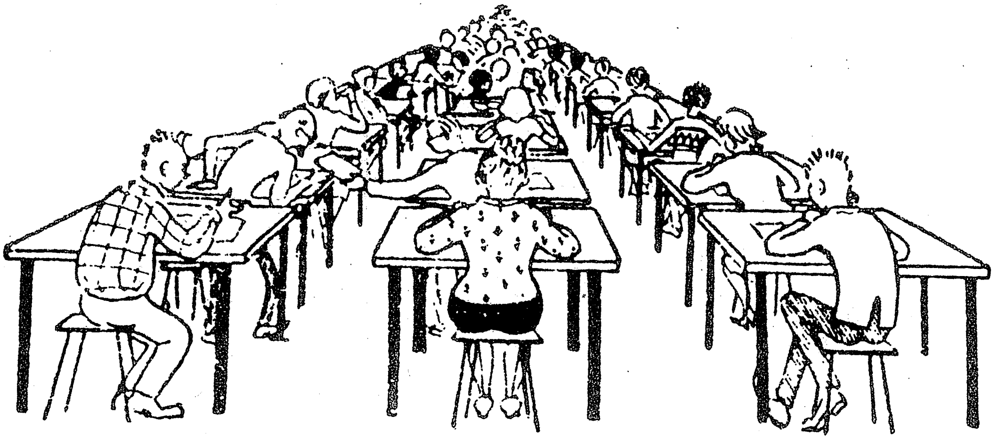
Earlier, however, Biggles enters a room which is locked from the inside "but he was able to get in by the old trick of sliding a sheet of paper under the door and pushing the key out of the lock - again using the blade of his penknife - so that it fell on the paper. The paper was then withdrawn, bringing the key with it. Another moment the door was open, and they had reached their objective - Preuss's office."

Preuss, one of the gang, is not in the office and so, to avoid arousing his suspicion, they "left the office as they had entered, locking the door behind them and leaving the key in the lock."

They *what*?

Suffering catfish! As Biggles was wont to say.

Another thing - but this is one Captain Johns does get right. How was the door locked from the inside and Preuss not in his office? The answer is on p. 15.



A Russian artist's concept of Richardson's
fantasy weather forecast factory.

FOURIER SPIRALS USING LOGO

Colin Fox, Melbourne CAE

In the February 1986 issue of *Function* the front cover showed some spirals related to the function

$$f(x) = \begin{cases} pi/4 & \text{if } 2n\,pi < x < (2n+1)pi \\ -pi/4 & \text{if } (2n+1)pi < x < 2(n+1)pi \\ 0 & \text{otherwise.} \end{cases}$$

These spirals arise when analysing this function by Fourier methods. (See pages 3 to 7 of *Function*, February 1986). This analysis yields the infinite series

$$f(x) = \sin x + (1/3)\sin 3x + (1/5)\sin 5x + \ldots$$

To write a computer program to draw these spirals, LOGO is a (or, perhaps, the) most natural programming language to use. Page 5 of February's *Function* contains details of the construction of the spirals. In particular, notice that the method involves repeating infinitely often the pair of movements:

move a certain distance in a straight line,

then turn left (or right) through a certain angle.

The distance decreases with each repetition while the turning angle remains constant.

A Logo procedure to accomplish this can be written as follows. If you are unsure about writing Logo procedures on your school's microcomputer, ask a fellow student or teacher who is not. If there is no such person at your school, get in touch with me and I will help you become your school's Logo expert!!

```
TO FOURIER.SPIRAL :ANGLE :COEFFICIENT
  FD ONE/:COEFFICIENT LT 2*:ANGLE
  FOURIER.SPIRAL :ANGLE :COEFFICIENT+2
END
```

This procedure contains a subprocedure, ONE, which will output the screen distance corresponding to one unit on the graph.

If you are using an Apple Logo type:

```
TO ONE
  OP 75
END
```

If you are using Commodore Logo, type 90 instead of 75. For BBC Logos use 300.

Having typed in these procedures, you can draw a spiral by typing, for example,

FOURIER.SPIRAL 45 1 (and press RETURN key).

N.B. The procedure will run forever! Press CTRL-G (Apple and Commodore) or ESCAPE (BBC) to give the turtle a rest.

Notice that we use degrees not radians. Notice also that we need to do some more programming if the February front cover is to appear on the screen. The next two procedures are one way to achieve this.

```
TO START :ANGLE
  AXES
  PU HOME PD
  SETH 90 LT :ANGLE
  FOURIER.SPIRAL :ANGLE 1
END
```

On Apple Logos, continue by typng:

```
TO AXES
  PU SETPOS [-135 0] PD SETX 135
  PU SETPOS [0 95] PD SETY -95
END
```

For BBC Logos, replace each 135 with 600 and each 95 with 300. For Commodore Logo, replace SETPOS [-135 0] with SETXY -155 0. Replace SETPOS [0 95] with SETXY 0 110. Replace 135 with 155 and -95 with -110.

