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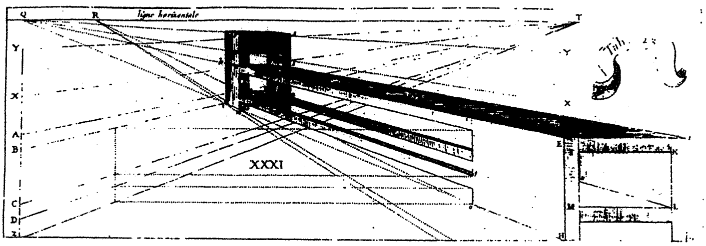

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Mathematics finds, as we point out in our editorial statement opposite, many applications, some of them in fields that do not at first sight look "mathematical".

Professor Praeger's article on weaving (p.7) is a case in point, for the very simple question of how to tell if a certain weave, produced by a designer, will hang together or fall apart when it is actually woven, leads to some interesting and, in its details, quite difficult, mathematics.

Perspective drawing is another such field. For the mathematics this has produced, see Dr Stillwell's article on p.14.

Then mathematics may be applied to economics, and here the name of Vilfredo Pareto is much invoked, to industrial design (folding chairs) and to the calendar we use every day.

THE FRONT COVER

This issue's front cover diagram shows a chair drawn from an unusual perspective. See pp.14 - 22.

CONTENTS

VILFREDO PARETO (1848-1923) A DISTRIBUTION AND A USEFUL OBSERVATION

Neil S. Barnett,
Footscray Institute of Technology

Vilfredo Pareto was trained in the physical sciences and practised as an engineer for twenty years before becoming interested in the application of mathematics to economics. Although born Italian he spent much of his working life in Switzerland; is first major work, 'Cours d'Economie Politique' was based on a series of lectures that he had given at Lausanne. Others of his works include the 'Manuale di Economia Politica' and 'Traité de sociologie générale'.

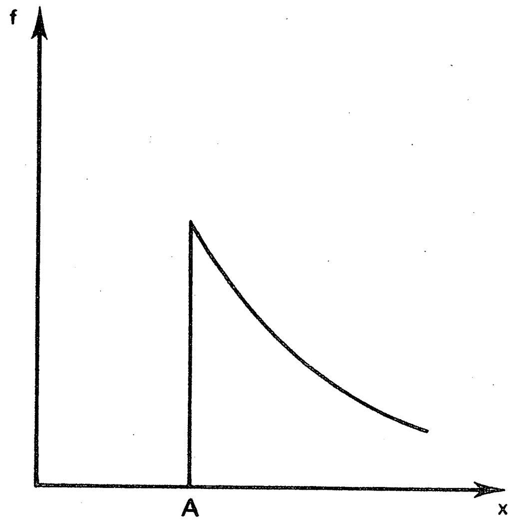
One of the issues dealt with in the first of his works concerned the so-called, 'law of income distribution'. He concluded that income distribution shows a high degree of constancy for different times and countries. If the distribution is plotted on a logarithmic scale it appears as a line with negative slope, the slope being remarkably consistent from population to population. This 'law' has received much criticism on various grounds but none-the-less seems a reasonable approximation in many instances. It can be quantified as follows:

Let the proportion of individuals of a population whose income exceeds x be given by p_x, then $p_x = \frac{B}{x^{\beta+1}}$, hence $\operatorname{Log} p_x = \operatorname{Log} B - (\beta + 1)\operatorname{Log} x$. ($B$ and β are population constants B, $\beta > 0$). If the formula is applied only to those above subsistence incomes it seems to be fairly generally applicable; under such circumstances $\beta \approx 0{\cdot}5$.

There is a theoretical distribution that has evolved from Pareto's observation, not surprisingly called the Pareto distribution. A continuous random variable, X, is defined (in fact replacing $\beta + 1$ above by α) such that the probability density function (p.d.f.) is

$$f(x) = \begin{cases} \dfrac{B}{x^{\alpha+1}} & \text{where } \alpha > 0 \text{ and } x \geqslant A \\ 0 & \text{elsewhere.} \end{cases}$$

See the graph below.



