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FUNCTION

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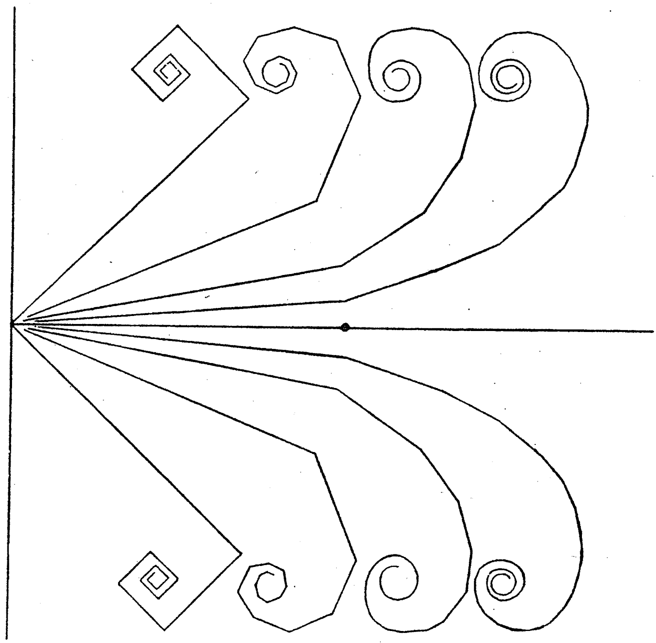

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Function is a mathematics magazine addressed principally to students in the upper forms of schools, and published by Monash University.

It is a "special interest" journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. *Function* is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

Function does *not* aim directly to help its readers pass examinations. It does guarantee to enrich and inform their mathematical lives, to provide high quality articles and stimulating reading for the mathematically inclined.

It is hoped that the readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you. Address them to:

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Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

(continued on back cover)

CONTENTS

ABOUT *FUNCTION*

Function is a journal of mathematics. It aims to present interesting mathematics and applications of mathematics which can be understood by students in their last years of secondary school.

Usually there are a number of articles on some aspect of mathematics. Such articles can deal with mathematics itself (pure mathematics), with some application of mathematics, or with that separate branch of mathematics known as statistics. Then too there can be articles on aspects of the computer, its hardware or software. Last year, for example, saw articles on Projective Geometry, Mathematics in Tennis, Quality and Statistics and Turing Machines.

Each issue has a feature cover, and usually there is an article to go with it. We try to pick one that is interesting both visually and mathematically. This issue's shows an old, but still valid, way of demonstrating a rather surprising result, which is itself only a small part of a very important chapter in modern mathematics.

Another regular feature of *Function* is the Problem Section. This is the aspect of our journal that most seeks interaction with you, the reader.

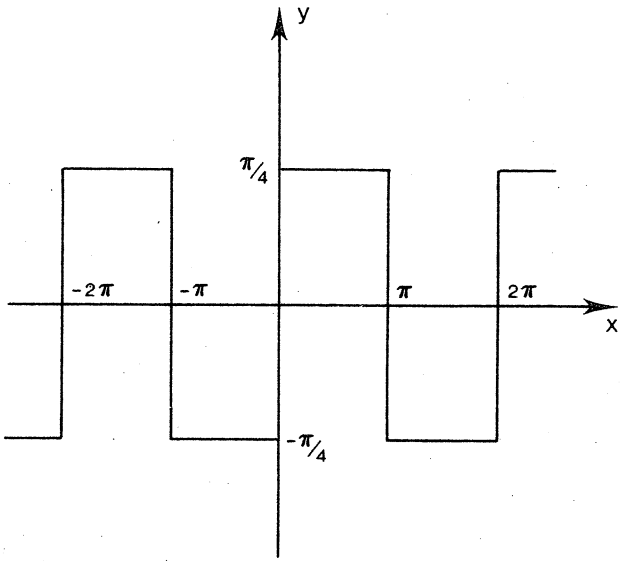
We are always very pleased to receive material from our readers. Over the last nine years, readers, especially school students,have sent us articles, covers, cartoons, letters and computer programmes. Especially, however, they have sent us problems and solutions. Many readers have found that this is the way in which they can best interact with *Function*.

The key to benefiting from *Function* is this aspect: interaction. Even when you read an article, you get most out of it when you contribute most. There is an active way to read an article - check the derivations, for example; ask your teacher about difficult points; if exercises are set, try them.

This active involvement with the material leads to much greater rewards. You will understand more, and you will be developing skills that will stand you in good stead for later work in mathematics.

THE FRONT COVER

The graph below shows a function whose values, in successive intervals of length π, are alternately $\pm \pi/4$. This function is *periodic* in that its behaviour in the interval $-\pi$ to π is repeated exactly in each successive such interval of length 2π, whether to the right or to the left.



Periodic functions can be broken down into components which are either sines or cosines, the function itself being in each case the sum of such components (in most cases infinitely many of them). This result is due to the French mathematician and physicist *Jean-Baptiste Joseph Fourier* (1768-1830), who developed it in the course of his studies on heat conduction. The breakdown of periodic functions into sine and cosine components is nowadays termed *Fourier analysis* in honour of Fourier.

The function drawn above, *f(x)* let us call it, has the following Fourier analysis:

$$f(x) = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots \quad . \qquad (*)$$

The infinite series on the right is called a *Fourier series*.

When

$$0 < x < \pi$$

the sum of the infinitely many terms will always be $\pi/4$. If we approximate by taking finitely many the sum will be approximately $\pi/4$, the approximation being (as a general rule) the better the more terms we take.

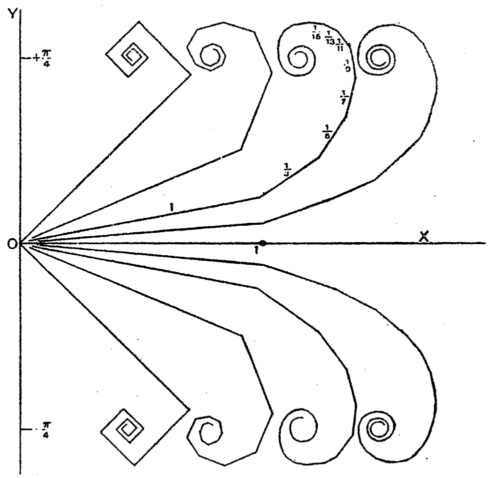
When x takes on any other value (except for multiples of π) the sum of the series will be $\pm\ \pi/4$, as may readily be deduced from the previous paragraph and the properties of the sine function.

When $x = 0$, or any other multiple of π, the terms on the right are all zero and hence the series sums to zero, the mean value or average of the values $\pm\ \pi/4$ to right and left of such points.

Our cover shows a graphical illustration of this result for eight values of x in the range $-\pi < x < \pi$, specifically

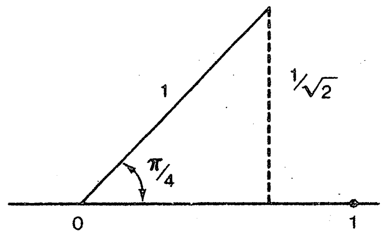
$$\pm\frac{\pi}{32},\ \pm\frac{\pi}{16},\ \pm\frac{\pi}{8},\ \pm\frac{\pi}{4}$$

(reading from right to left). Here is that picture again with some more detail shown. (It comes from Carslaw's *Plane Trigonometry*, 1909 - a once widely-used school text.)