Now, clear the screen (type CS and press RETURN) and draw the spiral for *pi*/4 by typing

START 45 (and press RETURN key).

If this produces the top left spiral from February's front cover, then you can complete the top half of the picture by typing START 22.5, START 11.25 and START 5.625. If it doesn't, carefully check your typing or ask your local Logo expert for assistance.

How will you reproduce the bottom half of the picture?

Finally, you may like to edit the FOURIER.SPIRAL procedure so that it draws the spirals shown below. These spirals come from the infinite series

$\sin x + (1/2)\sin 2x + (1/3)\sin 3x + (1/4)\sin 4x + \ldots$.

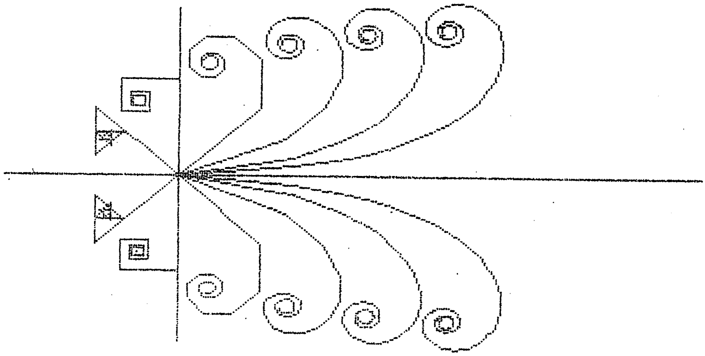


fig. 1

This screendump is from Logotron Logo (on the BBC microcomputer). I changed the value in ONE from 300 to 200 and altered the position of the origin from [0 0] on the screen to [-300 0]. That is, in my AXES procedure, I changed SETPOS [0 350] to SETPOS [-300 350].

For Apple, change 75 to 50 in ONE and SETPOS [0 95] to SETPOS [-70 95] in AXES.

For Commodore, change 90 to 60 in ONE and SETXY 0 110 to SETXY -80 110 in AXES.

I leave the editing (changing) of the FOURIER.SPIRAL procedure to you.

PROOFS OF PYTHAGORAS' THEOREM

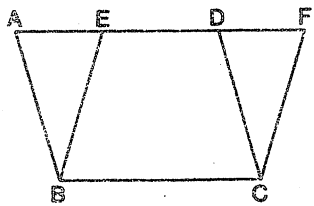
The six diagrams on pages 16-21 allow construction of a number of proofs of Pythagoras' Theorem. They show the equality of the hatched area and the grey area by demonstrating that they may exchange places.

One proof uses the sequence (a), (b), (c), (d), (e), (f), but others are also possible. A little explanation is called for here, but the main purpose of this article is to stimulate you to explore matters for yourself.

In Figure (c), the grey "wedge" has moved up, displacing part of each of the hatched areas, formerly above it in Figure (b). It is not difficult to show that the area displaced is equal to that *replaced* below the wedge. For example, that displaced from the small square of Figure (b) equals that at the bottom and to the left of the dashed vertical line in Figure (c).

Figure (e) shows the "wedge" splitting up into two parallelograms, but retaining the same area. This depends on a theorem. *Parallelograms on the same base and between the same parallel lines have the same area.*

So in the diagram below,



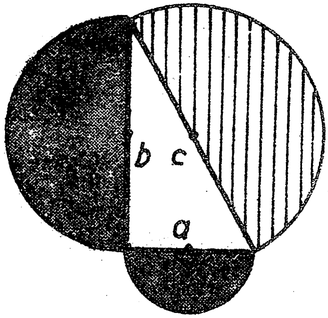
the parallelograms ABCD, EBCF satisfy the condition. This is proved by noting that the triangles ABE, DCF are in fact congruent (i.e. superposable) and so have the same area. The area of the trapezium EBCD is then adjoined to each to produce the result.

Figures (a) - (f) appeared first in an article by R.Rubinow in the Soviet journal *Quant*. This article was adapted and translated into German by H.Begander, who published it in the East German journal *Alpha*, with which *Function* has an exchange agreement.

There are many hundreds of proofs of Pythagoras' Theorem, but the one referred to above was new to us. The diagrams shown may also be used to construct others and we encourage you to try your hand at this. (E.g. consider the role of the white triangles in Figures (a), (b), (d), (f).)