Since $f(x)$ is a p.d.f., then

$$1 = \int_{x=A}^{\infty} f(x)dx = B\left[\frac{x^{-\alpha}}{-\alpha}\right]_{x=A}^{\infty} .$$

Now, since $\alpha > 0$, this gives

$$\frac{B}{\alpha A^{\alpha}} = 1$$

which means that $B = \alpha A^{\alpha}$ and

$$f(x) = \frac{\alpha A^{\alpha}}{x^{\alpha+1}}, \quad \alpha > 0, \quad x \geqslant A$$

and so the probability that $x \geqslant X$ is

$$\int_{X}^{\infty} f(x)dx = \left(\frac{A}{X}\right)^{\alpha} \quad \text{for} \quad X \geqslant A .$$

The mean and variance of Pareto's distribution can be shown to be $E(X) = \frac{\alpha A}{\alpha - 1}$ if $\alpha > 1$

and $V(X) = \frac{A^2 \alpha}{(\alpha - 2)(\alpha - 1)^2}$ if $\alpha > 2$.

It is interesting to note that:

(i) if $0 < \alpha \leqslant 1$ then the distribution has neither a finite mean nor finite variance,

(ii) if $1 < \alpha \leqslant 2$ the distribution has a finite mean but no variance.

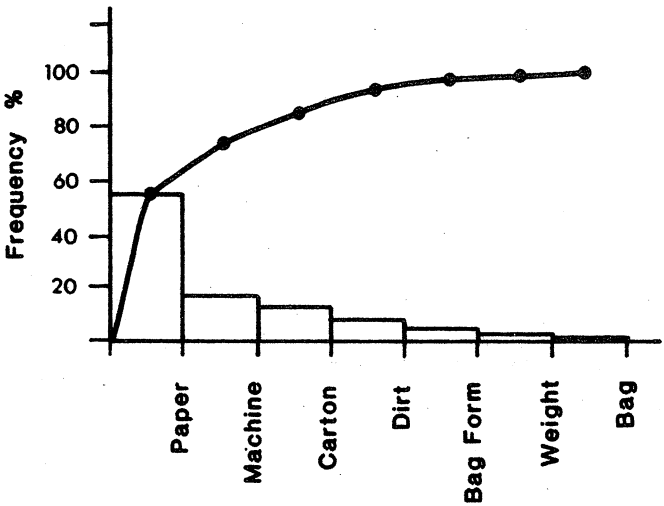
Although many of Pareto's ideas on social and economic matters were controversial, his 'law' of income distribution developed from the very simple, indisputable observation, at the time, that just a few people held most of the national wealth. In this observation he seems to have struck upon a principle relevant to many cause-and-effect relationships. Put simply, it has generally been found that for many observed effects, of all the contributing causes, only a few contribute most to the effect. This may sound vague and rather trite but it has proved a very useful concept in helping solve many industrial problems.

To be more specific, it is often the case that approximately 80% of a company's profit is made on 20% of the different items that it sells and approximately 80% of product defects will be the result of approximately 20% of the total fault types that can occur. This latter notion that most problems stem from a few causes can be used to give direction to the problem solving process. The 80%/20% guide is of course not rigid; it is merely an approximation, but in many instances it is a remarkably good one.

An example of the application of Pareto's principle was passed on to me recently by the Production Management Team at Bradford University. As I understand it, a student was working in conjunction with a company manufacturing tea-bags that was plagued with quality problems. The student, by studying the manufacturing process and liaising with those involved in the manufacture, was able to characterize different quality problems into seven disjoint categories. These were weight problems, bag problems, dirt problems, machine problems, bag formation problems, carton problems and paper problems. All quality problems arising in a two week production run were recorded in detail and by type. The following data were obtained:

Problem Type	Frequency %	Cumulative Frequency %
Paper	55·95	56
Machine	17·98	74
Carton	11·22	85
Dirt	6·85	92
Bag Formation	4·93	97
Weight	1·91	99
Bag	1·16	100

From this table the following cumulative frequency diagram was drawn. The problem types are drawn in rank order (from highest contributor to overall problems to the lowest contributor).



The cumulative frequency diagram pictures what is immediately apparent from the table (as it is displayed in rank order), that 74% of the problems are caused by 29% of the primary causes. It is immediately apparent then that tackling paper and machine problems will have the greatest effect on the total number of problems arising. In this particular case both paper and machine problems proved relatively easy to solve. Prior to this simple analysis management were unaware that minor adjustments would have a major impact on the reduction of production quality problems.

It must of course be acknowledged, when tackling industrial problems of this nature, that a business exists to make profit. Manufacturing problems cause financial loss so a company wants to know what are the most costly problems. The analysis of problem costs in the above cited example showed carton problems to represent the largest proportion of problem cost followed by machine faults. Paper faults were the third major contributor. Between them carton and machine problems (29% of problem types) represented 79% of the total cost.