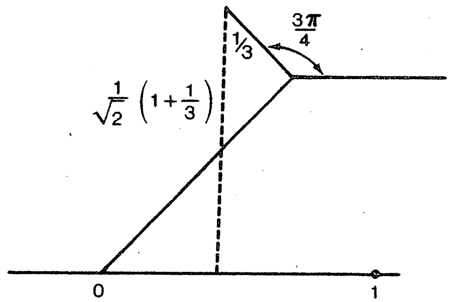


To see how it is constructed, take the top left-hand curve.

Draw a line of length 1 making an angle of $\pi/4$ (45°) with the x-axis. The right-hand end of this line will be sin $\pi/4$ ($\frac{1}{\sqrt{2}}$) above the x-axis.



Now starting at this point, take a line of length 1/3 making an angle of $\frac{3\pi}{4}$ with the right-ward horizontal thus:



In this way, we reach, as the diagram shows, a height of $\frac{1}{\sqrt{2}}\ (1 + \frac{1}{3})$.

Continuing in this way, we find the spiral shown on the cover. Note that we have also proved the relation

$$\frac{\pi}{4} = \frac{1}{\sqrt{2}}\ (1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \frac{1}{15} + \ldots),$$

which you might care to check on a computer or calculator.

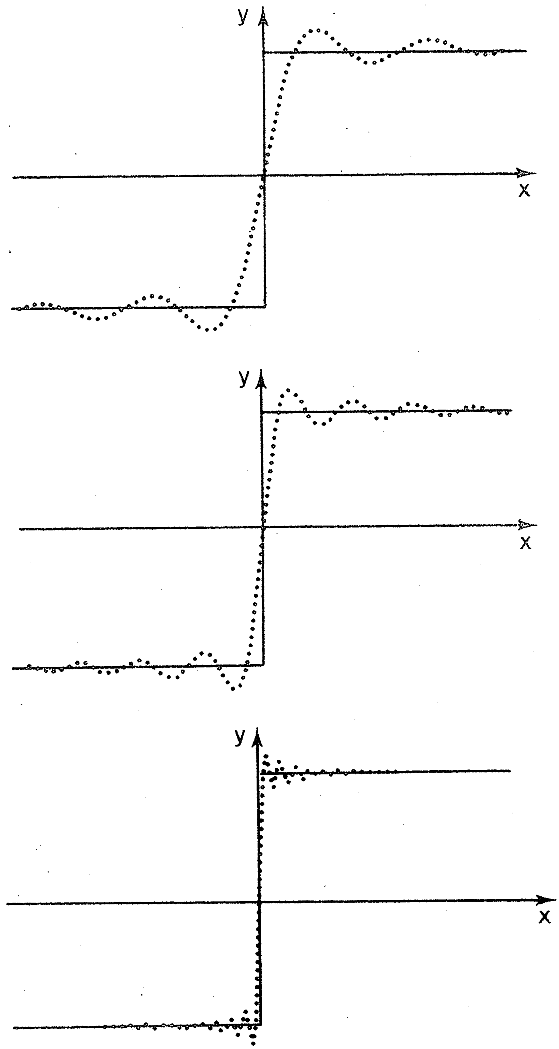
Similar formulae for π come from the other spirals; what are they? What about other values, like $\pi/6$, $\pi/3$ or $\pi/2$? (This last is particularly interesting : the result is usually referred to as *Gregory's series* - see *Function, Vol*.4, *Part* 1.)

As another interesting exercise, consider the horizontal projections of the curves - what do we get from these? (Such an investigation could make a nice project for (say) the Maths Talent Quest - or you could write it up and send it to us.)

There is a lot more to this topic - just to show you one more aspect, we give the result of a computer graphic approach to Equation (*).

The series (*) was summed by a computer, taking in the first instance twelve, in the next case twenty-four, and finally one hundred terms.

The results were compared with the value of f(x), particular attention being paid to the region near x = 0, where, because of the "corners", the approximation is least accurate. It is readily seen that quite good approximations are achieved (relatively) quickly, and in the final case, only the near neighbourhood of the corner shows appreciable error in the approximation.



FUNCTIONS OF SEVERAL VARIABLES

Colin Fox, Melbourne C.A.E.

Given a square of side length ℓ units, the measure of its area A is given by the formula $A = \ell^2$. That is, *A is a function of ℓ*. Using function notation, this is written $A = f(\ell)$.

Now consider a rectangle ℓ units by w units. To calculate its area, we use the formula $A = \ell w$. In this case, *A is a function of ℓ and w*. That is, A is a function of *two* variables and this can be written $A = f(\ell, w)$. So, for a rectangle with width 18cm and length 25cm

$$\begin{aligned} A &= f(25,\ 18) \\ &= 25 \times 18 \\ &= 450\ . \end{aligned}$$

Another familiar function of two variables is the volume of a cylinder:

$$V = \pi r^2 h\ .$$

Here, we can write $V = f(r, h)$.

The simple interest formula provides an example of a function of *three* variables:

$$I = \frac{PRT}{100}\ .$$

$$I = f(P, R, T)\ .$$

For example, the interest earned if \$600 is invested for two years at 12% per annum can be calculated as follows:

$$\begin{aligned} I &= f(600,\ 12,\ 2) \\ &= \frac{600 \times 12 \times 2}{100} \\ &= 144\ . \end{aligned}$$

In many applications of mathematics, the functions involved have two or more variables. For some nice examples of this, see the article 'The Mathematics of Measurement' in this issue of function.

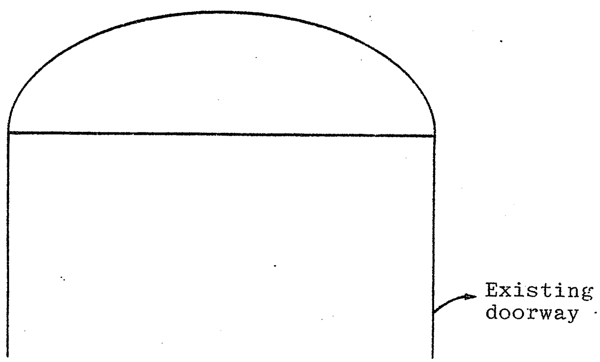
DESIGNING AN ARCH

Neil S. Barnett,

Footscray Institute of Technology

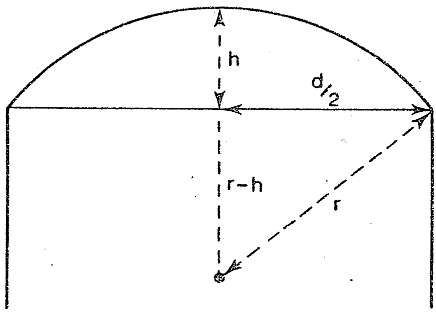
Recently a member of the Institute's maintenance crew 'dropped in' to obtain help in designing an arch that was required to be built above an *existing* doorway (the wall above the door had to be cut out). He related that as an apprentice he vaguely remembered a technique for drawing the shape of an arch using a bit of chalk and a piece of string; however, he could not recall the method exactly. My visitor was quite adamant that he didn't want a circular arch but rather an 'oval one'.

I assumed that his not wanting a circular arch meant rather that he didn't want a *semi*-circular one (although with the dimensions he gave this was not possible) and that a circular cap would be an acceptable possibility. Given the width of the doorway and the height to which the arch should reach I set about providing some alternatives for design. He was not of course interested in the detail, just the end result and an easy method of execution. This is an important principle that should always be kept in mind when offering a mathematical solution to a practical problem. In most instances the end user doesn't want to know (and probably wouldn't understand anyway) all the 'gory' mathematical detail - just the relevant procedure and a rough method to see if the result is reasonable. In this particular instance the drawing is evidence enough of whether or not the method will provide a satisfactory result.



Armed with the door width d and the required arch height h , I provided the following methods for drawing the arch.