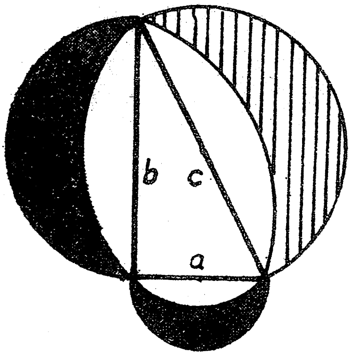
VARIATIONS ON A THEME

$$\frac{\pi}{8}a^2 + \frac{\pi}{8}b^2 = \frac{\pi}{8}c^2$$



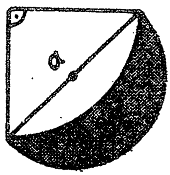
a

$$\frac{\pi}{4}a^2 + \frac{\pi}{4}b^2 = \frac{\pi}{4}c^2$$



b

$$A = \frac{a^2}{4}$$

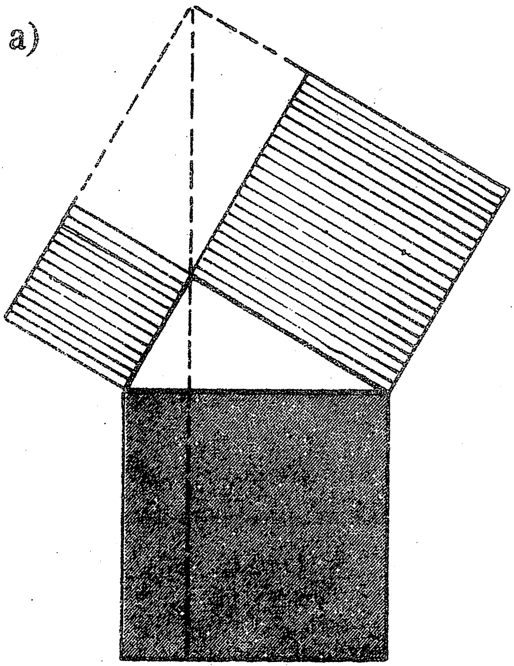


See also p.32.

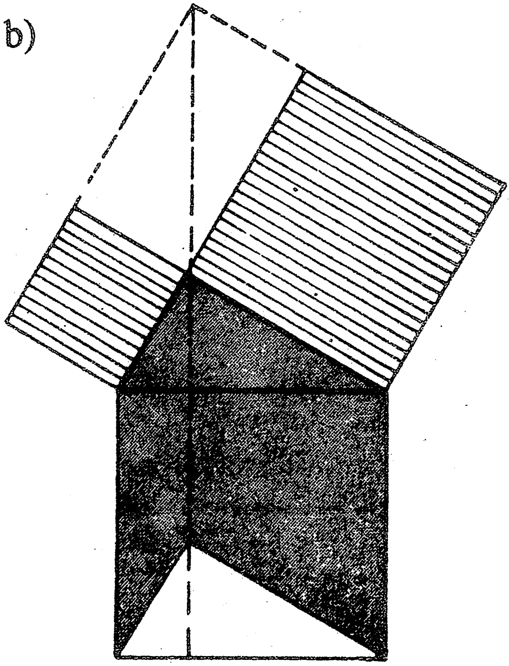
oo oo oo oo oo

In answer to the problem posed on p.9, Preuss's office had two doors. Biggles burgled one. Preuss had left by the other, locking it and pocketing the key in the usual way.

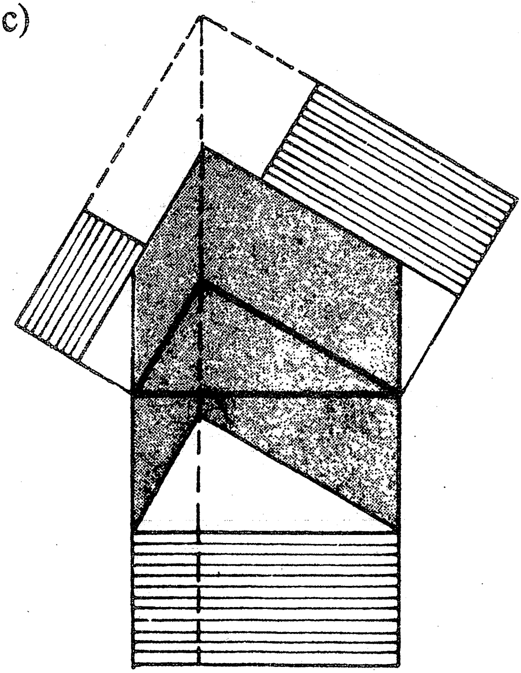
a)