Thus, presented with two cumulative frequency diagrams, one relating problems to types and one relating cost to problem types, management had at its disposal information on where most profitably and effectively to concentrate its effects on improvement.

This systematic search for the main components in an 80%/20% relationship is often called a Pareto analysis, in acknowledgement of Pareto's observation of wealth distribution last century. Perhaps national and international economies have become sufficiently complex that in this realm Pareto's principle is no longer a useful one. None-the-less, in many other areas his principle has proved to be a very useful observation.

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TWO MORE BREAKTHROUGHS

Two further long-standing conjectures have now been resolved.

The first is the so-called Mertens conjecture which has been disproved. This was first proposed by Stieltjes (whose discoveries in integration were described in the previous issue of *Function*) in 1885, and reformulated by Mertens in 1897. This was a rather technical result in number theory, whose truth would have implied the truth of another, much more famous conjecture, known as the Riemann hypothesis.

It is now known that the Mertens conjecture is false however. This was discovered by Andrew Odlyzko of Bell Laboratories (USA) and Herman Te Riele of Amsterdam. They have not discovered a specific counterexample, but have demonstrated that one must exist.

The Riemann hypothesis itself is now believed, however, to be true, despite this apparent set-back. The proof is due to a Japanese mathematician, Hideya Matsumoto, now resident in Paris.

There is just one word of warning here, however. The full proof is not yet published, and, till it is, there is still some ground for scepticism.

For more on this, see *New Scientist*, 18 April 1985.

MATHEMATICS AND WEAVING: I. FABRICS AND HOW THEY HANG TOGETHER†

Cheryl E. Praeger, University of Western Australia

Although weaving is one of the oldest activities of mankind, and much has been written about it, it is only in the last five years that the theoretical aspects of weaving have received much attention. In this paper I will give a mathematical description of a fabric and then discuss the problem of when a fabric hangs together.

1. *Description of a fabric.*

As you probably know a roll of woven fabric has some threads, called *warp* threads, running along the length of the fabric and other threads, called *weft* threads running across the fabric. However I want to give a precise mathematical description of a fabric. This is most easily done diagrammatically. The real fabric (Figure 1(a)) is idealized by considering a warp thread as a *vertical strand* (a strand is the set of points in the plane lying strictly between two infinite parallel lines) and a weft thread as a *horizontal strand*; these horizontal and vertical strands are *woven* under and over each other to form an interlacement pattern (Figure 1(b)). The intersection of a vertical strand and a horizontal strand, called a *square*, is coloured black if the vertical strand is on top and white if the vertical strand is underneath (Figure 1(c)). This diagram of black and white squares is called by weavers a design, diagram, draft, drawdown diagram or *point diagram* and is the way a weaver would usually describe a weaving pattern. However we (and the weaver) will not be considering a point diagram covering the

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†This article is based on the Professor Praeger's Hanna Neumann memorial lecture delivered to the Fifth International Conference on Mathematics Education, August 1984.

whole plane. All fabric diagrams which we shall consider will be *periodic*, that is they will consist of a finite block of squares which is simply repeated as we move across and/or down the fabric. A block of squares of smallest size with this property is called a *fundamental block* for the fabric. There may be more than one fundamental block but they all have the same "size"; see Figure 2(a) for the fundamental blocks for the "plain weave" fabric represented in Figure 1. In practise the point diagram is taken as either a fundamental block of the fabric or some union of fundamental blocks. To be able to use some mathematics in investigating problems in weaving we take one further step: we replace all black squares by a 1 and all white squares by a 0 so that the point diagram becomes a binary matrix D, that is a matrix with entries 0 or 1.

2. *Hanging together.*

Now a moment's thought should convince you that not all binary matrices will represent a fabric which *hangs together*, that is a fabric such that *no* proper subset of warp and weft threads can be completely lifted off the rest of the fabric. For example a whole column of zeros or ones means that the corresponding warp thread is not woven into the fabric. However there are many examples of a fabric's not hanging together which are much less obvious than these trivial examples (where an all-zero or an all-one row or column is present). I am told that in the past the usual way in which a weaver decided whether or not a fabric would hang together was to do a trivial weave. However in 1980 the British mathematician C.J.C. Clapham produced a mathematical algorithm to answer this question.