Method 1: Circular Cap



The problem is to find r in terms of the known quantities d and h (r is the radius of the circle of which an arc is a cross-section of the arch).

By Pythagoras' Theorem

$$(r - h)^2 + \left(\frac{d}{2}\right)^2 = r^2 .$$

So

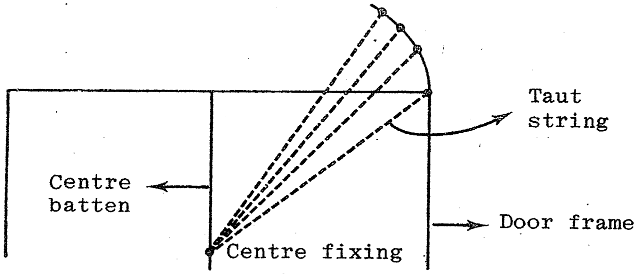
$$r^2 - 2rh + h^2 + \frac{d^2}{4} = r^2 ,$$

and we conclude that

$$r = \frac{4h^2 + d^2}{8h} .$$

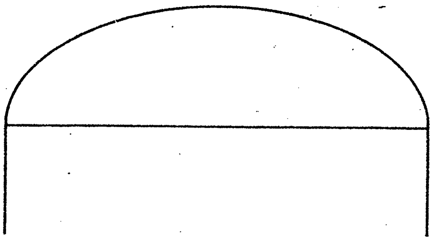
The specified values were h = 20cm, d = 90cm, and so r = 60.6 cm.

The drawing of this arch requires the fixing of a batten from the middle of the top door frame to the floor below in a vertical position. A string of length 606 mm with chalk or pencil attached can then be held taut so that one end is in contact with the centre batten and the other end in contact with either of the top corners of the door frame. Fixing the string at this point to the centre batten enables the other end of the string (still held taut) to be drawn across the top of the door with the chalk describing the form of the arch.



Method 2: Elliptical Arch

It is possible to construct the arch to be the top half of an ellipse.

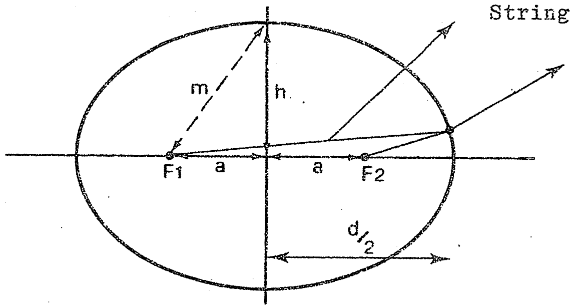


Using an ellipse with major axis d and minor axis $2h$ as the arch (both are known) the equation of it is :

$$\frac{x^2}{\left(\frac{d}{2}\right)^2} + \frac{y^2}{h^2} = 1$$

where the middle of the top of the door frame, the centre of the ellipse, is taken to be the origin.

An ellipse can be defined in a number of ways. One of these is that it is the path traced out by a point that moves in a plane so that the sum of its distances from two fixed points in that plane is constant. This means that if we take a length of string and attach its ends to separate points (ensuring that the string is slack) a pencil or chalk looped into the string and the whole pulled so that the string is taut will, when moved round, describe an ellipse. The two points at which the string is attached are called the foci. From the practical point of tracing an elliptical arch the only problem remaining is to find the exact location of these foci. With these and using the above technique the elliptical arch can easily be drawn.



We need to find the distance a .

Let the string length be $2m$, clearly then:

$$2m = (\frac{d}{2} + a) + (\frac{d}{2} - a) \qquad \text{i.e.} \quad 2m = d$$

thus the string length needs to be the same as the door width.

When the pencil or chalk is vertically above the centre at its maximum height the geometry dictates that:

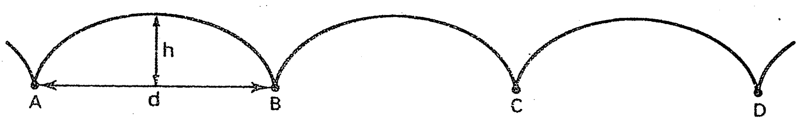
$$h^2 + a^2 = m^2 \qquad \text{i.e.} \quad a = \sqrt{\frac{d^2}{4} - h^2} \; .$$

Hence, for h = 20 cm and d = 90 cm, $a \doteq 40.3$ cm .

The two foci are thus 4.7 cm from either side of the door frame. The string, 90 cm long, is attached to the top of the door frame 4.7 cm from the side, keeping the string pulled tight the chalk can then be drawn across the top of the doorway scribing the form of the appropriate semi-elliptical arch.

Method 3: Cycloidal Arch

Another simple method (although a little more sophisticated than Methods 1 and 2) is to use a rolling circle to draw an arch. If a circle of prescribed diameter is rolled on a horizontal surface, a pen attached to a fixed point on its circumference will draw the following (called a cycloid):



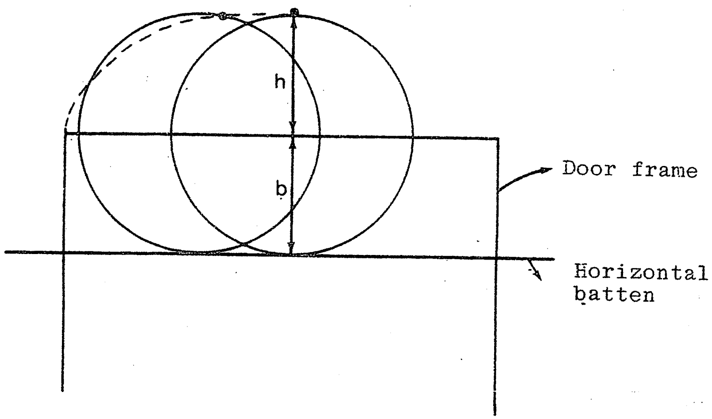
The height of each arch is equal to the diameter of the rolling circle and each corresponds to one complete revolution of it. Thus AB, BC, CD are all equal to the circumference of the rolling circle. (It can be shown that the length of one arch is 4 times the diameter of the rolling circle.) Constructing an arch in this manner means that there is a fixed relationship between the height of the arch and the width, namely:

$$\pi h = d \quad .$$

So if d = 90 cm then h must be 28.6 cm.

If we can persuade the carpenter to settle for a height of 28.6 cm rather than the requested 20 cm, then an arch can be traced using a rolling circle of diameter 28.6 cm with a pen attached to a fixed point on its circumference and the whole rolled along the top of the door frame starting with the pen point at one end of the frame.

It is of course possible to draw a variety of arches using different diameters for the rolling circle giving the arch height and width *exactly* as specified. This can be accomplished by placing a horizontal batten below the top of the door frame and rolling the circle along it rather than along the top of the doorway as in the foregoing.



The cycloid approach necessitates making a circular template for practical use, so the method is somewhat restrictive. It should be pointed out, however, that for regular construction of arches over *standard* size doorways and openings a template of the arch would be the easiest tool to use. This would certainly be the case if the arch had to be built rather than cut out, in fact you will often see bricklayers doing this very thing. The templates themselves have to be constructed and so the foregoing methods can be used to construct these too.

So a little mathematics can indeed be a very useful thing.

LETTER TO THE EDITOR

THE GREGORIAN OVER-CORRECTION

After reading 'The Calendar' by S.Rowe, *Function, Vol.9, Part* 4, I wish to suggest that the Gregorian correction over-corrects the perfect calendar by $1\frac{1}{2}$ days every 5000 years. According to Rowe's figures, the number of days in 1 year is 365.242196.

