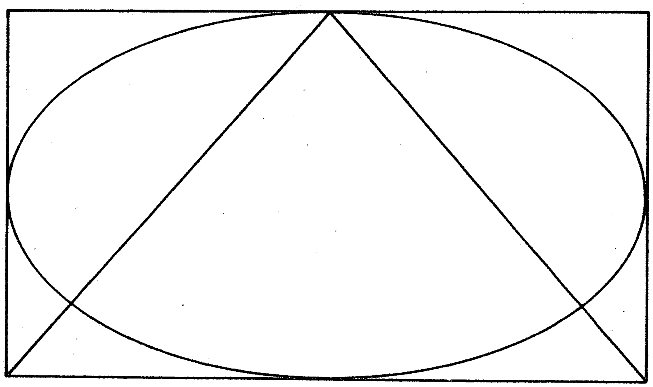
ISSN 0313 - 6825

FUNCTION

Volume 9 Part 5 October 1985



A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
Function,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

The magazine is published five times a year, appearing in February, April, June, August, October. Price for five issues (including postage): \$8.00*; single issues \$1.80. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

*\$4.00 for *bona fide* secondary or tertiary students.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

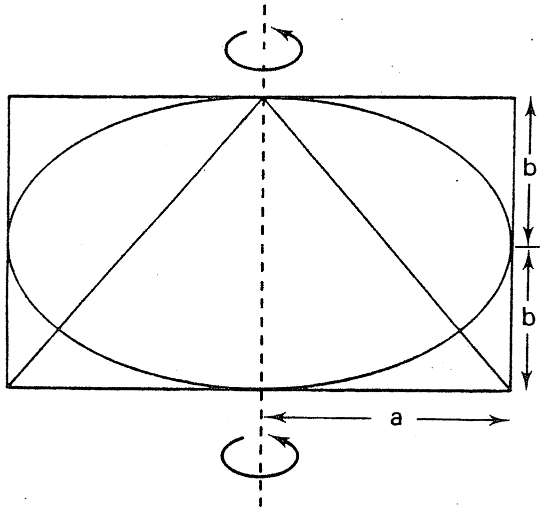
Registered for posting as a periodical - "Category B"

Another year comes to a close for us all. Next year will be *Function's* 10th year of publication, and to make it an even better magazine, please send us your articles, problems, etc. We enclose your subscription form.

For this issue, we continue with John Stillwell's historical articles, this time on Complex Numbers, and conclude the articles by Professor Cheryl Praeger, who now discusses how to set up a loom to weave a particular design. As usual, we also pinch some articles from our sister magazines. Well, they pinch from us, too! But we didn't pinch the Rambam from Rambo; no fear.

CONTENTS

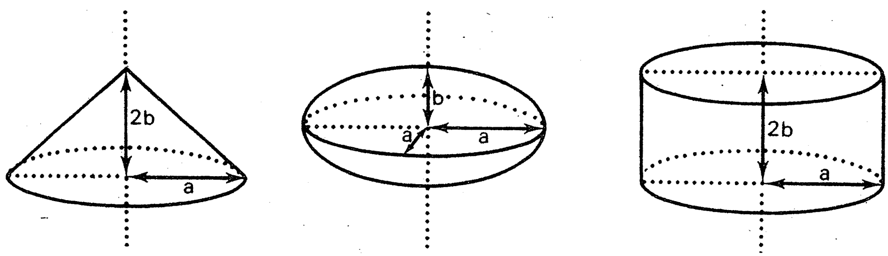
THE FRONT COVER



If the Front Cover diagram is rotated about its central vertical axis, the triangle sweeps out a 3-dimensional region which is a cone, the ellipse sweeps out an ellipsoid, and the outer rectangle sweeps out a cylinder. The volumes of these figures are in the ratio

Cone : Ellipsoid : Cylinder = 1 : 2 : 3

which is surprisingly neat.



To see this,

Volume of cone $= \frac{1}{3}$ area of base × height $= \frac{1}{3}\pi a^2(2b) = \frac{2}{3}\pi a^2 b$,
Volume of ellipsoid $= \frac{4}{3}\pi a^2 b$,

Volume of cylinder = area of base × height

$$= \pi a^2 (2b) = 2\pi a^2 b,$$

which are, indeed, in the ratio 1 : 2 : 3 .

The Front Cover design is incorporated in a sculpture at San José State University, U.S.A.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

KAPREKAR'S CONSTANT

Take any number of four digits, not all equal, say 5080. Write its digits in decreasing order (8500) and in increasing order (0058) and subtract (to get 8442 in this case). Repeat the process: 8442-2448 = 5994.

Repeat again :	9954 - 4599 = 5355
And again :	5553 - 3555 = 1998
And again :	9981 - 1899 = 8082
And again :	8820 - 0288 = 8532
And again :	8532 - 2358 = 6174 .

We have applied the process seven times to the original number and have reached the number 6174. If we apply the process to this number, we get

$$7641 - 1467 = 6174$$

and so on for ever.

It is a remarkable fact that no matter what four-digit number we start with (as long as it does not have all its digits equal), we end up with (after at most seven applications of this process) the number 6174.

This was first discovered by D.R.Kaprekar, who has written several times for *Function*. The process is now termed the *Kaprekar process* and the number 6174 *Kaprekar's constant*.

Projects: check the result with a computer program; explore the situation for two-, three-, five- and six-digit numbers.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

COMPLEX NUMBERS IN ALGEBRA†

John Stillwell, Monash University

1. Introduction

The introduction of complex numbers clarifies and unifies many different areas of mathematics. They throw light on what would otherwise be mysterious relationships between these areas. That complex numbers do all this, and more, is one of the miracles of mathematics. At the beginning of their history, complex numbers $a + b\sqrt{-1}$ were considered to be "impossible numbers", tolerated only in a limited algebraic domain because they seemed useful in the solution of cubic equations. But their significance turned out to be geometric, and ultimately led to the unification of algebraic functions with the theory of map projection, the study of gravitation and electromagnetism, and another "impossible" field, noneuclidean geometry. This resolution of the paradox of $\sqrt{-1}$ was so powerful, unexpected and beautiful that only the word "miracle" seems adequate to describe it.

This article will show how complex numbers emerged from the theory of equations and enabled its fundamental theorem to be proved - at which point it became clear that complex numbers had meaning far beyond algebra. The fundamental theorem of algebra states that each polynomial equation $p(x) = 0$ has a solution x in the complex numbers. Its proof was achieved, after considerable struggle, through a geometric understanding of complex numbers and polynomial functions.

2. Quadratic Equations

The usual way to introduce complex numbers in a mathematics course is to point out that they are needed to solve certain quadratic equations, e.g. the equation $x^2 + 1 = 0$. However, this did not happen when quadratic equations first appeared, since at that time there was no *need* for all quadratic equations to have solutions. Many quadratic equations are implicit in Greek geometry, as one would expect when circles, parabolas, etc. are being investigated, but one does not demand that every geometric problem have a solution. If one asks whether a

---

† For related articles, see *Function*, Volume 3, Part 5 and Volume 5, Part 3.

particular circle and line intersect, say, then the answer can be yes or no. If yes, the quadratic equation for the intersection has a solution; if not, no solution. An "imaginary solution" is quite uncalled for in this context.

Even when quadratic equations appeared in algebraic form, with Diophantus and the Arab mathematicians, there was initially no reason to admit complex solutions. One still wanted to know only whether there were real solutions, and if not the answer was simply - no solution. This is plainly the appropriate answer when quadratics are solved by geometrically completing the square as was still done up to the time of Cardano (1501-1576). A square of negative area did not exist in geometry. The story might have been different had mathematicians used symbols more, and dared to consider the symbol $\sqrt{-1}$ as an object in its own right, but this did not happen until quadratics had been overtaken by cubics, at which stage complex numbers became unavoidable, as we shall now see.

3. Cubic Equations

The solution of the cubic equation

$$y^3 = py + q$$

was discovered by Scipione del Ferro of Bologna early in the 16th century. After being kept secret for some time, the solution was revealed in Cardano's *Ars Magna* (1545).