To see the problem more clearly, let $D = (d_{ij})$ be an $m \times n$ binary matrix, that is the row i column j entry d_{ij} is 0 or 1. Suppose that the fabric corresponding to D does not hang together - say a set T of vertical strands and a set S of horizontal strands lifts off the rest of the fabric. We can regard T as a subset of $\{1,2,...,n\}$ and S as a subset of $\{1,2,...,m\}$. Now the strands of S and T lift off the rest of the fabric if and only if

(i) every vertical strand *not* in T goes under every horizontal strand in S, and

(ii) every horizontal strand *not* in S goes under every vertical strand in T.

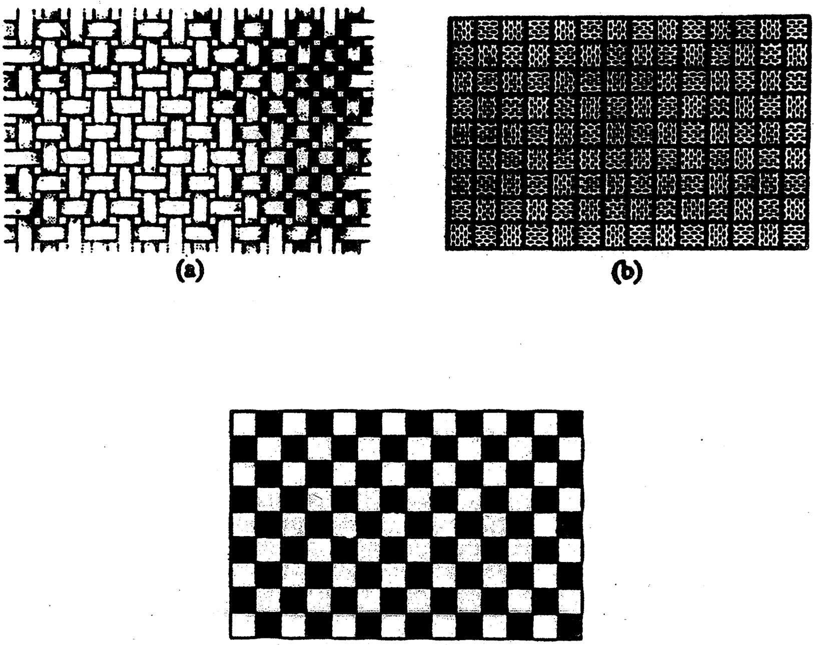
These two conditions are equivalent to

(i)' $d_{ij} = 0$ for all $i \in S$, and $j \notin T$,

and

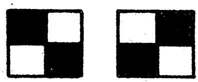
(ii)' $d_{ij} = 1$ for all $i \notin S$, and $j \in T$.

Thus the fabric will not hang together if and only if there are subsets T,S of $\{1,2,...,n\}$ and $\{1,2,...,m\}$ respectively (with at least one a proper non-empty subset)



(a) (b) (c)

Figure 1



(a)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(b)

Figure 2

such that (i)' and (ii)' are true. In other words the fabric will not hang together if and only if it is possible to rearrange the rows and columns of D in such a way that the resulting matrix has the form

$$\begin{pmatrix} X & 0 \\ J & Y \end{pmatrix}$$

where 0 is an all-zero matrix, J is an all-one matrix (with at least one of 0 and J non-vacuous) and X and Y are not restricted. However to run through all possible sets S,T to see whether they have this property would take far too long when m and n are large. Clapham introduced an alternative method, whose details are somewhat technical for *Function*, but interested readers can find it in the *Bulletin of the London Mathematical Society, Vol.*12 (1980), pp.161-164. Essentially it consists of the evaluation of n numbers. Of these one is equal to zero necessarily, but if any of the others is zero, then the fabric does not hang together. Otherwise it does. This algorithm is easily implemented on a computer.

The only theoretical advance I know of on Clapham's work is due to T.C. Enns and was published in the technical journal *Geometrica Dedicata, Vol.*15 (1984), pp.259-260. He has given an efficient algorithm for determining whether or not a fabric woven with any number of sets of parallel threads (not just one set of warp threads and one set of weft threads) will hang together.

For more on the mathematics of weaving, see the article by B. Grunbaum and J.C. Shepherd "Satins and twills: an introduction to the geometry of fabrics" in *Mathematics Magazine, Vol.*53 (1980), pp.139-161. In a sequel to the present article, I shall discuss the mathematics involved in the actual setting up of the loom.