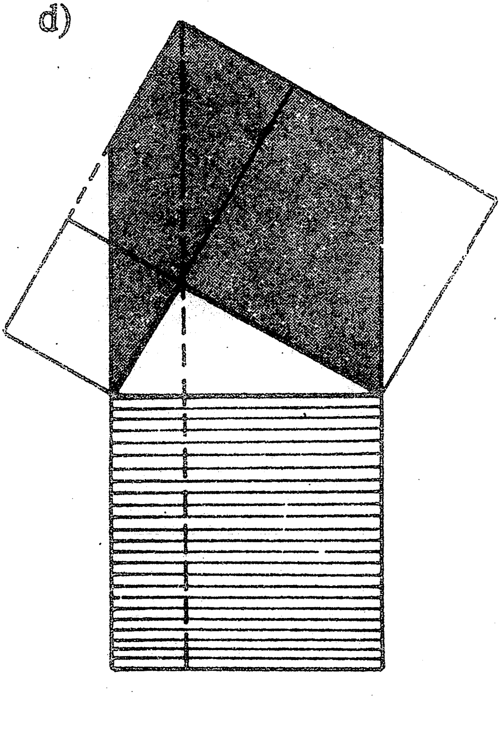
b)



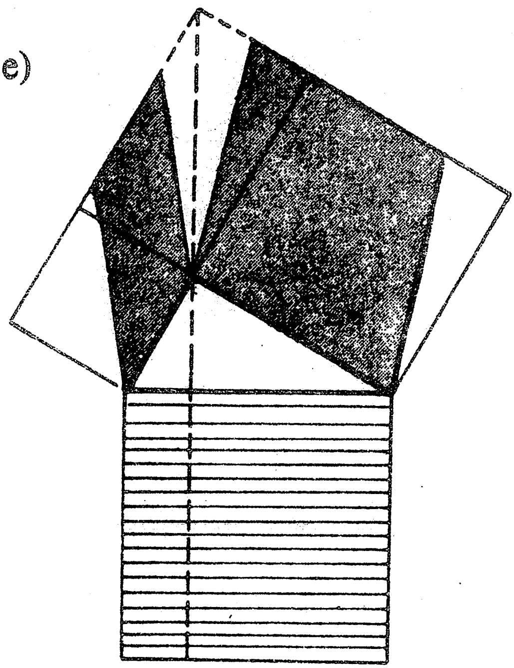
c)



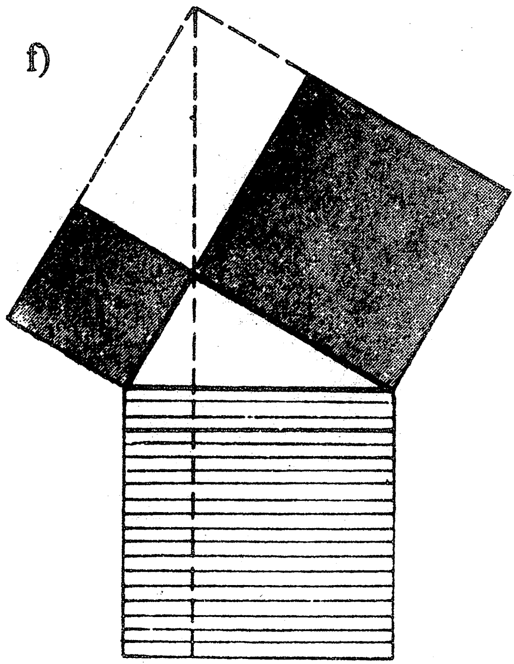
d)



e)



f)



LETTERS TO THE EDITOR

MORE NUMBER PATTERNS

<u>Sums of Consecutive Squares</u> <u>Pattern 1</u>

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$
$$55^2 + 56^2 + \dots + 59^2 + 60^2 = 61^2 + 62^2 + \dots + 65^2$$

<u>The product of the sum of two squares by the sum of two squares is always the sum of two squares</u>

<u>Pattern 2</u>

$$\begin{aligned}(x^2 + y^2)(a^2 + b^2) &= (ax + by)^2 + (by - ay)^2\\ &= (ax - by)^2 + (bx + ay)^2\\ (6^2 + 7^2)(4^2 + 2^2) &= (6.4 + 7.2)^2 + (2.7 - 4.6)^2\\ &= (38^2) + (-16^2)\\ &= 1700\end{aligned}$$