When we work on a 365 day year, we are neglecting 0.242196 of a day each year.

$$0.2422 = \frac{1211}{5000} \quad \text{(neglecting 0.000004 of a day).}$$

This means that the 'perfect' calendar has to be corrected 1211 days over 5000 years (or 121098 days over 500000 years). This may be accomplished by making every fourth year a leap year of 366 days. (This adds 1250 days in 5000 years) unless it occurs at the end of century. (From the 1250 days added, we now subtract 50 days leaving 1200 additional days every 5000 years), except that every fifth century is to be counted as a leap year. (Thus 10 days are added to the 1200 making 1210). At the end of every 5000 years there would have to be a double leap year. (There would appear a February 30.) So much for the 'perfect calendar (neglecting 0.000004 of a day in 5000 years).

However, the Gregorian correction to our calendar is as follows: Every year the number which is divisible by 4 is a leap year, excepting the last day of each century, which would not be a leap year unless perfectly divisible by 400. This Gregorian correction requires that 970 days be added every 4000 years.

When 970/4000 is converted to a fraction with a denominator of 5000, there is obtained 1212.5/5000.

Thus the Gregorian correction over-corrects the 'perfect' calendar by $1\frac{1}{2}$ days every 5000 years (neglecting .000004 of a day).

Garnet J. Greenbury, Brisbane.

THE MATHEMATICS OF MEASUREMENT

Michael Deakin, Monash University

In February, 1871, *Scientific American* carried a report of a highly successful heavier-than-air flying machine. It was steam-powered, and proved itself capable of rising to a height of 200 feet. It travelled horizontally for a distance of "about a block" at a public display in San Francisco. This performance was considerably better than many logged as "flights" by pioneer aero clubs. It must be remembered, however, that the machines involved there were much larger - large enough, indeed, for the machine's operator to become airborne along with his machine. The quest for heavier-than-air flight involved not only the production of a flying machine, but one that was capable of carrying its pilot (at minimum) aloft.

The small toy demonstrated a hundred years ago was not large enough for this. The production of a larger model based on the same design has never proved feasible. Indeed, despite the efforts of a few enthusiasts, practical steam-planes seem likely to remain beyond the realms of possibility (and certainly beyond those of economic sense). Airplanes in the strict sense became feasible only after the advent of the internal combustion engine.

The difficulty with the steamplane is one of *scaling*. We are all familiar with the idea of a scale drawing, or a scale model. All the geometric features of one system may, by use of these devices, be reproduced (except for size) in another. Who has not taken delight in those intricately constructed dolls' houses that go to such pains to reproduce exactly all the appearances of a large mansion? Perhaps we are partly captivated by the scale involved, but also the intricacy of the workmanship holds our attention. For the construction of a really good scale model demands enormous care over details - the paint, for example, needs to be suitably thin.

The idea that some modelling principle may apply beyond the realms of geometry (as that term is popularly applied) is an attractive one. The simple-minded approach, however, of relying on geometrical scaling alone breaks down, as the case of the steamplane shows. That an extension of the geometrical laws of scaling (known to the Greeks via the theory of similar figures) did not hold in the obvious manner is perhaps an insight we owe to Galileo.

Modern theory in this area begins with an analysis of measurement and what it entails.

Whenever we measure a quantity, this measurement must be made with reference to some agreed unit. We have units of length, such as the metre, units of mass, such as the kilogram, units of time, such as the second, and so on.

The three units just listed are, in fact, three of the basic units of the very widely accepted SI (for Systeme Internationale). Australia has, for about a decade now, adopted SI units (with some minor modifications) for its everyday life, industry and commerce, and indeed so have almost all countries - the only major exception being the U.S.A.

Furthermore, as we shall see, the three units already mentioned suffice (in a way to be explored later) to deal with all measurements in the science of mechanics (the response, in terms of changes of observed motion, that bodies display when they are acted on by forces). Other aspects of measurement require further units for quantities like temperature, electric current, and so on.

But let us get back to mechanics. There is nothing particularly sacrosanct about the SI units. They are convenient and widely accepted. But other systems exist - until recently we used another ourselves. The choice of a system of units is a social convention - no more. Until recently, we used an older, "Anglo-Saxon", system and (with minor differences) this is still in use in the U.S.A. today.

What is required, if science is to progress, is that researchers using different systems be able to talk to one another. It has to be possible for two investigators each to translate their measurements into the other's units and to reach agreement as to results.

This is trivially easy, as long as we stick to basic measurements such as those of length, mass and time. All we need are basic conversion tables, such as:

$$\left.\begin{array}{r}1\text{ mile} = 1609\text{ m}\\ 1\text{ pound} = 0.454\text{ kg}\\ 1\text{ hour} = 60\text{ s}\end{array}\right\} \qquad (1)$$

Nor is the matter greatly more complicated when we consider the so-called "derived" quantities. "Derived" because they are derived from the basic quantities. Consider the case of speed.

Take two towns 70 km apart and suppose a car undertakes this journey in the time of three-quarters of an hour. In SI units, it has travelled 70000 m in 2700 s and so its speed has been 25.9 ms^{-1}. But an American would say that it has travelled 43.5 miles in three-quarters of an hour and so it has gone at a speed of 58 mph. So

$$25.9\ ms^{-1} = 58\ mph.$$

Let us now generalise this situation. Take two observers A, B. Let ℓ_A be A's measure of the length of the journey and ℓ_B be B's measure of the same length. Similarly, let t_A, t_B be the corresponding measures of time. Now each observer will derive, from his two measurements, a speed and this speed will be a description, from each observer's viewpoint, of an event witnessed by both.

Observer A finds a speed v_A, where

$$v_A = \ell_A / t_A \quad , \tag{2}$$

while observer B notes a speed v_B, where

$$v_B = \ell_B / t_B \quad . \tag{3}$$

But, as long as we have a conversion table for our basic units, these matters can be reconciled. Suppose $\ell_A = L\,\ell_B$, where L is an appropriate conversion factor like the 1609 of Display (1) above. Similarly, we can suppose $t_A = T\,t_B$. We now wish to find the appropriate conversion factor, V say, for for speed. I.e. if

$$v_A = V\,v_B \tag{4}$$

what is V ?

But this is very easy. A little manipulation of Equations (2,3,4) produces

$$V = L/T = LT^{-1} \quad . \tag{5}$$

This equation is often put into words by saying that speed has the *dimensions* of length over time. It comes out in everyday language in our use of phrases like "metres per second" or shorthand like ms^{-1} .

Other, more complicated, quantities also have *dimensions*, i.e. appropriate relations to the basic units of measurement. Take, for example, g, the acceleration due to gravity at the earth's surface. This quantity is not precisely constant, but is almost so, and its value is normally quoted as $9.81\ \mathrm{m\,s}^{-2}$. How would an American measure it?

Well ms^{-2} translates into LT^{-2}, where L is the number of feet (the basic unit our American would use) in a metre, and T (the number of seconds in a second) is one. We find, from Display (1) and a bit of work, that $L = 3.28$ and this means that $g = 32.2\ \mathrm{fs}^{-2}$ (f standing for feet). This is in fact close to the accepted value in the USA, though it is frequently rounded to 32, the nearest integer.

Force is connected to our basic units by an empirical relation, Newton's second law. If F represents force,

m mass and a acceleration, this reads

$$F = ma \ , \tag{6}$$

So force is measured in units of mass times acceleration. Its dimensions, commonly denoted by $[F]$, are thus given by

$$[F] = MLT^{-2} \ , \tag{7}$$

where L, T have their previous meanings and M is a similar ratio, referring to mass.