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BEWARE OF MATHEMATICIANS

Canon XXXVI of the Council of Laodicea, held some time in the period 343-381 AD, reads:

They who are of the priesthood, or of the clergy, shall not be magicians, enchanters, mathematicians, or astrologers; nor shall they make what are called amulets, which are chains for their own souls. And those who wear such, we command to be cast out of the Church.

This looks as if several eminent mathematicians who were also in holy orders, were in breach of Canon Law. However, a note explains the apparent anomaly.

"Mathematicians" are they who hold the opinion that the celestial bodies rule the universe, and that all earthly things are ruled by their influence.

THE CALENDAR†

S. Rowe, Student,
Swinburne Institute of Technology

Historically, all measurements of time have depended on astronomical observations - the day is measured from the rotation of the Earth, the week approximates the changing phases of the moon, the month is measured from the revolution of the moon around the Earth and the year is measured from the revolution of the Earth around the Sun.

Many ancient civilizations, particularly the Babylonian, based their calendars on the cycles of the moon, and the lunar measurement of years has been preserved in the modern Jewish, Chinese and Moslem calendars. Against this, the Egyptians based their calendars on the Sun (which also figured prominently in their religion). The Egyptian civilization depended upon the seasonal rising of the Nile, which was closely associated with the solar cycle. Ancient peoples determined the solar year by observing the rising of a bright star after it had been invisible because of its proximity to the Sun. A common star used for this purpose was Sirius. By averaging many such observations, the solar year was found to be very close to 365 days.

In ancient Rome, months were based on lunar cycles. The pontifices watched for the first appearance of the thin crescent moon after the new moon so that they could declare the beginning of the newmonth. This first day, shouted from the steps of the Capitol, was termed Kalendae, which means the calling. Our word calendar is derived from this term.

Unfortunately for our measurement of time, the lunar cycle is not a whole number of days, nor is the time Earth takes to complete an orbit of the Sun relative to the stars. The Moon's cycle is 29.53059 days, while the Earth's orbit around the Sun takes 365.242196 days. So 12 months are short of a year, and 13 months would give us a year that is too long. And our seven day week (which is based on religion), although close to the lunar phases, is not a factor of the lunar period, the month or the year.

When the Romans adopted the Egyptian solar year at the time of Julius Caesar, their own lunar-solar calendar was very much in error. Introduced to Rome by an astronomer, the Egyptian calendar was ordered into official Roman use by Julius Caesar in 45 BC. It was called the Julian calendar, and was based on a solar year of 365.25 days. The year was divided into months, of which eleven contained 30 or 31 days

† This article first appeared in the Bulletin of the Lions Club of Belgrave, March, 1985.

and the twelth had 28 days only. The first month was March and the last was February. July is named after Julius Caesar and August after Augustus Caesar, both months being allocated the full 31 days, as befitted a Caesar. The seventh month was named September, the eighth October, the ninth November and the tenth December after the Latin septem, octo, novem and decem for seven, eight, nine and ten respectively.

The Julian calendar lost approximately one-quarter day each year. This loss was corrected by adding an extra day to the twelth month (February) every fourth year, which was the leap year. Nevertheless, this calendar gradually became out of step with the seasonal position of the Sun relative to the stars. The year of the Julian calendar was actually 11 minutes 4 seconds longer than the time it takes the apparent Sun to revolve to precisely the same position. By 1500 the error amounted to approximately 11 days. Christian religious festivities based on Easter assumed a fixed vernal equinox of March 21, and as a consequence they were becoming gradually out of step with the seasons. Accordingly, Pope Gregory XIII entrusted a reformation of the calendar to a German Jesuit whose latinized name is Clavius. Clavius used a scheme devised by a Neapolitan astronomer in which centuries would not be leap years unless perfectly divisible by 400. To correct the calendar, Pope Gregory ordered that October 15, 1582, should follow October 4. Despite protests from angry mobs, who thought that ten days of their lives were being stolen, the correction was made and the new calendar was called the Gregorian calendar. The new calendar also moved the beginning of the year from March 25 to January 1.

The Gregorian calendar was adopted by most of the Roman Catholic countries and by Denmark and the Netherlands in 1582. But it was nearly two centuries before it was generally accepted. During that time, a traveller could leave England in February 1679, for example, and find that it was February 1680 in some parts of Europe and Scotland. The day of the month was also different between England and some parts of Europe.