It is

$$y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} .$$

The formula involves complex numbers when $\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3 < 0$.

However, it is not possible to dismiss this as a case with no solution, *because a cubic always has at least one real root* (since $y^3 - py - q$ is positive for sufficiently large positive y , and negative for sufficiently large negative y). Thus this formula raises the problem of reconciling a real value, found by inspection say, with an expression of the form

$$\sqrt[3]{a + b\sqrt{-1}} + \sqrt[3]{a - b\sqrt{-1}} .$$

Cardano did not face up to this problem in his *Ars Magna* He did, it is true, once mention complex numbers, but in connection with a quadratic equation, and accompanied by the comment that these numbers were "as subtle as they are useless".

The first mathematician to take complex numbers seriously, and achieve the necessary reconciliation, was Bombelli in 1572. Bombelli worked out the formal algebra of complex numbers, with the particular aim of reducing expressions $\sqrt[3]{a + b\sqrt{-1}}$ to the form $c + d\sqrt{-1}$. His method enabled him to show the reality of some expressions resulting from the formula. For example, the solution of

$$x^3 = 15x + 4$$

is $x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}$,

according to the formula. On the other hand, inspection gives the solution $x = 4$. Bombelli had the hunch that the two parts of x in the formula were of the form $2 + n\sqrt{-1}$, $2 - n\sqrt{-1}$, and found by cubing these expressions formally (using $(\sqrt{-1})^2 = -1$) that indeed

$$\sqrt[3]{2 + 11\sqrt{-1}} = 2 + \sqrt{-1}$$

$$\sqrt[3]{2 - 11\sqrt{-1}} = 2 - \sqrt{-1} \quad ,$$

hence the formula also gives $x = 4$.

Much later, Hölder (in 1896) showed that any algebraic formula for the solution of the cubic must involve square roots of quantities which become negative for particular values of the coefficients.

4. Wallis' Attempt at Geometric Representation

Despite Bombelli's successful use of complex numbers, most mathematicians regarded them as impossible, and of course we call them "imaginary" even today, and use the symbol i for the imaginary unit $\sqrt{-1}$. The first attempt to give complex numbers a concrete interpretation was made by Wallis in 1673. This attempt was unsatisfactory, as we shall see, but nevertheless an interesting "near miss". Wallis wanted to give a geometric interpretation to the roots of the quadratic equation which we shall write as

$$x^2 + 2bx + c^2 = 0$$

where $b, c \geqslant 0$. The roots are

$$x = -b \pm \sqrt{b^2 - c^2}$$

and hence real when $b \geqslant c$. In this case the roots can be represented by points P_1, P_2 on the real number line which are determined by the geometric construction in Figure 1.

When $b < c$, lines of length b attached to Q are too short to reach the number line, so the points P_1, P_2 "cannot be had in the line", and Wallis seeks them "out of that line... (in the same Plain)" (sic). He is on the right track, but arrives at unsuitable positions for P_1, P_2 by sticking too closely to his first construction. Figure 2 compares his

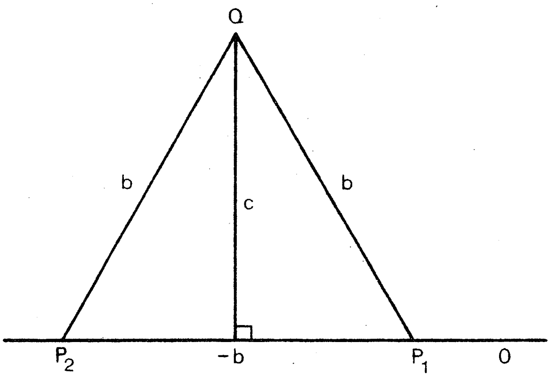


Figure 1.

representation of $P_1P_2 = -b \pm i\sqrt{c^2 - b^2}$ when $b < c$ with the modern representation.

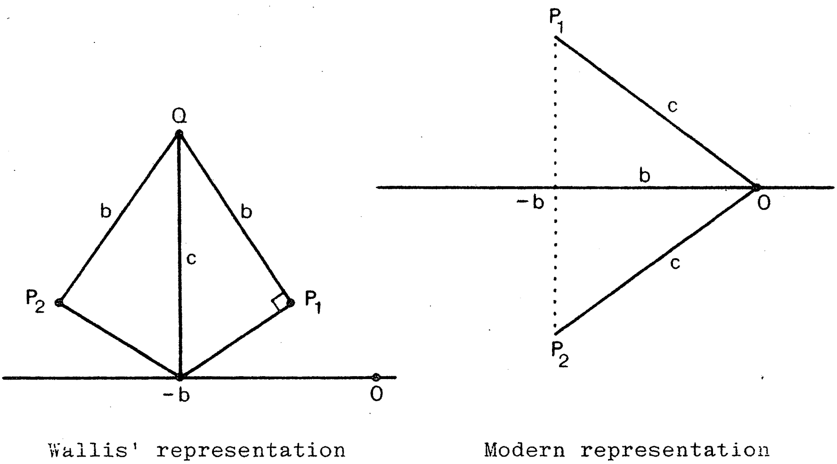


Figure 2.

Apparently Wallis thought + and - should continue to correspond to "right" and "left", though this has the unacceptable consequence that $i = -i$ (let $b \to 0$ in his representation). This was an understandable oversight, since in Wallis' time even negative numbers were still under suspicion, and there was confusion about the meaning of $(-1) \times (-1)$, for example. Confusion was compounded by the introduction of square roots, and as late as 1770 Euler gave a "proof" in his Algebra that $\sqrt{-2} \times \sqrt{-3} = \sqrt{6}$.

5. The Fundamental Theorem of Algebra

The assertion that an nth degree algebraic equation has n roots, and the consequent acceptance of imaginary roots, is usually credited to Girard (1629). Descartes, in 1637, observed that if $x = a$ is a root of $p(x) = 0$, then $p(x) - p(a)$ is divisible by $x - a$, hence $p(x) = (x-a)q(x)$ where $q(x)$ is a polynomial of lower degree, and one can recursively factor any polynomial $p(x)$ into linear factors given a *single* root of each algebraic equation. The fundamental theorem can therefore be restated as the assertion that any algebraic equation has a root, and the existence of n roots is a corollary.

The first attempt at a proof was made by D'Alembert in 1746. In fact, D'Alembert usually receives more credit for making the attempt than actual execution, though his idea was eventually developed into a rigorous proof by Weierstrass in 1891. A better, though still unsatisfactory, attempt was made by Euler in 1749. It was Gauss who, in 1799, gave the first reasonably satisfactory proof.

This proof, in Gauss' doctoral dissertation, departed radically from previous approaches by employing topological arguments. That is, Gauss was concerned with *qualitative* properties of curves, such as whether they intersect or not, rather than *quantitative* properties such as the position of intersection. His topological arguments were plausible rather than rigorous, nevertheless he correctly judged that the fundamental theorem of algebra is best viewed topologically. By basing his proof on qualitative arguments he was able to show the *existence* of a root to any polynomial equation, without having to produce an algebraic formula for calculating it. This was a shrewd move because such formulae do not exist for equations of degree $\geqslant 5$, as Gauss himself suspected, and Ruffini attempted to prove, in 1799. (The proof that there is no algebraic solution of 5th degree equations was completed by Abel in 1826.)

The other striking aspect of Gauss' proof is its full use of the geometric nature of complex numbers. The idea of the complex plane had crystallised around 1800 in the works of Wessel (1797) and Argand (1806), in addition to Gauss' much deeper contribution, but Gauss was unaware of Wessel's work and evidently believed his contemporaries were not ready for the complex plane (as he later believed they were unready for noneuclidean geometry). He concealed the complex elements in his proof by working with real and imaginary parts, and only much later (in 1849) did he rework the proof to make the complex numbers explicit.

The gist of Gauss' proof is as follows.