We can do the same thing with other equations, too. Take Newton's law of gravitation

$$F = G\, m_1 m_2 / r^2 \ , \tag{8}$$

The left hand side, as we have just seen, has dimensions MLT^{-2}. So, therefore, for consistency, has the right. But its dimensions are clearly $[G]M^2L^{-2}$, where $[G]$ stands for "the dimensions of G". Hence

$$[G]M^2L^{-2} = MLT^{-2}$$

or

$$[G] = M^{-1}\, L^3\, T^{-2} \tag{9}$$

Similarly, we may find the dimensions of any mechanical quantity.

One point does, however, need to be made - about angles. The radian measure of an angle is the ratio of two distances, arc-length divided by radius. So, in fact, are all other measures of angle (degrees, etc.). These others, however, measure the two distances in different units, a complication best avoided. So we will measure all angles in radians and say they are dimensionless. Their dimensions are $L\,L^{-1}$ or I, as it is written. I behaves like 1 does in ordinary algebra. Angles are, in fact, measured as pure numbers.

Now all this has been quite straightforward and you might quite legitimately ask why we make such a fuss about it. We now move on and see that such considerations can actually be used to deduce physical laws.

Suppose, for example, we wish to know the speed with which a particle, dropped from rest at a height h, will strike the earth. Neglecting air resistance, we write down all the relevant physical quantities, with their dimensions:

v (the required speed) : LT^{-1}
h (the given height) : L
m (the particle's mass): M

and we mustn't forget

g (the gravitational constant) : LT^{-2} .

Our physical law must be of the form

$$v = f(h,m,g) \ . \tag{10}$$

where f is some function† of the three variables given. This must have the dimensions LT^{-1} , as it is to give us a velocity. Immediately we see that mass cannot be involved, and almost as immediately we see that h,g must combine as $\sqrt{hg}$. Thus

$$v = k \sqrt{hg} \tag{11}$$

where k is some constant.

Theoretically, the value of k is $\sqrt{2}$, and this could be learned approximately by one experiment, instead of the many we might imagine we need to determine f in Equation (10).

To proceed further, we repeat the calculation in apparently more complicated form. Rewrite Equation (11) as

$$v/\sqrt{hg} = k \ . \tag{12}$$

Here both sides have dimension I ; they are purely numerical. The left-hand term is called a *dimensionless ratio*. It is usual to proceed by forming from the mechanical quantities involved, all the different dimensionless ratios that can be formed. If n quantities are believed to be involved and if there are r basic quantities (in mechanics, $r = 3$) , then there are $n-r$ such ratios. As these are designated $\Pi_1, \Pi_2, \ldots, \Pi_{n-r}$ with the capital Greek letter pi, and as it was first stated by the physicist Buckingham, this result is called *Buckingham's pi theorem.*

This theorem goes on to state that the physical law sought must have the form :

$$f(\Pi_1, \Pi_2, \ldots, \Pi_{n-r}) = 0 \ . \tag{13}$$

If we apply this to our case, we see that $n = 4$, $r = 3$ and so there is only one dimensionless ratio, namely $v/\sqrt{hg}$. Thus Equation (13) becomes

$$f(v/\sqrt{hg}) = 0 \ . \tag{14}$$

Assume Equation (14) has a root, k say, and so reach

$$v/\sqrt{hg} = k \ ,$$

i.e. Equation (12).

† See the article "Functions of Several Variables" in this issue of *Function*.

As a further example of the use of the pi theorem, we deduce Kepler's third law of planetary motion. The quantities involved, with their dimensions are :

The time taken to orbit the sun, τ : T

Newton's gravitational constant, G : $M^{-1} L^3 T^{-2}$

The mass of the sun, m_s : M

The mass of the planet, m_p : M

The average distance of the planet from the sun, a : L

The eccentricity (or deviation from the circular) of the planet's orbit, e : I .

These six quantities combine into three dimensionless ratios that can be found by inspection. They are

$$\Pi_1 = G M_s \tau^2 a^{-3} , \quad \Pi_2 = m_p/m_s , \quad \Pi_3 = e .$$

So we have

$$f(G M_s \tau^2 a^{-3} , m_p/m_s , e) = 0 . \tag{15}$$

This is an equation in the three quantities in the parentheses; suppose we can solve for the first of these. This yields

$$G m_s \tau^2 a^{-3} = \phi(m_p/m_s , e) \tag{16}$$

for some function ϕ .

Now in our solar system, m_p/m_s and e are both small in almost every case, and so we may approximate them by zero and write $\phi(0,0) = A$ (say) . Equation (16) now becomes, after some rearrangement,

$$\tau \simeq \sqrt{\frac{Am_s}{G}} \; a^{3/2}$$

or, to a good approximation ,

$$\tau \propto a^{3/2} , \tag{17}$$

which is Kepler's third law.

More exact analysis gives

$$\phi(m_p/m_s , e) = 4\pi^2/(1 + m_p/m_s). \tag{18}$$

(e turns out not to be involved after all.)

One of the most spectacular such calculations concerned the first atomic test in 1944. The U.S. military wished to keep secret the actual energy of this blast, but in 1947 they declassified a film of it. On the basis of an analysis he had carried out in 1941, the British physicist G.I.Taylor was able to compute the still supposedly secret energy. So too was the Soviet physicist L.I.Sedov.

Here is how they did it. As before, we begin with a list of the quantities involved and their dimensions. We have

The energy of the explosion, E : ML^2T^{-2}

The time since detonation, t : T

The radius of the fireball, R : L

The density of air outside the fireball, ρ_o : ML^{-3}

The density of air inside the fireball, ρ_i : ML^{-3}

The pressure of the air outside the fireball, P_o : $ML^{-1}T^{-2}$

The pressure of the air inside the fireball, P_i : $ML^{-1}T^{-2}$.

So $n = 7$ and $r = 3$. There are four dimensionless ratios, which are not too hard to find. We get

$$\Pi_1 = E\rho_o^{-1}R^{-5}t^2 \ , \ \Pi_2 = P_o^5 t^6 E^{-2}\rho_o^{-3}, \ \Pi_3 = \rho_i/\rho_o \ , \ \Pi_4 = P_o/P_i$$

We can now write, applying the Buckingham pi theorem and the argument used to reach Equation (16),

$$\Pi_1 = \phi(\Pi_2,\Pi_3,\Pi_4) \quad . \tag{19}$$

Now, very plausibly, Π_3 and Π_4 are very small. Π_2 is a little harder, as it involves the unknown energy E. However, $P_o^5 \, t^6 \, \rho_o^{-3}$ is measured in units of energy squared and all its components refer to quite normal pressures, densities and times. So we would expect it to be the square of a normal energy - i.e. much smaller than an energy released by an atomic explosion. So we assume Π_2 is small also. We thus write

$$E\rho^{-1}R^{-5}t^2 \simeq \phi(0,0,0) \quad . \tag{20}$$

Taylor wrote, so he could test it, this formula as

$$R = A\,t^{2/5} \tag{21}$$

where

$$A = \left(\frac{E}{\rho_o\ \phi(0,0,0)}\right)^{1/5} . \tag{22}$$

He now took the logarithm of Equation (21) and so rewrote it as

$$\mathsf{R} = \mathsf{A} + \frac{2}{5}\,\mathsf{T} \tag{23}$$

where

$$\mathsf{R} = \log R,\ \mathsf{A} = \log A\ ,\ \mathsf{T} = \log t$$

so a plot of R versus T should be a straight line with slope 2/5. By analysing the film frame by frame, Taylor (and presumably Sedov also) was able to verify this. The formula is exceptionally accurate for times less than about a tenth of a second.