Finally, other countries began to accept the new calendar. The Protestants in Germany and Switzerland adopted it in 1700, Britain and the American colonies in 1752 (omitting the eleven days between September 2 and 14), Prussia began to use it in 1778, Ireland in 1782, Russia in 1902. Following the French Revolution, a new calendar was adopted in France, the first day of the year being September 22, 1792. This calendar was used until December 1805, when France accepted the Gregorian calendar again.

Other calendars are still in use, however, particularly in regard to religious events. The Jewish calendar uses a lunar cycle and a solar cycle. The months are lunar months, but they are about 11 days short of a solar year. A thirteenth month periodically has to be intercalated to maintain some synchronism with the solar cycle. The Moslem calendar ignores the solar cycle completely and is tied to lunar cycles with alternate months of 30 and 29 days. The year begins at different seasons over a 32.5 year cycle.

Prior to World War II there was an attempt among some business in Europe to introduce a 13 month calendar in which all months would have four weeks. This business calendar would have allowed more meaningful financial comparisons, but it did not receive wide acceptance.

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SUSAN'S WORLD FIRST

We have all heard of the prestigious Rhodes scholarships. Each year, one is awarded from each of the Australian states and from other parts of the former British Empire (including, quaintly, the U.S.A.). But how many of us have heard of the Rhodes post-doctoral fellowship? Yes, fellowship, not fellowships. There is only one, and this is offered world wide each year.

The holder for 1986 will be Susan Scott, the first Australian ever to win the award.

Dr Scott graduated from Monash University in 1979 and continued her studies at post-graduate level at the University of Adelaide, where she has been conducting research in mathematical physics.

The $40,000 award will cover her fares, accommodation and research expenses and will take her to Oxford for two years from January 1986.

Two of the areas she explores in her work are the origin of the universe and the nature of black holes. "I am fascinated by physics," Dr Scott said, "I just want to know as much about physics as I can."

Dr Scott also looks at whether there are other universes beyond our own.

"It's quite possible there are other universes," she said.

"If there are, we have no way of telling. It's possible we are unique."

Dr Scott has been enthusiastic about science since she was a teenager and is concerned about the small number of women involved in mathematical research.

She is keen to pursue her research abroad for a few years but eventually wants to teach so she can pass on her enthusiasm for science to future generations.

Based on an article by Jo McKenna, *The Herald* 22/7/85.

PROJECTIVE GEOMETRY

J.C. Stillwell, Monash University

Perspective.

Perspective may be simply described as the realistic representation of spatial scenes on a plane. This of course has been a concern of painters since ancient times, and some Roman artists seem to have achieved correct perspective by the first century BC. However, this may have been a stroke of

Figure 1

individual genius rather than the success of a theory, because the vast majority of ancient paintings show incorrect perspective. If indeed there was a classical theory of perspective, it was well and truly lost during the dark ages. Medieval artists made some charming attempts at perspective, but always got it wrong, and errors persisted well into the 15th century (Figure 1).

The discovery of a method for correct perspective is usually attributed to the Florentine painter-architect Brunelleschi (1377-1446), around 1420. The first published method appears in the treatise On Painting, Alberti (1436). The latter method, which came to be known as Alberti's veil, was to set up a piece of transparent cloth, stretched on a frame, in front of the scene to be painted. Then, viewing the scene with one eye, in a fixed position, one could trace the scene directly onto the veil. Figure 2 shows this method, with a peephole to maintain a fixed eye position, as depicted by Durer (1525).