If we write $z = x + iy$, then a polynomial function $f(z)$ can be separated into real and imaginary parts:

$$f(x + iy) = T(x,y) + i\,U(x,y) ,$$

where $T(x,y)$, $U(x,y)$ are polynomials obtained by multiplying out the powers of $(x + iy)$ in $f(x + iy)$. The points (x,y) in the plane for which $T(x,y) = 0$ form certain curves, as do the points for which $U(x,y) = 0$. We want to show there is a point where a $U = 0$ curve meets a $T = 0$ curve, because there we have

$$f(x + iy) = 0 + i0 = 0$$

and hence a solution $z = x + iy$ of the equation $f(z) = 0$. Gauss shows the existence of such a point by showing that there is a certain large circle which the $T = 0$ and $U = 0$ curves meet *alternately* , which implies they meet each other somewhere inside the circle (Figure 3).

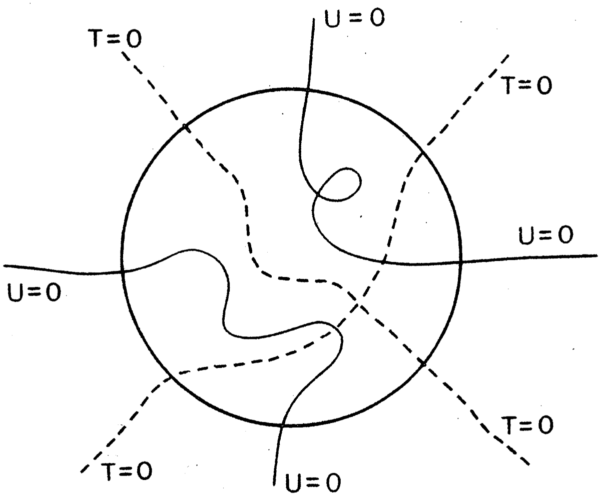


Figure 3.

SAME SUM, SAME PRODUCT[†]

Do you see something special in the following four triples?

(14, 50, 54)
(15, 40, 63)
(18, 30, 70)
(21, 25, 72)

No?

Then quickly get your calculator (of course you can also use pencil and paper!) and add the numbers of each triple.

You get a nice result but of course that's no big deal. You're quite right.

Now for multiplication.

Again calculate the product of each separate triple. You have already seen the result in the title of this article. If you want to know if this really *is* something special, consider the following questions.

- Can you find a *fifth* triple with the same sum and the same product as the triples shown above?
- Or to start simply: can you find *two* triples which have equal sums and equal products but different from the above sums and products?
- Three triples?
- Is it perhaps easier with pairs?
- Or with quadruples?

To show that the example shown above is not so special, here is a solution with five triples.

(6, 480, 495)
(11, 160, 810)
(12, 144, 825)
(20, 81, 880)
(33, 48, 900).

[†] This article is a translation from the Dutch by A.-M. Vandenberg. It first appeared in the journal *Pythagoras*, *Vol.* 24, and is reproduced under an exchange agreement.

To our great astonishment, in a booklet on mathematical puzzles we found another such example interwoven into one large magical square! You can see this 8 by 8 square below. All rows, all columns and both the diagonals, all eighteen of them, have a sum 840 and also all have the same product. We could only believe it ourselves after we had completely checked it out - and now let's hope no printing error has crept in.

46	81	117	102	15	76	200	203
19	60	232	175	54	69	153	78
216	161	17	52	171	90	58	75
135	114	50	87	184	189	13	68
150	261	45	38	91	136	92	27
119	104	108	23	174	225	57	30
116	25	133	120	51	26	162	207
39	34	138	243	100	29	105	152

You are probably rather reluctant about checking it. The addition is fairly manageable but multiplication seems a lot more tricky. The product becomes a number of sixteen digits, much too large for your calculator.

But it isn't necessary at all to "figure out" such a product : it's much neater to factorize all the numbers of the square into prime factors and then to compare the products of those factors. For each row, column and diagonal it has to be : $2^7 . 3^8 . 5^3 . 7 . 13 . 19 . 23 . 29$.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

BROCHURE

The Australian Mathematical Society has published a brochure called "Mathematics Graduates are Highly Employable". The brochure discusses the advantages of a Mathematics education, employment facts and figures, a list of who employs mathematicians, and an outline of what Mathematics graduates do. For information about the brochure, write to Dr J.D. Gray, School of Mathematics, University of N.S.W., P.O. Box 1, Kensington, N.S.W. 2033.

The brochure records, for instance, the information that out of 34 subject areas, mathematics has the 8th highest employment rate (below medicine, health sciences, law, accountancy, but above computer science, other sciences and economics.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

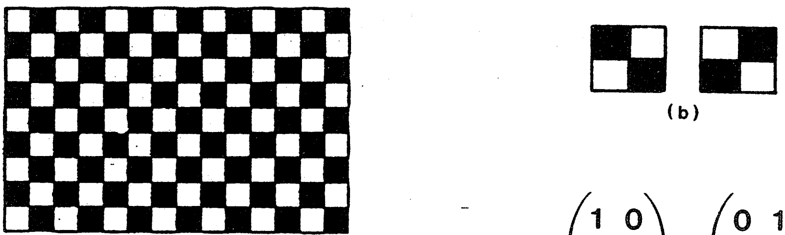
MATHEMATICS AND WEAVING: II. SETTING UP THE LOOM AND FACTORIZING MATRICES†

Cheryl E. Praeger,
University of Western Australia

The aim of this paper is to describe mathematically how a loom can be set up to weave a given design.

1. Designs

A fabric is described mathematically as follows. Vertical strands represent warp threads and horizontal strands represent weft threads (where a *strand* is the set of points in the plane lying strictly between two infinite parallel lines). The intersection of a vertical strand and a horizontal strand, called a *square*, is coloured black or white when the vertical strand passes over or under the horizontal strand respectively. All the fabrics considered in this paper are *periodic*, that is they consist of a finite block of squares which is simply repeated as we move across or down the fabric. Such a block of squares is called by weavers a *design*, diagram, draft, draw-down diagram or point diagram and is the way a weaver would usually describe a weaving pattern. We associate with each design a matrix of 0's and 1's that is a *binary matrix*, by replacing each black square by a 1 and each white square by a 0 , see for example Figure 1.



(a) (b)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(c)

Figure 1. Plain weave.

(a) A design for the fabric. (b) The smallest possible designs.

(c) Binary matrices corresponding to the designs in (b).

---

† This article continues Professor Praeger's discussion of weaving; see *Function*, Vol.9, Part 4 for the earlier part.

So when we talk of a design D we will mean either an array of black and white squares or its corresponding matrix. Our problem is : given a design D , describe mathematically how to set up a loom to weave D .

2. Looms

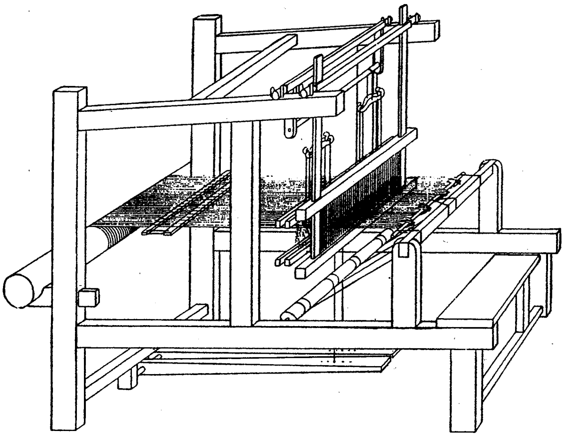


Figure 2.

A loom (see Figure 2) can be regarded as a device the purpose of which is to hold the warp threads parallel to each other and under tension. There is usually a mechanism for separating the warp threads into two layers so that the weft threads can be inserted in the space between the two layers; this space is called the *shed* (see Figure 3). Thus the weft threat passes over some warp threads and under others. Before the weft thread is passed across the warp threads each time, the composition of the two layers is changed to produce the desired pattern. For the simplest woven pattern, that of plain weave depicted in Figure 1, there are just two warp-weft interlacement sequences which alternate as we move down the fabric, namely under-over-under-over ... and over-under-over-under. How do we go about setting up the loom to weave the fabric represented by a design D ?