A, or log A, is the intercept of the graph and this can be read off, so that A is known. From experience with conventional explosives, $\phi(0,0,0)$ could be found, and so Equation (22) completes the calculation.

What this analysis achieves is the scaling up of an experiment involving conventional explosives and relatively low energies to one involving atomic bombs and their high energies. To appreciate the question of this scaling - the step in development from model to prototype - consider a simpler case. Let us consider again our opening example.

If now we scaled the model plane up by multiplying all its linear dimensions by a factor of L, flight (for sufficiently large L) would cease to be possible. The reason for this is that the lift on the plane - the force that holds it up - is roughly proportional to the area of the wings, and this increases by a factor of L^2 in the scaling up. The downward force, the weight, however, increases by a factor of L^3, and since, in scaling up, $L > 1$, $L^3 > L^2$, so that this downward force will ultimately become too great for the lift to overcome it. When this happens, flight becomes impossible.

This effect can be seen in nature. Small birds are capable of extremely agile and acrobatic flight, whereas the largest birds are flightless. The heaviest flying bird, the South American condor, only manages to stay in the air by using updraughts to keep it aloft. The largest bird capable of sustained power flight is probably the trumpeter swan.

This same effect - a scaling of volume as L^3 and a scaling of area as L^2 - appears in other areas as well. Figure 1 shows the graphs of kL^2 and KL^3. Initially the second of these lies below the first, but ultimately it overtakes it. In the example, the power required for flight comes to exceed the power available for flight.

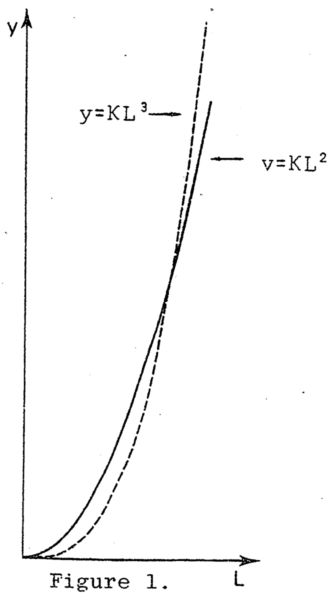


Figure 1.

Suppose a horse were enlarged to the size of an elephant. The weight increases as L^3 while the structural strength of its legs increases only as L^2. There are animals the size of elephants, but they have comparatively thicker legs than horses.

Another example comes from atomic physics. If a mass of plutonium is collected together, some of the neutrons generated in its interior will trigger the production of further neutrons by nuclear fission. The number of plutonium atoms available for fission is proportional to the volume, i.e. to L^3. Other neutrons escape out the surface of the plutonium - this number is proportional to the surface area, i.e. to L^2.

From Figure 1, the first process must eventually predominate if we increase L sufficiently. This is why plutonium has a critical mass. Above this, the neutrons are captured in large enough numbers to overcome the leakage and a chain reaction occurs.

from The Loves of the Triangles

Stay your rude steps, or o'er your feet invade
The Muses' haunts, ye sons of War and Trade!
Nor you, ye Legion Fiends of Church and Law,
Pollute these pages with unhallow'd paw!
Debased, corrupted, grovelling, and confined,
NO DEFINITIONS touch *your* senseless mind;
To *you* no POSTULATES prefer their claim,
No ardent AXIOMS your dull souls inflame;
For *you* no TANGENTS touch, no ANGLES meet,
NO CIRCLES join in osculation sweet!
For *me*, ye CISSOIDS, round my temples bend
Your wandering Curves; ye CONCHOIDS extend;
Let playful PENDULES quick vibration feel,
While silent CYCLOID rests upon her wheel;
Let HYDROSTATICS, simpering as they go,
Lead these light Naiads on fantastic toe;
Let shrill ACOUSTICS tune the tiny lyre;
With EUCLID sage fair ALGEBRA conspire;
The obedient pulley strong MECHANICS ply,
And wanton OPTICS roll the melting eye!

John Hookham Frere: 1798

RADICAL EQUATIONS†

When we solve an equation each line of the solution is usually a statement equivalent to the statement made on the line above it. That is to say the solution set is preserved. Sometimes, however, we need to use steps that do not possess this property, but are "weaker" and merely say that the new solution set will include the original solution set. When this happens we say that the step is *irreversible*.

To illustrate this point, consider

$$x = 3 .$$

This is a very simple equation and the solution set is clearly $\{3\}$.

If, however, we square both sides we get

$$x^2 = 9 ,$$

whose solution set is $\{-3,3\}$, which is not the earlier solution set, but includes it. The squaring operation is irreversible.

These considerations become important with *radical equations* - those involving a square root or "radical" sign. Consider for example

$$x - \sqrt{3x-2} = 4 . \qquad (1)$$

To solve this equation, put $y = \sqrt{3x-2}$. Then

$$y^2 = 3x - 2 ,$$

i.e.

$$x = \frac{y^2+2}{3} .$$

Equation (1) now becomes

$$\frac{y^2+2}{3} - y = 4$$

i.e.

$$y^2 - 3y - 10 = 0 .$$

† This article is based on one by Bill Bompart, Augusta College, Georgia, USA, which appeared in *The Two-Year College Mathematics Journal* (now *The College Mathematics Journal*), Vol.13, No.3 (June 1982) pp.198-199. We thank Professor Bompart for his permission to use this material in *Function*.

This has the roots 5, -2 .

We thus have either

$$\sqrt{3x - 2} = 5 \tag{2}$$

or

$$\sqrt{3x - 2} = -2 . \tag{3}$$

But Equation (3) has no solution, as the left-hand side (with no sign attached to the square root sign) is taken to be positive and so cannot equal -2. Thus we must have Equation (2) and this has the clear solution $x = 9$. This is readily seen to satisfy Equation (1).

Suppose we tried to solve Equation (3). We might square both sides and simplify to reach $x = 2$. This does not satisfy Equation (3), as it gives a left-hand side of +2, not -2. Nor does it satisfy Equation (1), but rather the related equation

$$x + \sqrt{3x - 2} = 4 . \tag{4}$$

The more usual technique for solving radical equations like Equation (1) is as follows. From Equation (1), we find

$$\sqrt{3x - 2} = x - 4 . \tag{5}$$

Square both sides :

$$3x - 2 = (x - 4)^2 , \tag{6}$$

i.e.

$$x^2 - 11x + 18 = 0 ,$$

i.e. $$x = 9 \quad \text{or} \quad 2 .$$

It is now necessary to check these by substitution back into Equation (1). We find that $x = 9$ is a solution, but $x = 2$ is not.

This method, by which the square root is isolated to allow a squaring, does not distinguish, after that squaring, between Equation (5), equivalent to Equation (1), and

$$- \sqrt{3x - 2} = x - 4 , \tag{7}$$

equivalent to Equation (4). When Equation (5) is squared, the result is Equation (6), and when Equation (7) is squared, the result is also Equation (6). If we take the square root of Equation (6), we get

$$\pm \sqrt{3x - 2} = x - 4 . \tag{8}$$

The process of squaring Equation (5) is *irreversible* - i.e. if we try to undo the effect of the squaring, by taking a square root, we do not return to Equation (5), but reach the ambiguous Equation (8). This is the reason for the extraneous root $x = 2$ found by the usual approach. This is a root of Equation (7), or equivalently Equation (4), not of the equation we set out to solve.

Compare the logic of the first method. Here, *because we know* $\sqrt{3x - 2}$ *is positive*, we reject Equation (3). Had this read $-\sqrt{3x - 2} = -2$, we would reach $x = 2$, which is a valid solution of *this* equation, but not of the equation we set out to solve. Thus we reject, without its ever really arising, the line of reasoning that leads to the spurious solution.