Also

<u>Pattern 3</u>

$$1^3 = 1^2$$
$$1^3 + 2^3 = (1 + 2)^2$$
$$1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2$$
$$1^3 + 2^3 + 3^3 + 4^3 = (1 + 2 + 3 + 4)^2$$
$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = (1 + 2 + 3 + 4 + 5)^2$$
$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = (1 + 2 + 3 + 4 + 5 + 6)^2$$

and

<u>Pattern 4</u>

$$1 + 0 = 0 + 1 = 1$$
$$(1 + 2) + 1 = 1 + 3 = 4$$
$$(1 + 2 + 3) + (1 + 2) = 3 + 6 = 9$$
$$(1 + 2 + 3 + 4) + (1 + 2 + 3) = 6 + 10 = 16$$
$$(1 + 2 + 3 + 4 + 5) + (1 + 2 + 3 + 4) = 10 + 15 = 25$$

Pattern 5

$$1^2 + 2^2 + 3^2 + \dots\dots 24^2 = 4900 = 70^2$$

$$18^2 + 19^2 + 20^2 + \dots\dots + 28^2 = 5829 = 77^2$$

$$25^2 + 26^2 + 27^2 + \dots\dots + 50^2 = 38025 = 195^2$$

$$38^2 + 39^2 + 40^2 + \dots\dots + 48^2 = 20449 = 143^2$$

$$456^2 + 457^2 + 458^2 + \dots\dots + 466^2 = 1529^2$$

$$854^2 + 855^2 + 856^2 + \dots\dots + 864^2 = 2849^2$$

Digital Invariants

Order 3

$153 = 1^3 + 5^3 + 3^3$

$370 = 3^3 + 7^2 + 0^3$

$371 = 3^3 + 7^3 + 1^3$

$407 = 4^3 + 0^3 + 7^3$

Order 4

1634; 8208; 9474

Order 5

4150; 4151; 54748; 92727; 93084; 194979

Order 6

548834

Order 7

1741725; 4210818; 9800817; 9926315; 14459929

Order 8

24678050; 24678051; 88593477

Order 9

146511208; 472335975; 534494836; 912985153

Order 10

4679307774

Miscellaneous Digital Invariants

$43 = 4^2 + 3^3$

$63 = 6^2 + 3^3$

$89 = 8^1 + 9^2$

$135 = 1^1 + 3^2 + 5^3$

$175 = 1^1 + 7^2 + 5^3$

$518 = 5^1 + 1^2 + 8^3$

$598 = 5^1 + 9^2 + 8^3$

$3435 = 3^3 + 4^4 + 3^3 + 5^5$

$438{,}579{,}088 = 4^4 + 3^3 + 8^8 + 5^5 + 7^7 + 9^9 + 0^0 + 8^8 + 8^8$

$1306 = 1^1 + 3^2 + 0^3 + 6^4$

$1676 = 1^1 + 6^2 + 7^3 + 6^4$

$2427 = 2^1 + 4^2 + 2^3 + 7^4$

Square Numbers containing the Nine Digits Unrepeated

$11826^2 = 139854276$

$12363^2 = 152843769$

$12543^2 = 157326849$

$14676^2 = 215384976$

$15681^2 = 245893761$

$15963^2 = 254817369$

$18072^2 = 326597184$

$19023^2 = 361874529$

Consecutive Numbers

123 − 45 − 67 + 89 = 100
123 + 4 + 5 + 67 − 89 = 100
123 + 45 − 67 + 8 − 9 = 100
1 + 2 + 34 − 5 + 67 − 8 + 9 = 100
1 + 23 − 4 + 5 + 6 + 78 − 9 = 100
(1)(2)(3)(4) + 5 + 6 + (7)(8) + 9 = 100
(1 + 2 − 3 − 4)(5 − 6 − 7 − 8 − 9) = 100
98 − 76 + 54 + 3 + 21 = 100
9 − 8 + 76 − 5 + 4 + 3 + 21 = 100
9 − 8 + 7 + (6 + 54 + 32)(1) = 100

Difference of Two Squares containing the Nine Digits

$11113^2 - 200^2 = 123458769$

$31111^2 - 200^2 = 967854321$

$11117^2 - 200^2 = 123547689$

$11356^2 - 2000^2 = 124958736$

$12695^2 - 6017^2 = 124958736$

$16260^2 - 11808^2 = 124958736$

$12372^2 - 300^2 = 152976384$

Square Numbers containing the Nine Digits Unrepeated

$25059^2 = 627953481$
$25572^2 = 653927184$
$25941^2 = 672935481$
$26409^2 = 697435281$
$26733^2 = 714653289$
$27129^2 = 735982641$
$27273^2 = 743816529$
$29034^2 = 842973156$
$29106^2 = 847159236$
$30384^2 = 923187456$

Square Numbers containing the Ten Digits Unrepeated

$32043^2 = 1026763849$
$32286^2 = 1042385796$
$33144^2 = 1098524736$
$36172^2 = 1237069584$
$39147^2 = 1532487609$
$45624^2 = 2081549376$
$55446^2 = 3074258916$
$68763^2 = 4728350169$
$83919^2 = 7042398561$

Garnet J. Greenbury
Brisbane.