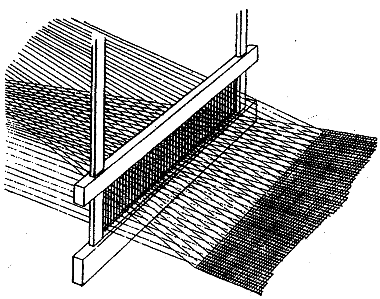


Figure 3.

The description given below is based on work by mathematician and weaver, Janet Hoskins. She has also investigated the similar problems of weaving a given pattern on two types of multiply threaded looms. We will show that D is the matrix product $D = AB^TC$ of three binary matrices A, B^T, C ; where A is called the *shed matrix*, B is called the *tie up matrix*, C is called the *threading matrix*, and B^T is the transpose of B, (that is the matrix obtained from B by interchanging the rows and columns of B, for example the transpose of

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}).$$

Each of A, B, C provides significant information about the processes of weaving the design D, and A, B and C together give enough information to allow a loom to be set up to weave D.

The loom is prepared as follows. The warp threads are connected to objects called *harnesses* (see Figure 4). Each warp thread is connected to exactly one harness. Certain combinations of harnesses are then connected or "tied up" to objects called *treadles* ; (think of a treadle as a foot-pedal). When a treadle is operated those warp threads whose harnesses are connected to it move downwards to produce a shed through which the weft thread may move. Thus we require (i) one harness for each warp thread - noting that a given harness may be tied to several warp threads, and (ii) one treadle for each row in the design D - noting that when several rows of D are the same then the same treadle can be used for each of them.

On some looms, operation of a treadle causes its harness to move *upwards*. From now on, we will assume we have such a loom.

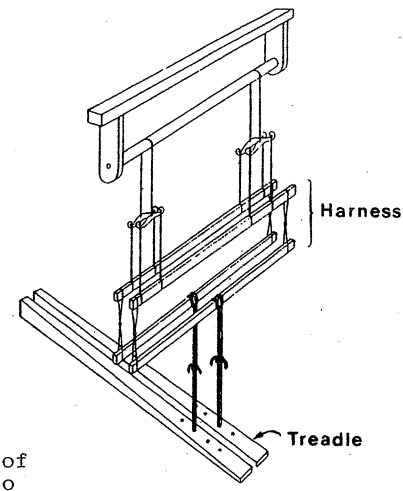


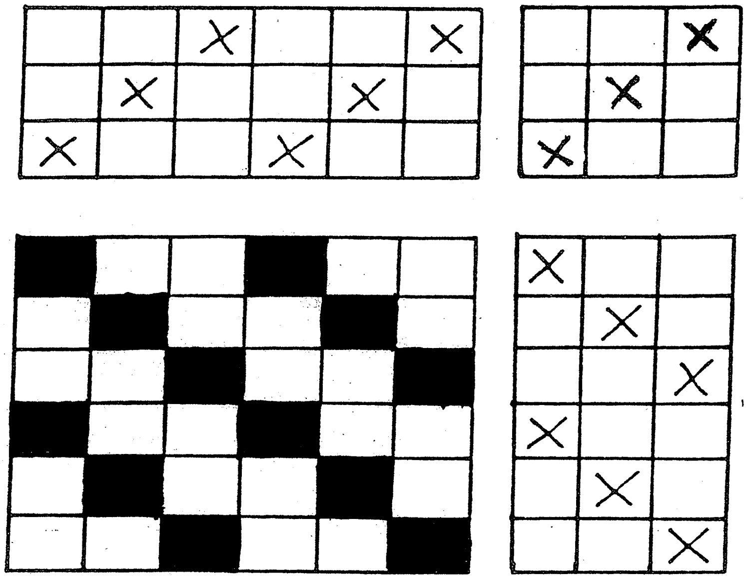
Figure 4.

3. Factorizing D

We now define the matrices A, B and C with particular reference to the example given in Figure 5.

First we number the harnesses $1, \ldots, h$ in some way. The threading "matrix" C as shown in Figure 5 has one column for each warp thread, (the warp threads correspond to the columns of D remember), and one row for each harness; there is a cross × in the row i column j position if warp thread j is tied to harness i, otherwise the position is blank. This threading "matrix" C is converted to a binary matrix by substituting a 1 for each × and a 0 for each blank. Thus $C = (c_{ij})$ has exactly one 1 in each column.

C threading "matrix" **B** tie-up "matrix"



A shed "matrix"

D design

Figure 5.

Next we number the treadles $1, \ldots, t$ in some way. The tie up "matrix" B as shown in Figure 5 has one row for each harness and one column for each treadle; there is a cross × in the row i column j position if harness i is tied up to treadle j, otherwise this position is blank. This tie up "matrix" B is converted to a binary matrix $B = (b_{ij})$ by substituting a 1 for each × and a 0 for each blank.

Finally the shed "matrix" A has one column for each treadle and one row for each row of D, that is for each weft thread; there is a cross × in the row i column j position if the shed for weft thread i is created by operating treadle j, otherwise this position is blank. As for the threading matrix, this shed "matrix" A is converted to a binary matrix by substituting a 1 for each × and a 0 for each blank. Thus $A = (a_{ij})$ has exactly one 1 in each row.

For the example given in Figure 5 we obtain the following binary matrices A, B, C and D and a little checking shows that $D = AB^TC$; we shall show that this equation always holds.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

We show that for all designs, $D = AB^TC$. We denote the row i column j entry in A,B,C,D by $a_{ij}, b_{ij}, c_{ij}, d_{ij}$ respectively. The entry d_{ij} of D is 1 (that is, weft thread i passes under warp thread j), if and only if the treadle k which is operated for row i is connected to the harness ℓ to which warp thread j is tied. In terms of the entries in the matrices : $d_{ij} = 1$ if and only if the unique non-zero entry a_{ik} in row i of A, and the unique non-zero entry $c_{\ell j}$ in column j of C are such that the ℓk-entry $b_{\ell k}$ of B is also non-zero, that is,

$$a_{ik}b_{\ell k}c_{\ell j} = 1 \ .$$

Note that all of the expressions $a_{ix}b_{yx}c_{yj}$ as $x (x = 1,2,\ldots, t)$ and $y (y = 1,2, \ldots h)$ run over all allowable values, the only one which has a chance of being non-zero is $a_{ik}b_{\ell k}c_{\ell j}$ and this one is non-zero if and only if $b_{\ell k} = 1$. Thus

$$d_{ij} = 1 \text{ if and only if } \sum_{x,y} a_{ix}b_{yx}c_{yj} = 1 \ ,$$

where the summation is over all allowable x and y. Moreover $d_{ij} = 0$ if and only if this sum is zero. This result, in terms of matrix products, is precisely the equation $D = AB^TC$, because the sum

$$\sum_{x,y} a_{ix}b_{yx}c_{yj}$$

is the (i,j) element in AB^TC, by the definition of matrix product.

4. Methods of factorizing D

Thus we have seen that the factorization of D as the product of A, B^T and C gives us enough information to set up a loom and weave the design D, provided of course that the the number of rows of C (that is the number of harnesses required) is at most the number of harnesses available on the loom, and the number of columns of A (that is the number of treadles required) is at most the number of treadles available on the loom. The problem therefore reduces to the following: given D, find an efficient method of factorizing D as the product of three binary matrices A, B^T and C such that each column of C contains exactly one 1 and each row of A contains exactly one 1. A solution of this problem was published in 1981 by J.A. and W.D. Hoskins. Their method involves first partitioning the set of columns of D into classes such that two columns are in the same class if and only if they are equal. Each class corresponds to a harness, that is a row of C, and the matrix C is easily constructed as follows: the rows of C correspond to harnesses and the columns of C correspond to columns of D; the row i column j entry of C is 1 if and only if column j of D lies in the class corresponding to harness i. This part of the method was well known and used by weavers. Partitioning the rows of D in a similar manner produces A. Finally a particularly easy computer solution of the equation $D = AB^TC$ was possible which yielded B^T and hence B. A computer is helpful because a typical matrix D in practice could have of the order of 1000 rows or columns!