This technique depends on two simple facts. Let a, b, c represent constants and x the unknown quantity. Then:

(1) Squaring both sides of the equation

$$\sqrt{ax + b} = c \quad , \tag{9}$$

where c *is known to be positive, is* a reversible operation;

(2) The equation

$$\sqrt{ax + b} = c \tag{10}$$

where c *is known to be negative,* has no real roots.

The method given first, that is to say the new, as opposed to the usual, method, is designed to make systematic use of these facts and to reach, in a systematic way, equations like Equation (9) or Equation (10).

Let us now see this method in operation in a more complicated situation. Consider

$$\sqrt{2x + 3} = 1 + 2\sqrt{x - 7} \ . \tag{11}$$

First replace each radical with a new quantity. This introduces the *defining equations*

$$y = \sqrt{2x + 3} \tag{12}$$

$$z = \sqrt{x - 7} \tag{13}$$

and rewrites the original equation as

$$y = 1 + 2z \ . \tag{14}$$

Equations (12), (13), (14) now form a system of three simultaneous equations in three unknowns. To solve this system, square each of Equations (12), (13) to reach

$$y^2 = 2x + 3$$

$$z^2 = x - 7$$

which allow us to eliminate x and write

$$y^2 = 2z^2 + 17 .$$

But Equation (14) gives

$$y^2 = 4z^2 + 4z + 1$$

and so

$$4z^2 + 4z + 1 = 2z^2 + 17$$

or, after simplification,

$$z^2 + 2z - 8 = 0 .$$

This gives

$$z = 2 \quad \text{or} \quad -4 .$$

But we know, from our second simple fact, that z, defined by Equation (13), must be positive, and so cannot equal -4. We thus have $z = 2$ and so find, from Equation (13) and the first simple fact, that $x = 11$ and this is the only root of the original equation.

Admittedly the procedure for this second example is a little involved, but compare it with the usual approach. This begins by squaring Equation (11) :

$$\begin{aligned} 2x + 3 &= (1 + 2\sqrt{x - 7})^2 \\ &= 1 + 4(x - 7) + 4\sqrt{x - 7} \\ &= 4x - 27 + 4\sqrt{x - 7} . \end{aligned}$$

So we find

$$- 2x + 30 = 4\sqrt{x - 7} ,$$

that is

$$- x + 15 = 2\sqrt{x - 7} .$$

A second squaring will now eliminate the radical sign.

$$x^2 - 30x + 225 = 4(x - 7) .$$

I.e.

$$x^2 - 34x + 253 = 0 .$$

The roots of this equation are 11, 23 and these must now be tested in Equation (11) . It is found that the first of these works, but not the second, which satisfies the related equation

$$-\sqrt{2x+3} = 1 - 2\sqrt{x-7} \ .$$

PROBLEM SECTION

Each issue of *Function* contains a number of problems which readers are invited to attempt. We are always pleased to receive problems and/or solutions from our readers. Many find that this is a way in which they can interact constructively with *Function*.

We begin by solving some of those still outstanding.

SOLUTION TO PROBLEM 9.3.2

This problem asked for a proof that a triangle with angles A,B,C is equilateral if and only if

$$\cot C + \cot B + \cot C = \sqrt{3} \ .$$

Devon Cook (Urrbrae Agricultural High School, Netherby, S.A.) sent us his solution, which follows.

(1) If the triangle is equilateral, $A = B = C = \frac{\pi}{3}$ (60°) and so $\cot A + \cot B + \cot C = 3/\sqrt{3} = \sqrt{3}$.

(2) If $\cot A + \cot B + \cot C = \sqrt{3}$,
$\cot A + \cot B - \cot(A+B) = \sqrt{3}$, since $C = \pi - (A+B)$.

Now we let $x = \cot A$, $y = \cot B$ and

$$\cot(A+B) = \frac{\tan A \tan B - 1}{\tan A + \tan B} = \frac{1-xy}{x+y} \ , \quad \text{thus}$$

$$x + y + \frac{1-xy}{x+y} = \sqrt{3} \quad \text{so}$$

$$y^2 + y(x-\sqrt{3}) + (x^2 - \sqrt{3}x + 1) = 0$$

where the discriminant is $-(\sqrt{3}x - 1)^2$.

Since the discriminant is always negative unless $x = 1/\sqrt{3}$ and therefore by substitution, $y = 1/\sqrt{3}$, we have unique solutions $\cot A = 1/\sqrt{3}$ and $\cot B = 1/\sqrt{3}$; thus A,B are $\pi/3$. Thus $C = \frac{\pi}{3}$ and the triangle must be equilateral.

The proposer, John Barton, of Drummond Street, North Carlton, also supplied a solution. We give this also.

Prove that a triangle with angles A,B,C is equilateral if $\cot A + \cot B + \cot C = \sqrt{3}$.

For the present, keep C fixed, and use the fact that $B = \pi - A - C$.

$$\begin{aligned}
\text{Let } f(A) &= \cot A + \cot B + \cot C \\
&= \cot A - \cot(A + C) + \cot C . \\
\text{Then } f'(A) &= -\operatorname{cosec}^2 A + \operatorname{cosec}^2 (A + C) \\
&= -\operatorname{cosec}^2 A + \operatorname{cosec}^2 B \\
&= \frac{\sin^2 A - \sin^2 B}{\sin^2 A \sin^2 B} \\
&= \frac{\sin A + \sin B}{\sin^2 A \sin^2 B} \cdot 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2} .
\end{aligned}$$

Since $\sin A$, $\sin B$, $\cos \frac{1}{2}(A + B)$ are all positive, $f'(A)$ has the sign of $\sin \frac{1}{2}(A - B)$ and hence that of $\frac{1}{2}(A - B)$, since $\frac{1}{2}(A - B)$ is in $(-\frac{1}{2}\pi , \frac{1}{2}\pi)$. It follows that $f'(A)$ is negative for $A < B$, positive for $A > B$.

Hence $f(A)$ is an absolute minimum when $A = B = \frac{1}{2}\pi - \frac{1}{2}C$ and then

$$\begin{aligned}
f(A) &= 2 \tan \frac{C}{2} + \cot C \\
&= \frac{3 \tan^2 \frac{1}{2}C + 1}{2 \tan \frac{1}{2}C} \\
&= \frac{3t^2 + 1}{2t} , \text{ where } t \text{ denotes } \tan \tfrac{1}{2}C .
\end{aligned}$$

The domain of t is $(0, \infty)$ since $0 < C < \pi$.

Now $\dfrac{3t^2 + 1}{2t}$ has absolute minimum value $\sqrt{3}$, when $t = \dfrac{1}{\sqrt{3}}$. (Simple exercise.)

It follows that, of all triangles (of whatever shape), $\cot A + \cot B + \cot C$ has its minimum value when $A = B$ and $\tan \frac{1}{2}C = \frac{1}{\sqrt{3}}$, so that $C = \frac{\pi}{3}$, and then $A = B = \frac{\pi}{3}$, that is, the triangle is equilateral. Since for any triangle which is not equilateral $\cot A + \cot B + \cot C > \sqrt{3}$, it follows that if $\cot A + \cot B + \cot C = \sqrt{3}$, then the triangle is equilateral.

MORE ON PROBLEM 9.3.1

We also had some more correspondence on Problem 9.3.1. This consisted of three parts:

(i) Show that any positive integral power of the product of the first four odd numbers leaves a remainder 1 when divided by 8 or 13.

(ii) Find the set of numbers, such that any one of them when divided into $(5 \times 7 \times 11)^n$, where n is any positive integer leaves a remainder of 1.