[For Pattern 3, several proofs are available, one due to a 13-year-old schoolgirl, Jeanette Hilton. This was discussed in Function, Vol. 5, Part 2. The 'digital invariants' are sometimes referred to as Armstrong Numbers (we don't know why, or who Armstrong was). Problem 5.3.1 concerned Armstrong numbers. Readers may like to check the above list. We haven't! The 'miscellaneous digital invariants' satisfy less stringent conditions than do the Armstrong numbers. Eds.]

∞ ∞ ∞ ∞

1986

Earlier this year 11M1 at Katoomba High School, as a class project, devised a list of the first 100 positive integers made up from the digits of the number 1986, used in order.

The operations and functions used were

$+, -, \times, \div, \sqrt{\ }, (\ ), !,$ powers, C_r, P_r.

Also used for four numbers (68, 69, 97, 99) were trig. and inverse trig. functions. I am not happy with this for obvious reasons. The students are still working on it.

Perhaps you would like to issue a challenge to other students to provide a list of their own.

Enclosed is our effort.

1	1^{986}	51	$-(-1 + (\sqrt{9})!) + 8! \div 6!$
2	$1 + \sqrt{9} - 8 + 6$	52	$-(1 + \sqrt{9}) + 8! \div 6!$
3	$-1 + \sqrt[9]{8^6}$	53	$-1 + 9! \div 8! \times 6$
4	$(1 + \sqrt{9})! - 8 + 6$	54	$1 \times 9! \div 8! \times 6$
5	$19 - 8 + 6$	55	$1 + 9! \div 8! \times 6$
6	$1 - 9 + 8 + 6$	56	$(1 + \sqrt{9})(8 + 6)$
7	$-1 + \sqrt[\sqrt{9}]{8} + 6$	57	$1^9 + 8! \div 6!$
8	$1 + 9 - 8 + 6$	58	$(-1 + 9) \times 8 - 6$
9	$1 + \sqrt[\sqrt{9}]{8} + 6$	59	$1 \times \sqrt{9} + 8! \div 6!$
10	$-1 + 9 + 8 - 6$	60	$(1 + 9! \div 8!) \times 6$
11	$-1 + 9 \times 8 \div 6$	61	$-1 + (\sqrt{9})! + 8! \div 6!$
12	$1 + 9 + 8 - 6$	62	$(1 + (\sqrt{9})!) \times 8 + 6$
13	$1 + \sqrt{\sqrt{9}} \times \sqrt{8} \times \sqrt{6}$	63	$-1 + (\sqrt[\sqrt{9}]{8})^6$
14	$\sqrt{-1 + 9} \times \sqrt{8} + 6$	64	$1 \times (\sqrt[\sqrt{9}]{8})^6$
15	$1^9 + 8 + 6$	65	$-1 + 9 \times 8 - 6$
16	$-1 + \sqrt{9} + 8 + 6$	66	$1 \times 9 \times 8 - 6$
17	$19 - 8 + 6$	67	$1 + 9 \times 8 - 6$
18	$1 + \sqrt{9} + 8 + 6$	68	$\tan^{-1}(1) + 9 + 8 + 6$
19	$-1 + (\sqrt{9})! + 8 + 6$	69	$\cos^{-1}(\sin 19) - 8 + 6$
20	$1 \times (\sqrt{9})! + 8 + 6$	70	$(-1 + 9) \times 8 + 6$
21	$19 + 8 - 6$	71	$-1 + {}^{9}P_{(8-6)}$
22	$-1 + 9 + 8 + 6$	72	$(1 + \sqrt{9} + 8) \times 6$