5. Conclusion

In this paper I have shown how to obtain mathematically enough information from a design to set up a loom to weave the design. I remind the reader that not all designs when woven produce a fabric which hangs together, but that this can be tested using the method discussed in the earlier article, *Function* Vol.9, Part 4.

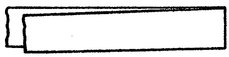
∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

The largest prime number so far discovered is $2^{216091} - 1$, according to the 'Los Angeles Times' of 18th September. This number, of 65050 digits, was discovered by an oil company in Houston, Texas, not by drilling below the ground, but during trials of their new Cray computer, which works at 400 million calculations per second.

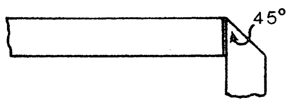
∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

FOLDING A 60° ANGLE†

How would you fold an angle of 60° in a strip of paper? Folding a right angle is easy - just fold the paper back on itself and crease.

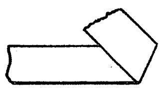


Folding an angle of 45° is also easy: fold the strip down along the first crease.



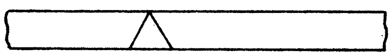
How would you fold an angle of 60° ? The following procedure is due to Professor Jean Pedersen, of the University of Santa Clara in California.

Start with a strip of paper about 25cm long and 4½cm wide (a piece of gummed brown paper tape is ideal) and fold any angle you like at the left hand end by folding the right hand end up.



Crease firmly and unfold.

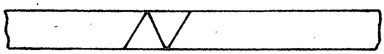
Now fold the right hand end of the strip down along the first crease (similar to the second diagram above), crease firmly and unfold.



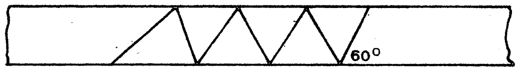
---

† This article is reprinted under an exchange agreement with *Mathematical Digest* (RSA) in which it first appeared (No.59, April 1985).

Now fold the right hand end of the strip up along the second crease, crease firmly and unfold.



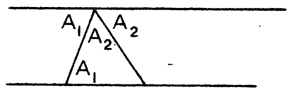
Continuing in this way, folding alternatively up and down along the previous crease, you obtain a pattern of creases along the strip.



Now for the surprise: whatever the angle you folded at the first crease, the angle at the seventh crease will turn out to be 60° when you check it with a protractor! If you don't believe it (scepticism is a very healthy mathematical attitude), do it again with a new strip of paper, starting with a different angle. The angle at the seventh crease will again be 60°, according to your protractor.

What is the explanation? Suppose the first angle folded was A_1 and the second A_2. Since the second crease bisects the angle between the first crease and the edge of the paper, we have

$$A_1 + 2A_2 = 180^{\circ}$$



Now write $A_1 = 60^{\circ} + E_1$. The angle E_1, which could be negative, measures the amount by which A_1 differs from 60°. It follows that

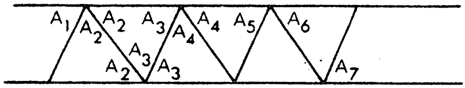
$$\begin{aligned} A_2 &= \tfrac{1}{2}(180^{\circ} - A_1) \\ &= \tfrac{1}{2}(180^{\circ} - 60^{\circ} - E_1) \\ &= 60^{\circ} - \tfrac{1}{2}E_1 . \end{aligned}$$

So $A_2 - 60^{\circ} = -\tfrac{1}{2}E_1 = -\tfrac{1}{2}(A_1 - 60^{\circ})$.

This shows that A_2 is closer to 60° than A_1. Similarly, if the angle at the third crease is A_3, we find that $A_3 - 60^{\circ} = -\tfrac{1}{2}(A_2 - 60^{\circ}) = \tfrac{1}{4}(A_1 - 60^{\circ})$, and A_3 is closer to 60° than A_2.

At each crease, the angle is closer to 60° than the angle at the previous crease. In fact, the difference between the angle and 60° is halved at each stage.

Take a specific case, with $A_1 = 76^\circ$. Then $A_2 = 52^\circ$, $A_3 = 64^\circ$, $A_4 = 58^\circ$, $A_5 = 61^\circ$, $A_6 = 59^\circ 31'$, $A_7 = 60^\circ 15'$.



This illustrates why your protractor measured the angle at the seventh fold as 60°. The angle was not exactly 60°, but the amount by which it differed from 60° was too small to be detected.

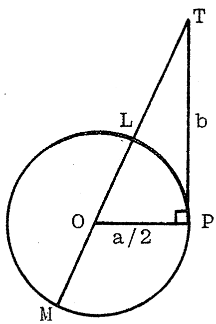
∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

SOLVING QUADRATICS

Garnet J. Greenbury, Brisbane

To solve $x^2 - ax = b^2$, Descartes is said to have proceeded as follows.

Draw a circle centre O, and radius $OP = a/2$. Draw $PT = b$, perpendicular to OP. Join TO cutting the circle at M and L. Then $TM = x$.



Proof.

By Pythagoras' theorem

$$TO^2 = TP^2 + OP^2 ,$$

i.e. $$\left(TM - \frac{a}{2}\right)^2 = b^2 + \left(\frac{a}{2}\right)^2 ,$$

i.e. $$(TM)^2 - a(TM) = b^2 .$$

This shows that $x = TM$ is a solution of the equation.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

THE RAMBAM

Hans Lausch, Monash University

> *Those who appear to be talented and to have capacity for the higher method of study, i.e. that based on proof and on true logical argument, should be gradually advanced towards perfection, either by tuition or by self-instruction.*
>
> *More nevukhim 1,33*

This quotation is a guide to our learning community today as much as it was 850 years ago. 1135 was the year when Henry I died after having lost his Matilda, but it was also the year when Rambam was born. Who was the Rambam? There is an extremely brief answer to this question: he was Abū ʿImrān Mūsā ibn Maimūn ibn 'Abdallāh al Qurṭubī al-Isrā'īlī. He became better known by his proper Hebrew name R(abbi) Mosche ben Maimon, hence the acronym Rambam, but also as Moshe hazman (the Moses of his time), Moses Maimuni, Moses Aegyptius, Hanesher hagadol (The Big Eagle), and especially Maimonides. He was born in Qurtuba in the country of al-Andalus, i.e. Cordoba in Spain, on the eve of Pesach (Passover). He died in Fustat (Old Cairo) on 13 December, 1204 and was buried in Tiberias in the Holy Land, where his tomb can still be seen. His family lived in Cordoba until 1148/9 when the North African Almohades conquered Spain and began their rule with religious persecutions. Moshe's family left Cordoba and is believed to have lived for a dozen years in various places in Spain; in 1160 they settled in the Moroccan city Fez, but had soon to move again, reaching Fustat in 1165 after a brief stay in the Crusader Kingdom of Outremer. Moshe's father soon died and Moshe practised medicine to support the family. He spent the rest of his life in Cairo and became a famous physician. He was patronized by Ṣalāḥ al-dīn's (Saladin, King Richard Coeur de Lion's chivalrous opponent) wazīr, al-Qāḍi al-Fāḍil al-Baisānī; later he became physician to Saladin himself and his son. From 1177 he was nagid, i.e. leader of the Jewish community in Egypt. Maimonides' most popular work is "Dalālat al-ḥā'irīn", better known by the name of its Hebrew translation "More nevukhim" which means "teacher of the confused" or "guide of the perplexed".

In a *Function* article, it is perhaps not quite appropriate to depict Maimonides as the old philosopher with turban and beard, nor is there any need for doing so: tradition has it that Moshe wrote the important work "Maqāla fi ṣinā 'at al-manṭiq", a treatise in the art of logic, in fourteen chapters, when he was about sixteen. It is partly lost in Arabic.