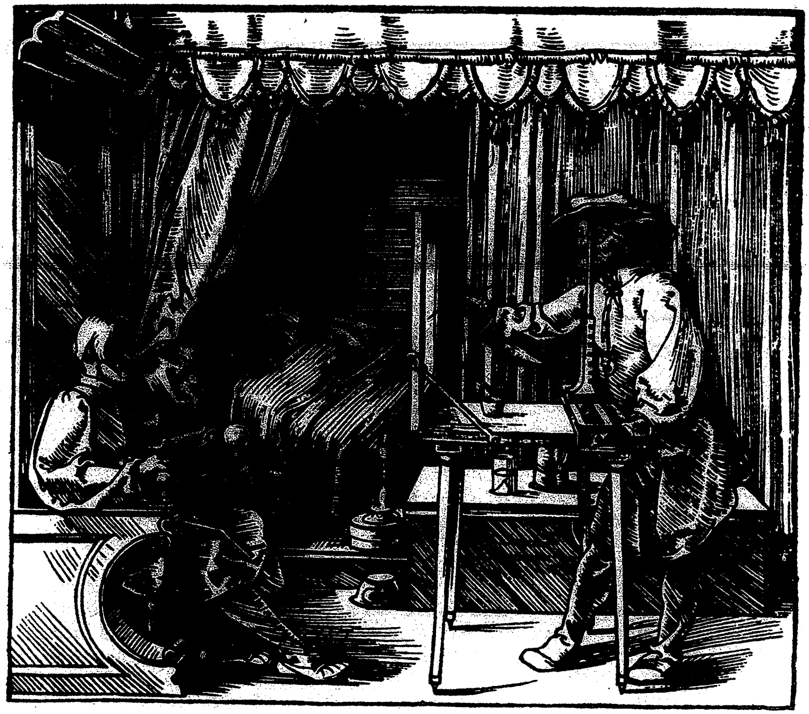


Figure 2

Alberti's veil was fine for painting actual scenes, but to paint an imaginary scene in perspective some theory was required. The basic principles used by Renaissance artists were:

(i) a straight line in perspective remains straight
(ii) parallel lines either remain parallel or converge to a single point (their vanishing point).

These principles suffice to solve a problem artists frequently encountered : the perspective depiction of a square-tiled floor. Alberti (1436) solved the special case of this problem in which one set of floor lines is horizontal, i.e. parallel to the horizon. His method, which became known as the *costruzione legittima*, is indicated in simplified form in Figure 3.

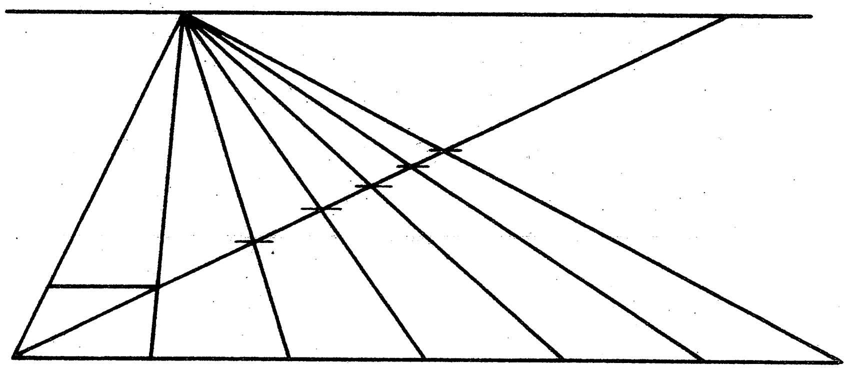


Figure 3

The non-horizontal floor lines are determined by spacing them equally along the base line (imagined to touch the floor) and letting them converge to a vanishing point on the horizon. The horizontal floor lines are then determined by choosing one of them arbitrarily, thus determining one tile in the floor, and then producing the diagonal of this tile to the horizon. The intersections of this diagonal with the non-horizontal lines are the points through which the horizontal lines pass. This is certainly true on the actual floor (Figure 4), hence it remains true in the perspective view.

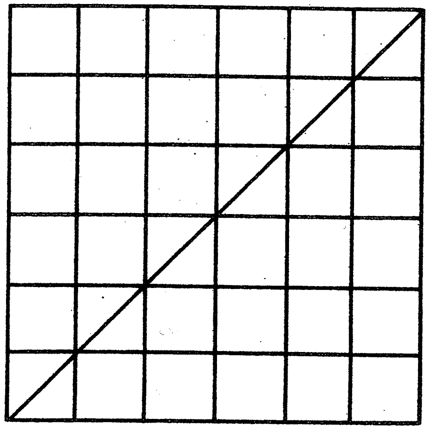


Figure 4

The same principles in fact suffice to generate a perspective view of a tiled floor given an arbitrarily situated tile.

Anamorphosis.

It is clear from the Alberti veil construction that a perspective view will not look absolutely correct except when seen from the viewpoint used by the artist. Experience shows, however, that distortion is not noticeable except from extreme viewing positions. Following the mastery of perspective by the Italian artists, an interesting variation developed, in which the picture looks right only from one, extreme, viewpoint. The first known example of this style, known as anamorphosis, is an undated drawing by Leonardo da Vinci from the Codex Atlanticus (compiled between 1483 and 1518). Figure 5 shows part of this drawing, a child's face which looks right when viewed with the eye near the right hand edge of the page.