23 $1 \times 9 + 8 + 6$

24 $1 + 9 + 8 + 6$

25 $(-1 + (\sqrt{9})!)^{(8-6)}$

26 $(1 + \sqrt{9})! + 8 - 6$

27 $(-1)^{9} + {}^{8}C_{6}$

28 $(-1 + \sqrt{9})(8 + 6)$

29 $-1 + \sqrt{9} \times 8 + 6$

30 $1 \times \sqrt{9} \times 8 + 6$

31 $1 + \sqrt{9} \times 8 + 6$

32 $(1 + \sqrt{9})! \times 8 \div 6$

33 $19 + 8 + 6$

34 $1 \times (\sqrt{9})! + {}^{8}C_{6}$

35 $1 + (\sqrt{9})! + {}^{8}C_{6}$

36 $(1 - \sqrt{9} + 8) \times 6$

37 $1 \times 9 + {}^{8}C_{6}$

38 $(1 + \sqrt{9})1 + 8 + 6$

39 $-1 \times 9 + 8 \times 6$

40 $1 - 9 + 8 \times 6$

41 $-1 + (\sqrt{9})! \times 8 - 6$

42 $(1 \times \sqrt{9})(8 + 6)$

43 $1 + \sqrt{9}(8 + 6)$

44 $-(1 + \sqrt{9}) + 8 \times 6$

45 $-(1 \times \sqrt{9}) + 8 \times 6$

46 $(-1 + (\sqrt{9})!) \times 8 + 6$

47 $-(1 \times 9) + 8! \div 6!$

48 $1^{9} \times 8 \times 6$

49 $(1 + (\sqrt{9})!)^{8-6}$

50 $-(1 \times (\sqrt{9})!) + 8! \div 6!$

73 $1 + {}^{9}P_{(8-6)}$

74 $(1 + 9) \times 8 - 6$

75 $-(1 \times 9) + 86$

76 $-1 - 9 + 86$

77 $-1 + 9 \times 8 + 6$

78 $1 \times 9 \times 8 + 6$

79 $1 + 9 \times 8 + 6$

80 $-1 + 9^{(8-6)}$

81 $1 \times 9^{(8-6)}$

82 $1 + 9^{(8-6)}$

83 $-(1 + \sqrt{9}) + 86$

84 $((1 \times \sqrt{9})! + 8) \times 6$

85 $-1^{9} + 86$

86 $(1 + 9) \times 8 + 6$

87 $1^{9} + 86$

88 $-1 + \sqrt{9} + 86$

89 $1 \times \sqrt{9} + 86$

90 $1 + \sqrt{9} + 86$

91 $-1 + 98 - 6$

92 $1 \times 98 - 6$

93 $1 + 98 - 6$

94 $-1 + 9 + 86$

95 $1 \times 9 + 86$

96 $1 + 9 + 86$

97 $\sin^{-1} 1 + 9 - 8 + 6$

98 $(1 + (\sqrt{9})!)(8 + 6)$

99 $1 + \cos^{-1}(\sin((\sqrt{9})!)) + 8 + 6$

100 $(1 + 9)^{(8-6)}$

R.D.Coote
Head Teacher, Mathematics,
Katoomba High School.

PROBLEM SECTION

The chief editor, thinking he had solved PROBLEM 9.5.1, failed to recognise that he had dropped a factor of 2 and so he oversimplified the problem, ended up with the wrong answer, and is now hanging his head in shame. To make matters worse, David Shaw of Geelong West Technical School had solved the problem correctly, but either:

(a) he forgot to mail it,
or (b) Australia Post or Monash University lost it,
or (c) the editor lost it.

However, the matter was not irreparable; the solution was rewritten and here it is.

CORRECT SOLUTION TO PROBLEM 9.5.1.

The problem read:

I have N weights of 1 kg, 2 kg, ..., N kg respectively. Remove the m kg weight and require

Sum of (weights < m kg) = Sum of (weights > m kg).

For what values of m, N is this possible?

David Shaw's solution, like the chief editor's, reached the condition

$$\frac{m(m-1)}{2} = \frac{N(N+1)}{2} - \frac{m(m+1)}{2},$$

which he simplified (correctly) to

$$N^2 + N - 2m^2 = 0$$

$$\text{i.e.} \quad N = \frac{-1 \pm \sqrt{(1 + 8m^2)}}{2}$$

Thus, as N is integral, we require $1 + 8m^2$ to be a square number, say k^2. Then

$$k^2 - 8m^2 = 1.$$

Now, clearly $k = 3$, $m = 1$ is a solution of this equation, and indeed

$$k^2 - 8m^2 = (3^2 - 8 \times 1^2)^n \quad \text{for} \quad n = 1, 2, 3, \ldots .$$

Factorising, we find

$$(k + \sqrt{8}m)(k - \sqrt{8}m) = (3 + \sqrt{8})^n (3 - \sqrt{8})^n.$$

Now put

$$k + \sqrt{8}\, m = (3 + \sqrt{8})^n$$

$$\text{and} \quad k - \sqrt{8}\, m = (3 - \sqrt{8})^n.$$

Adding gives

$$2k = (3 + \sqrt{8})^n + (3 - \sqrt{8})^n ,$$

and subtracting gives

$$2\sqrt{8}\, m = (3 + \sqrt{8})^n - (3 - \sqrt{8})^n.$$

We can now generate as many solutions as required by giving n the values 1, 2, 3, ... , e.g.