Its first translation into Hebrew, "Milot hahigayon", is by Moses ibn Tibbon (1254), and amongst scholars, it became widely known in this form. Maimonides tells us in his introduction: "An eminent person, one of the masters of the juridical sciences and the possessor of clarity and eloquence in the Arabic language, has asked a man who studied the art of logic to explain to him the meanings of numerous terms frequently occurring in the art of logic: to interpret to him the technical language commonly adopted by the masters of the art, and to endeavour to do this with extreme brevity and not to indulge in details of meaning, lest the discourse become too long." He then proceeds to explain 175 logical terms. Maimonides' astonishing knowledge at the age of 16 is based on work by two great Moslem scholars: Abū Naṣr Muḥammad ibn Muḥammad ibn Tarkhān ibn Uzlagh al-Fārābī (873-951) and Abū Ḥamid Muḥammad ibn Muḥammad al-Tūsī al Shāfi'i al-Ghazzālī (1058-1111). Ultimately the treatise rests on Aristotle's eight books of logic. Here are a few "dictionary entries" drawn randomly from amongst the 175 terms ("Milot"): individual, accident, definition, categories, compound expression of information, distinct, mathematics, the art of logic, physics, laws, right, wrong, prior in excellence, together in order, correlation, opposites, difference, elements, agent, form, first ideas, syllogism, premise, conclusion, contradiction, binary sentence. Here are some of Maimonides' definitions:

"... the subject and the predicate together is called a proposition."

"Every proposition either affirms something of something, e.g., "Zayd is wise", "Zayd stands", or negates something of something, e.g., "Zayd is not wise", "Zayd does not stand". .."

"The affirmative proposition may affirm the predicate of all of the subject, e.g., "Every man is an animal'; and we call it a universal affirmative, and we call 'every' a universal affirmative sign."

"[The affirmative proposition] may affirm the predicate of a part of the subject, e.g., 'Some men write'; and this we call a particular affirmative, and we call 'some' a particular sign."

In chapter 6 and 7, Maimonides uses these and other explanations to tell us about syllogisms, i.e. correct management of propositions, the daily bread of the mathematician. The essence of what he explains can be illustrated by means of "more modern" examples which were invented by Lewis Carroll, best known through his "Alice's Adventures in Wonderland". You are invited to use your logical skills for finding correct conclusions, as a practical introduction into the art of syllogisms.

(1) Every one who is sane can do Logic;
(2) No lunatics are fit to serve on a jury;
(3) None of your sons can do logic.

Conclusion?

(Answer: If your sons were fit to serve on a jury, they would have to be sane (2). They would thus have to be able to do Logic (1). But they cannot do Logic (3). Conclusion: Your sons are not fit to serve on a jury.)

Now try these yourself. Please send your conclusions to the Editor.

A. *(1) Babies are illogical; (2) Nobody is despised who can manage a crocodile; (3) Illogical persons are despised.*

Conclusion?

B. *(1) Gentiles have no objection to pork; (2) Nobody who admires pigsties ever reads Hogg's poems; (3) No Mandarin knows Hebrew; (4) Every one who does not object to pork, admires pigsties; (5) No Jew is ignorant of Hebrew.*

Conclusion?

C. *(1) The only animals in this house are cats; (2) Every animal is suitable for a pet, that loves to gaze at the moon; (3) When I detest an animal, I avoid it; (4) No animals are carnivorous, unless they prowl at night; (5) No cat fails to kill mice; (6) No animals ever take to me, except what are in this house; (7) Kangaroos are not suitable for pets; (8) None but carnivora kill mice; (9) I detest animals that do not take to me; (10) Animals, that prowl at night, always love to gaze at the moon.*

Conclusion?

Another work of Maimonides which is of great interest to the mathematician and astronomer, deals with the calendar. But that is another story.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

"After a low of 14, tomorrow's top temperature should reach 23 degrees."

ABC weather forecast (3AR)

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

PROBLEM SECTION

SOLUTION TO PROBLEM 9.1.1.

A boy took a calculator with him when he went to the store. He bought four items and calculated correctly that their total cost was \$7.11. The only trouble was that he used the multiply key to reach his answer. What did the items cost?

Let a, b, c, d be the costs. Then we have

$$a + b + c + d = abcd = 7.11.$$

If we work in cents, let $A = 100a$, $B = 100b$, $C = 100c$, $D = 100d$. Then A, B, C, D are integers, and

$$A + B + C + D = 711 , \quad ABCD = 711 \times 10^8 = 79 \times 2^6 \times 3^2 \times 5^6 .$$

One of A, B, C, D must be a multiple of 79. If we try, for example, $D = 79$ we get $A + B + C = 632$ and $ABC = 2^6 \times 3^2 \times 5^6 = 9\ 000\ 000$. For all positive A, B, C, the "arithmetic" mean $\frac{1}{3}(A+B+C)$ has to be greater than (or equal) the geometric mean $(ABC)^{\frac{1}{3}}$, and indeed $\frac{1}{3}(632) > (9\ 000\ 000)^{\frac{1}{3}}$. So far so good. If, instead, we had tried $D = 79 \times 2$, or $D = 79 \times 2^2$, or $D = 79 \times 3$, then again $\frac{1}{3}(A+B+C) > (ABC)^{\frac{1}{3}}$ in each case, as has to be true. But if we tried any other available multiple of 79, e.g. $D = 79 \times 5$ or $D = 79 \times 2 \times 3$ etc., we find an impossible situation arises for A, B, C.

Whichever of the cases $D = 79$, or 79×2, or 79×2^2 or 79×3 we consider, at least one factor, say C, has to be a multiple of 25 (because 5^6 has to be shared between three factors). But if we tried, for instance, $D = 79$, $C = 25$, then $A + B = 607$ and $AB = 2^6 \times 3^2 \times 5^4 = 360\ 000$ and this combination violates the inequality $\frac{1}{2}(A + B) \geqslant (AB)^{\frac{1}{2}}$ which must always hold. Arguing in this way, we find there is only *one* combination of values which is possible,

$$A = 120 , \quad B = 150 , \quad C = 125 , \quad D = 316$$

except for rearrangements amongst themselves. The original amounts were \$1.20, \$1.50, \$1.25, \$3.16.

SOLUTION TO PROBLEM 9.1.3.

The decimal representations of $n/13$, $n = 1, 2, \ldots, 12$ each have a period of 6 and belong to one of two families (each of which has 6 members).. If $i/13$ belongs to one family, then $(13-i)/13$ also belongs to that family. Why is this so?

If you write out the decimal representations of the twelve numbers, you find six of them involve repetitions of 076923 and another six involve 153846.

When 10, 20, 30, ... are divided by 13, there is always a remainder, as follows:

10 divided by 13 goes 0 times with 10 remainder
20 divided by 13 goes 1 times with 7 remainder
30 divided by 13 goes 2 times with 4 remainder
40 divided by 13 goes 3 times with 1 remainder
50 divided by 13 goes 3 times with 11 remainder
60 divided by 13 goes 4 times with 8 remainder
70 divided by 13 goes 5 times with 5 remainder
80 divided by 13 goes 6 times with 2 remainder
90 divided by 13 goes 6 times with 12 remainder
100 divided by 13 goes 7 times with 9 remainder
110 divided by 13 goes 8 times with 6 remainder
120 divided by 13 goes 9 times with 3 remainder.

Each number from 1 to 12 appears as a remainder exactly once.

In converting $n/13$ into a decimal, one adds zeros and divides, going through the above table in a pattern that goes from the line with remainder n to the line that starts with $10n$. This procedure puts one into one of two endless loops, producing the two families of decimal representations.

1/13 and 12/13 are in the same family for the following reason:

$$1001 \times 1/13 = 77 = 1000 \times 0.\dot{0}7692\dot{3} + 1/13$$

$$= 76.\dot{9}2307\dot{6} + 1/13$$

And of course, $.\dot{9}2307\dot{6} = 12/13$. Thus 12/13 and 1/13 not only must be in the one family, but must be three digits apart in their representations. A similar argument can be given for any other pair $n/13$ and $1 - n/13$.

SOLUTION TO PROBLEM 9.3.1.

(Solution submitted by John Barton, North Carlton, and also by the proposer, Garnet J. Greenbury, Brisbane.)