(iii) Is there a similar result for $(7 \times 11 \times 13)^n$?

This problem was submitted to us by Garnet J. Greenbury of Taringa, Queensland. Both he and John Barton sent solutions and a composite of these was printed in Volume 9, Part 5. A similar solution, but with different notation, came from Devon Cook. We also received, too late to print, the following solution from David Dyte, then in Year 11, Scotch College.

(i) Show that any positive integral power of the product of the first four odd numbers (105) leaves a remainder 1 when divided by 8 or 13.

To solve this problem I now draw on modulus notation, i.e. instead of writing $\frac{a}{b}$ leaves remainder c , I write $a \equiv c \pmod{b}$.

I proceed by mathematical induction.

Firstly, we know:

$$105^1 = 105 \equiv 1 \pmod{8} \qquad\qquad 105^1 = 105 \equiv 1 \pmod{13}.$$

Now assume this is true in each case for 105^n .

$105^n \equiv 1 \pmod 8$	$105^n \equiv 1 \pmod{13}$
$\therefore 105^n + 7 \equiv 0 \pmod 8$	$\therefore 105^n + 12 \equiv 0 \pmod{13}$
$\therefore 105(105^n + 7) \equiv 0 \pmod 8$	$\therefore 105(105^n + 12) \equiv 0 \pmod{13}$
$\therefore 105^{n+1} + 735 \equiv 0 \pmod 8$	$\therefore 105^{n+1} + 1260 \equiv 0 \pmod{13}$
But $735 \equiv 7 \pmod 8$	But $1260 \equiv 12 \pmod{13}$
$\therefore 105^{n+1} \equiv 1 \pmod 8$	$\therefore 105^{n+1} \equiv 1 \pmod{13}$

So, if each case is true for 105^n, it is true for 105^{n+1}.

But we know each case is true for 105^1
we know each case is true for 105^2
we know each case is true for 105^3
and so on.

Thus each positive integral power of 105 leaves remainder 1 when divided by 8 or 13.

I note that the above modulo arithmetic works for $\frac{x^k}{y}$ if and only if $x(y - 1) \equiv y - 1 \pmod y$ $(y \neq 1)$.

$\therefore$ for $x = 105$, as above, $x^k \equiv 1 \pmod y$ for all k in N if $y \in \{2,4,8,13,26,52,104\}$.

(ii) and (iii):

Also, from the above condition,
for $x = 385$, $x^k \equiv 1 \pmod y$ for all k in N
if $y \in \{2,3,4,6,8,12,16,24,32,48,64,96,128,192,384\}$
$(x = 5 \times 7 \times 11)$
and for $x = 1001$, $x^k \equiv 1 \pmod y$ for all k in N
if $y \in \{2,4,5,8,10,20,25,40,50,100,125,200,250,500,1000\}$
$(x = 7 \times 11 \times 13)$.

More generally:

For a given x, the remainder when each positive integral power of x is divided by y is one iff the set of prime factors of y (repeated factors counted separately) is a subset or equal to the set of prime factors of $x - 1$.

MORE ON PROBLEM 9.3.3

This problem concerned a circle, centre O and a point M on its circumference. Two points A,B lie outside the circle and it was required to show that $AM + MB$ was maximised when OM bisected $\measuredangle AMB$.

The problem came from Hai Tan Tran, whose solution appeared in our last issue. Lack of space prevented our publishing there a letter from John Barton on the same topic. We include it here.

"If our starting point allows use of the principles of optics, including the principle of least (stationary) time of travel as well as the principles of reflection, the results follow immediately.

"Because, for stationary (either max or min) values of $AM + MB$, the incident ray, AM say, and the reflected ray MB make equal angles with the normal OM to the reflecting 'surface' (the cylinder whose cross-section is the given circle) and all three lines AM, MB, OM are coplanar.

"Otherwise, if we are required to give a 'pure' mathematical solution, we might perhaps argue as follows to find the stationary values of $AM + MB$.

"Let the circle be $x = a\cos\theta$, $y = a\sin\theta$ for $0 \leqslant \theta < 2\pi$, and let A be (f, g) and B be (h, k). Then

$$s = AM + MB = \{(f - a\cos\theta)^2 + (g - a\sin\theta)^2\}^{\frac{1}{2}}$$
$$+ \{(h - a\cos\theta)^2 + (k - a\sin\theta)^2\}^{\frac{1}{2}}$$

is to be made stationary, so that $0 = ds/d\theta$.

We have

$$\frac{ds}{d\theta} = \frac{1}{AM}\{(f - a\cos\theta)a\sin\theta - (g - a\sin\theta)a\cos\theta\}$$
$$+ \frac{1}{MB}\{((h - a\cos\theta)a\sin\theta - (k - a\sin\theta)a\cos\theta\}$$
$$= \frac{af\sin\theta - ag\cos\theta}{AM} + \frac{ah\sin\theta - ak\cos\theta}{MB}.$$

The required condition is thus, cancelling a,

$$\frac{f\sin\theta - g\cos\theta}{AM} = -\frac{h\sin\theta - k\cos\theta}{MB}.$$

If we write this in vector style, using $\underset{\sim}{u} = \underset{\sim}{i}\cos\theta + \underset{\sim}{j}\sin\theta$, we have

$$|\underset{\sim}{MB}|(f\underset{\sim}{i} + g\underset{\sim}{j}) \times \underset{\sim}{u} = -|\underset{\sim}{AM}|(h\underset{\sim}{i} + k\underset{\sim}{j}) \times \underset{\sim}{u}$$
$$= |\underset{\sim}{AM}|\underset{\sim}{u} \times (h\underset{\sim}{i} + k\underset{\sim}{j}).$$

Since $\underset{\sim}{u} \times \underset{\sim}{u} = \underset{\sim}{0}$ we can write this

$$|\underset{\sim}{MB}|\{-\underset{\sim}{u}+(f\underset{\sim}{i}+g\underset{\sim}{j})\}\times\underset{\sim}{u} = |\underset{\sim}{AM}|\underset{\sim}{u}\times\{-\underset{\sim}{u}+(h\underset{\sim}{i}+k\underset{\sim}{j})\} ,$$

that is

$$|\underset{\sim}{MB}|\underset{\sim}{MA}\times\underset{\sim}{u} = |\underset{\sim}{AM}|\underset{\sim}{u}\times\underset{\sim}{MB} .$$

The products of the magnitudes of the vectors on the two sides are clearly equal, each being $|AM|\,|MB|$, to the required condition that

$$\hat{\underset{\sim}{MA}}\times\underset{\sim}{u} = \underset{\sim}{u}\times\hat{\underset{\sim}{MB}} \qquad \text{[\^{} denotes a unit vector]}$$

that is the angles between AM and OM on the one hand, and OM and MB on the other be equal or supplementary.

If they are supplementary, AMB is a straight line, which presumably is the solution for A and B on opposite sides (one in, one out) of the circle. We require the other solution, where A and B are on the same side (both in, both out) of the circle."

We conclude with some new problems.

PROBLEM 10.1.1 (Submitted by D.R.Kaprekar)

A man had 115 dollars. He spent 40 of them and 75 were left. He went out again and spent 46, leaving 29. A third time he went out and spent 19 leaving 10. Finally he went out and spent the 10, leaving nothing. Here is a table.

	Spent	Left
	40	75
	46	29
	19	10
	10	0
Totals	115	114

The total at right is 114, not 115. Where is the missing dollar?

PROBLEM 10.1.2 (From the 1985 School Mathematics Competition)

Show that, given any 17 numbers, it is always possible to choose 5 of them so that their sum shall be divisible by 5 .

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