$n = 1$ gives $k = 3$, $m = 1$, $N = 1$ (trivial)
$n = 2$ gives $k = 17$, $m = 6$, $N = 8$
$n = 3$ gives $k = 99$, $m = 35$, $N = 49$
$n = 4$ gives $k = 577$, $m = 204$, $N = 288$

and so on. The numbers rapidly become very large, e.g.

$n = 10$ gives $k = 22619537$, $m = 7997214$, $N = 11309768$.

Another approach to Equation (*) uses the theory of infinite continued fractions. David Shaw also discussed this in terms not dissimilar from those of S.J. Newton's letters in *Function*, *Vol.8*, *Part* 2 and elsewhere. This approach allows a proof that the solutions given above are in fact the only ones.

We now move on to PROBLEM 10.1.1.

COMMENT ON PROBLEM 10.1.1.

The problem (submitted by D.R.Kaprekar) read:

A man had 115 dollars. He spent 40 of them and 75 were left. He went out again and spent 46, leaving 29. A third time he went out and spent 19 leaving 10. Finally he went out and spent the 10, leaving nothing. Here is a table.

	Spent	Left
	40	75
	46	29
	19	10
	10	0
Totals	115	114

The total at right is 114, not 115. Where is the missing dollar?

Careful thought shows that this is really a non-problem and so we do not offer a conventional solution. However, we had the following interesting comment from Garnet J. Greenbury of Brisbane.

I shall propose another problem with small numbers.

A man had 6 dollars. He spent 2 of them and 4 were left. He went out again and spent 3, leaving 1. Finally he went out again and spent 1 leaving nothing. Here is a table:

	Spent	Left
	2	4
	3	1
	1	0
Totals	6	5

The total at the right is 5, not 6. Where is the missing dollar?

A similar table is obtained if he spent 3, 1 and 2.

	Spent	Left
	3	3
	1	2
	2	0
Totals	6	5

Again, a lost dollar.

However, if he spent 3, 2 and 1

Spent	Left
3	3
2	1
1	0
6	4

There is a lost 2 dollars.

Now if he spent 1, 3, 2 or 2, 1, 3

Spent	Left
1	5
3	2
2	0
6	7

Spent	Left
2	4
1	3
3	0
6	7

he gains 2 dollars.

And if he spends 1, 2, 3

Spent	Left
1	5
2	3
3	0
6	8

he gains 2 dollars.

All the answers −1, −1, −2, +1, +1, +2 are obtained if the order of spending is changed and the loss or gain depends on that fact only.

So Kaprekar's problem can be worded in all permutations of n numbers that sum to 115. There is not only a lost dollar.

PROBLEM 10.3.1

Garnet J. Greenbury also sent us this problem, due to Lewis Carroll.

Write down any three-digit numbers whose digits are all different, say	471
Reverse the order of the digits	174
Calculate the difference	297
Reverse the order of these digits	792
Sum the last two numbers	1089

Whatever three-digit numbers we begin with − provided always that the digits are different − this last number will be 1089. Why?

PROBLEM 10.3.2

Consider the following infinite pattern of square roots

$$\sqrt{(1 + \sqrt{(1 + \sqrt{(1 + \sqrt{(1 + \sqrt{(1 + \ldots} } } } } \ .$$

What value should be assigned to it?

PROBLEM 10.3.3

Prove that the product of n consecutive numbers is divisible by $n!$ (i.e. 1.2.3.4. ... n).

HOW LONG IS A PIECE OF STRING?

Suppose a number of pieces of string, of various lengths, are knotted together in some way or other, with the only restriction that no loops are formed. An example is shown opposite.

The string figure is a physical realization of a mathematical abstraction known as a *tree*, which has great significance in areas such as computer programming and the analysis of complicated communication networks. An important problem in both these areas is to determine the *diameter* of a tree, which in a communications network would be the longest path any message could be expected to travel. In a computer program, a tree would represent steps in the program, with branches at each stage, but no loops, and the diameter of the tree would represent the maximum possible running time of the program. In our string model, the diameter of the tree is the length of the longest possible string path.

Suppose a communications engineer needs to know the diameter of a tree which represents a communications network with thousands of paths and junctions. How should he set about solving this problem? Thinking of a tree as a bunch of knotted string provides a solution.

Pick up the string tree at any point and let it dangle vertically. Then reach for the lowest point, and dangle the tree from it. The longest path in the string tree runs from the top point to the bottom point.

This two-step procedure always locates the longest path in any string tree. Can you prove that?

From *Mathematical Digest*

oo oo oo oo oo

Continued from p.15.

$$\frac{\pi}{4}a^2 + \frac{\pi}{4}b^2 = \frac{\pi}{4}c^2$$