(i) *Show that any positive integral power of the product of the first four odd numbers leaves a remainder 1 when divided by 8 or 13.*

Since $1.3.5.7 = 105 = 8.13 + 1$

$$= 13e + 1 \;, \quad \text{say,}$$

we have

$$(13e + 1)^n = (13e)^n + \binom{n}{1} (13e)^{n-1} + \ldots + \binom{n}{n-1} (13e) + 1$$

$$= M(e) + 1 \;,$$

where $M(e)$ denotes a multiple of e, that is, of eight.

This establishes the first result, and the second one (dividing by 13) is done in exactly the same way, by writing $105 = 8t + 1$.

(ii) *Find the set of numbers, such that any one of them when divided into $(5 \times 7 \times 11)^n$, where n is any positive integer leaves a remainder of 1.*

$$5 \times 7 \times 11 = 385 = 1 + 384 = 1 + 3.2^7 \;.$$

Bearing in mind the argument of *(i)*, it is clear that the required set is the set of factors of 384, viz.

2 4 8 16 32 64 128

3 6 12 24 48 96 192 384 .

(iii) *Is there a similar result for $(7 \times 11 \times 13)^n$?*

$$7 \times 11 \times 13 = 1001 = 1 + 10^3 = 1 + 2^3.5^3 \;.$$

The required set is the set of factors of 1000, viz.

2 4 8 5 25 125 10 50 250

20 100 500 40 200 1000 .

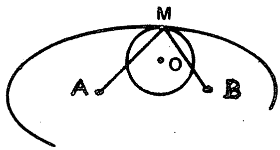
We note, concerning these sets, that they are complete, because, although there may be larger sets with the required property for values of $n > 1$, the requirement that n be any positive integer and so possibly =1 restricts the sets to the factors as given.

Solution to PROBLEM 9.3.3. *If O is the centre of a circle and M lies on its circumference and if A, B lie outside the circle, show that $AM + MB$ will be maximised if OM bisects $\measuredangle AMB$. Show the same property if this distance is to be minimised.*

This problem was submitted and solved by Hai Tan Tran, Plympton Park, S.A., and two other, very different, solutions were offered by John Barton. Here is Hai Tan Tran's solution.

Let $AM + MB = 2a$, the sum of the distances from M to the two fixed points A and B. Take A and B as foci, and draw the ellipse consisting of all points (including M) where distances from A and B sum to 2a.

For different positions M on the circle, we get different values of $2a$ and different ellipses. The ellipse which maximises $2a$ just touches the circle tangentially. But according to the properties of ellipses, MA and MB make equal angles with the tangent to the ellipse at M, which is also tangent to the circle. But for the circle, OM is perpendicular to this tangent, so that OM bisects $\measuredangle AMB$.



A similar line of reasoning can be given for the case of a minimum.

Here are some new problems to try:

PROBLEM 9.5.1 (submitted by John Mack, Sydney)
I have N weights weighing 1 kg, 2 kg, ..., N kg respectively. I wish to remove one weight of m kg so that:

Sum of (weights $< m$ kg) = sum of (weights $> m$ kg).

For what values of N, m is this possible?

PROBLEM 9.5.2 *A rectangular sheet of paper measuring a cm by b cm is folded so that one corner just reaches an opposite edge. Show how the paper must be folded in order that the length of the crease shall be a maximum.*

PROBLEM 9.5.3 *Two cylinders of equal radius, r, intersect at right angles. Find the volume common to the cylinders, assuming their axes of symmetry intersect.*

PERDIX

Question 6 of this year's international olympiad competition (*Function*, Vol.9, pt.4) has aroused much interest. The solution offered below is largely taken from written communications (for which many thanks) from John Barton, formerly of the mathematics department of the University of Melbourne, from Geoffrey Watterson, an editor of *Function*, and John Miller, both of the mathematics department at Monash University.

The problem is (No.6, IMO, July 5, 1985):

For every real number x_1, construct the sequence $x_1, x_2, \ldots$, by setting

$$x_{n+1} = x_n \cdot (x_n + \frac{1}{n}) \quad \ldots \quad (1)$$

for each $n \geqq 1$. Prove that there exists exactly one value of x_1 for which $0 < x_n < x_{n+1} < 1$ for every n.

Solution. Some preliminary results on inequalities.

LEMMA 1. *Let $P(x)$ be a non-zero polynomial with positive coefficients. Then $0 < a < b$ implies $P(a) < P(b)$.*

Proof. This follows because $0 < a < b$ implies that $a^t < b^t$ for any positive integer t, whence $ka^t < kb^t$ if $k > 0$.

Q.E.D.

COROLLARY. *Let $P(x)$ be as in lemma 1, and let $Q(x)$ be obtained from $P(x)$ by possibly increasing any or all of the (non-zero) coefficients. Let $0 < a < b$. Then $a\,P(a) < a\,Q(b)$.*

Proof. $P(a) < P(b)$, by lemma 1. $P(b) \leqq Q(b)$, by construction. Hence $P(a) < Q(b)$, whence $a\,P(a) < a\,Q(b)$.

Q.E.D.

Let us call sequences $x_1, x_2, \ldots$, determined from x_1 by the formula (1) of the problem, *C-sequences*. For this problem we can assume that $0 < x_1 < 1$; *we assume this for all C-sequences in what follows*. Notice that it then follows from (1) that every term of each C-sequence is positive.

LEMMA 2. *$x_n < x_{n+1}$ in a C-sequence if and only if $x_n > 1-\frac{1}{n}$; and $x_n = x_{n+1}$ if and only if $x_n = 1-\frac{1}{n}$.*

Proof. By (1) we have

$$x_{n+1} - x_n = x_n^2 + \frac{1}{n}x_n - x_n = x_n(x_n - (1-\frac{1}{n})) \quad ,$$

whence, since $x_n > 0$, the result follows.

Lemma 2 shows that there are two types of C-sequence :

Type I : such that there is a k (k may equal 1) such that

$$x_1 < x_2 < \ldots < x_k \geqq x_{k+1} > x_{k+2} > \ldots > x_{k+t} > \ldots \quad .$$

Type II: such that the sequence steadily increases, i.e.

$$x_1 < x_2 < \ldots < x_k < \ldots < x_n < \ldots < .$$

We say also that x_1 is of type I or of type II according as the C-series it starts is of type I or of type II, respectively.

LEMMA 3. *Let* x_1 *be of type* I . *Then* $x_n < \frac{k}{k+1}$ *for some fixed integer* k *and for all* x_n *in the* C*-sequence started by* x_1 .

Proof. Take k as in the C-sequence of type I exhibited above. Then $x_k \geqq x_{k+1}$; so by lemma 2 , $x_k \leqq 1 - \frac{1}{k}$. But $1 - \frac{1}{k} < 1 - \frac{1}{k+1} = \frac{k}{k+1}$. Thus x_k , the largest term of the C-sequence, is $< \frac{k}{k+1}$. Q.E.D.

LEMMA 4. *Let* a_1 *be of type* I *and* b_1 *be of type* II . *Then* $a_1 < b_1$.

Proof. Formula (1) ensures $a_1 \geqq b_1$ implies $a_2 \geqq b_2$, which in turn implies $a_3 \geqq b_3$; and so on, $a_n \geqq b_n$, for all n. By lemma 3 there then exists k such that $a_n < \frac{k}{k+1}$, for all n , whence $b_n < \frac{k}{k+1}$ for all n . In particular, then $b_{k+1} < 1 - \frac{1}{k+1}$, and so $b_{k+2} \leqq b_{k+1}$, by lemma 2. Hence b_1 is of type I , contrary to assumption. Q.E.D.

Denote the set of all type I numbers by A , and the set of all type II numbers by B . It is easily checked that $\frac{1}{4}$ is of type I and that $\frac{3}{4}$ is of type II . So, by lemma 4 , it follows that the numbers x such that $0 < x < 1$ have been divided into two non-overlapping sets A and B such that all the numbers in A are less than all the numbers in B . There must therefore be a number c , say, separating A and B with the property that $0 < x \leqq c$ for all x in A and $c \leqq x < 1$ for all x in B . [$c \doteq 0.645928$, found by trial and error with a computer.]

In which set does c *lie?*

We now show that c is the least element of B , and thence that c is the unique x_1 sought as the solution to the problem.

We need a couple of formulae.

Let x_1 start the C-series $x_1, x_2, \ldots,$ and y_1 start the C-series $y_1, y_2, \ldots$. Let $x_1 < y_1$. For each n set

$$\varepsilon_n = y_n - x_n . \quad \text{Then by (1)}$$

$$\begin{aligned}
\varepsilon_{n+1} = y_{n+1} - y_{n+1} &= (y_n - x_n)(y_n + x_n + \tfrac{1}{n}) \\
&= \left[(y_{n-1} - x_{n-1})(y_{n-1} + x_{n-1} + \tfrac{1}{n-1})\right] \\
&\qquad \times (y_n + x_n + \tfrac{1}{n}) \\
&= \ldots \\
&= (y_1 - x_1)(y_1 + x_1 + 1)(y_2 + x_2 + \tfrac{1}{2}) \ldots \\
&\qquad \times (y_n + x_n + \tfrac{1}{n})
\end{aligned}$$

i.e. $$\varepsilon_{n+1} = \varepsilon_1 (y_1 + x_1 + 1) \ldots (y_n + x_n + \tfrac{1}{n}) \quad \ldots\ldots \quad (2)$$

This is the first formula needed. We now put this formula into a different form.

From the definition of ε_1 , $y_1 = x_1 + \varepsilon_1$; so $y_1 + x_1 + 1 = \varepsilon_1 + 2x_1 + 1$. Hence

$$\begin{aligned}
y_2 = x_2 + \varepsilon_2 &= x_2 + \varepsilon_1 (y_1 + x_1 + 1) \\
&= x_2 + \varepsilon_1 (\varepsilon_1 + 2x_1 + 1) ,
\end{aligned}$$

and so

$$\begin{aligned}
y_2 + x_2 + \tfrac{1}{2} &= \varepsilon_1(\varepsilon_1 + 2x_1 + 1) + 2x_2 + \tfrac{1}{2} \\
&= \varepsilon_1^2 + (2x_1 + 1)\varepsilon_1 + 2x_2 + \tfrac{1}{2} .
\end{aligned}$$

Continuing in this way we see that $y_n + x_n + \frac{1}{n}$ is a polynomial in ε_1 , with coefficients that are polynomials in $x_1, x_2, \ldots , x_n$ with all non-zero coefficients positive. Hence the same is true for the product $(y_1 + x_1 + 1) \ldots (y_n + x_n + \frac{1}{n})$. Substituting in (2), we have

LEMMA 5. $$\varepsilon_{n+1} = \varepsilon_1 P_n(\varepsilon_1) \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

where $P_n(\varepsilon_1)$ is a polynomial in ε_1 with coefficients that are polynomials in $x_1, x_2, \ldots, x_n$, all of whose non-zero coefficients are positive.

LEMMA 6. *Let x_1 and y_1 be in B with $x_1 < y_1$. Then the C-sequence begun by y_1 has a term greater than 1 .*

Proof. We use formula 2. Since x_1 and y_1 are of type II, $x_k > 1 - \frac{1}{k}$ and $y_k > 1 - \frac{1}{k}$ for all k. Hence $y_k + x_k + \frac{1}{k} > 2 - \frac{1}{k}$, for all k.

Thus, for $k \geqq 2$, $y_k + x_k + \frac{1}{k} \geqq \frac{3}{2}$. Hence, in (2),

$$\varepsilon_{n+1} \geqq (y_1 - x_1)(y_1 + x_1 + 1)\left(\frac{3}{2}\right)^{n-1} .$$

Consequently, since $y_1 > x_1$, for some n, $y_{n+1} > 1$.

Q.E.D.

COROLLARY. *No element of B, except possibly c, generates a C-series for which $0 < x_n < x_{n+1} < 1$, for all n.*

Proof. Let $x_1 \neq c$ be in B. Then $c < \frac{c + x_1}{2} < x_1$, and, since c separates A and B, $\frac{c + x_1}{2}$ is also in B. By the lemma therefore, there exists a member x_{n+1} of the C-sequence started by x_1 such that $x_{n+1} > 1$.

Q.E.D.

LEMMA 7. *The number c does not lie in A.*

Proof. Suppose c lies in A, and so is the largest number of A. Set $x_1 = c$ and $y_1 = x_1 + \varepsilon_1$, where ε_1 is a positive number. Then for the C-series $x_1, x_2, \ldots,$ and $y_1, y_2, \ldots$, setting $\varepsilon_k = y_k - x_k$, for all k,

$$\varepsilon_{n+1} = \varepsilon_1 P_n(\varepsilon_1) ,$$

by formula (3). If ε_1 is less than 1, then $P_n(\varepsilon_1) < P_n(1)$, by lemma 1, since by lemma 5, P_n satisfies the conditions for lemma 1 to hold. Now here, $c = x_1, x_2, \ldots$ are constants, and $y_1, y_2 \ldots$ depend on the choice of ε_1. Hence $P_n(1)$ is a constant, depending on n, $P_n(1) = K_n$, say, and when $0 < \varepsilon_1 < 1$, $\varepsilon_{k+1} < \varepsilon_1 K_k$.

Now choose k as follows: x_1 is of type I and so, by lemma 3, we can choose an interger k such that $x_n < \frac{k}{k+1}$, for all n.

Next choose ε_1 small enough to make
$x_{k+1} < y_{k+1} = x_{k+1} + \varepsilon_{k+1} < \frac{k}{k+1}$ (for example

$\varepsilon_1 = \frac{1}{K_k} \left(\frac{k}{k+1} - x_{k+1}\right)$ will do). Thus y_1 is of type I; but this is a contradiction, since $y_1 > x_1 = c$ and c is the largest number of type I.

Consequently c cannot belong to A.

Thus c *is the least element of* B.

FINAL RESULT. *The* C*-series* $c_1, c_2, \ldots,$ *started by* $c = c_1$, *satisfies* $0 < c_n < c_{n+1} < 1$ *for all* n; *and* c *is the only number with this property.*

Proof. Consider the C-series of type II started by $y_1 = c$ and that started by $x_1 = y_1 - \varepsilon_1$, of type I since $x_1 < y_1 = c$, and so x_1 is in A. Of course we choose ε_1 so that $0 < x_1$.

By lemma 3 there exists an integer k such that $x_n < \frac{k}{k+1}$ for all n. Thus, setting

$$\varepsilon_{n+1} = y_{n+1} - x_{n+1} ,$$

$$y_{n+1} - \varepsilon_{n+1} < \frac{k}{k+1}$$

i.e.

$$y_{n+1} < \frac{k}{k+1} + \varepsilon_{n+1} ,$$

whence

$$y_{n+1} < 1 + \varepsilon_{n+1} , \qquad \text{for all } n .$$

But $\varepsilon_{n+1} = \varepsilon_1 P_n(\varepsilon_1)$, by lemma 5. Let $Q_n(\varepsilon_1)$ be obtained by replacing x_1 by y_1, x_2 by y_2, ..., and x_n by y_n. Since $P_n(\varepsilon_1)$ has the form described in lemma 5, by the corollary to lemma 1, choosing $\varepsilon_1 < 1$, we have

$$\varepsilon_{n+1} = \varepsilon_1 P_n(\varepsilon_1) < \varepsilon_1 Q_n(1) .$$

Here $Q_n(1)$ is a constant L_n, say, depending on the constants $y_1 = c$, $y_2, \ldots, y_n$. Thus

$$\varepsilon_{n+1} < \varepsilon_1 L_n ;$$

so ε_{n+1} can be made as small as we please, by choosing ε_1 sufficiently small. This is true for each n. Hence, for each n,

$$y_{n+1} < 1 + \varepsilon_{n+1}$$

says that y_{n+1} is less than any number greater than 1. So y_{n+1} cannot be bigger than 1. Hence

$$y_{n+1} \leqq 1 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

for all n.

But clearly no $y_n = 1$; for if $y_n = 1$, then $y_n > 1 - \frac{1}{n}$, and so by lemma 2, $y_{n+1} > y_n = 1$. This contradicts inequality (4). Hence $y_n < 1$ for all n.

This completes the proof.