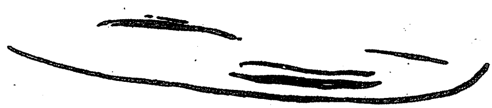


Figure 5

The idea was taken up by German artists around 1530. The most famous example occurs in Holbein's painting The two ambassadors (1533). A mysterious streak across the bottom of the picture becomes a skull when viewed from near the picture's edge. The art of anamorphosis reached its technically most advanced form in France in the early 17th century. It seems no coincidence that this was also the time and place of the birth of projective geometry. In fact the key figures in the two fields, Niceron and Desargues, were well aware of each others' work.

Niceron (1613 - 1646) was a student of Mersenne and, like him, a monk in the order of Minims. He executed some extraordinary anamorphic wall paintings, up to 55 metres long, and also explained the theory in a book La perspective curieuse, published in 1638. The cover illustration is one of his illustrations : anamorphosis of a chair.

The anamorphosis, viewed normally, shows a chair like none ever seen, yet from a suitably extreme point one sees an ordinary chair in perspective. This example encapsulates an important mathematical fact : a perspective view of a perspective view is not in general a perspective view. Iteration of perspective views gives what we now call a projective view, and Niceron's chair shows that projectivity is a broader concept than perspectivity. As a consequence, projective geometry, which studies the properties which are invariant under projection, is broader than the theory of perspective. Perspective itself did not develop into a mathematical theory, descriptive geometry, until the end of the 18th century.

Desargues' projective geometry.

The mathematical setting in which one can understand Alberti's veil is the family of lines ("light rays") through a point (the "eye"), together with a plane V (the "veil"), as in Figure 6.

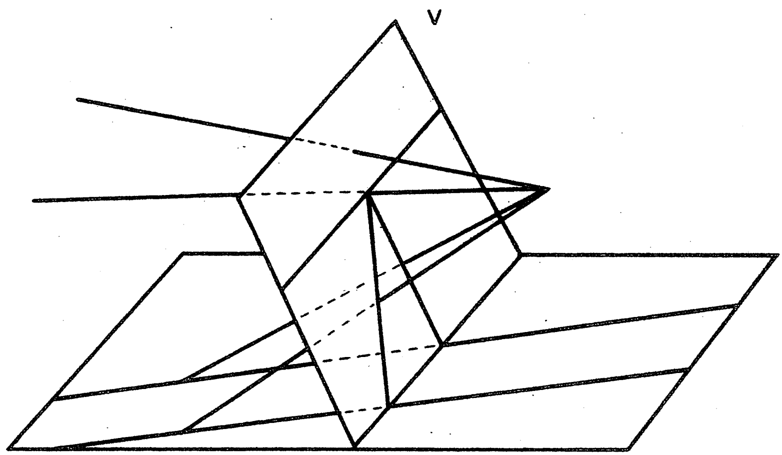


Figure 6.

In this setting, the problems of perspective and anamorphosis were not very difficult, but the concepts were interesting and a challenge to traditional geometric thought. Contrary to Euclid, one had

(i) points at infinity ("vanishing points") where parallels met,

and

(ii) transformations which changed lengths and angles (projections).

The first to construct a mathematical theory incorporating these ideas was Desargues (1591-1661), although the idea of points at infinity had already been used by Kepler in 1604.

Desargues' book <u>Brouillon project d'une atteinte aux événemens des recontres du Cone avec un Plan (1639)</u> (Schematic sketch of what happens when a cone meets a plane) suffered an extreme case of delayed recognition, being completely lost for 200 years. Fortunately, his two most important theorems, the so-called Desargues' theorem and the invariance of the cross-ratio, were published in a book on perspective by Bosse (1648).

Kepler and Desargues both postulated one point at infinity on each line, closing the line to a "circle of infinite radius". All lines in a family of parallels share the same point at infinity. Non-parallel lines, having a finite point in common, do not have the same point at infinity. Thus any two distinct lines have exactly one point in common - a simpler axiom than Euclid's. Strangely enough, the line at infinity was only introduced into the theory by Poncelet (1822), even though it is the most obvious line in perspective drawing, the horizon. Desargues made extensive use of projections in the Brouillon project; he was the first to use them to prove theorems about conic sections.

Desargues' theorem is a property of triangles in perspective illustrated by Figure 7.

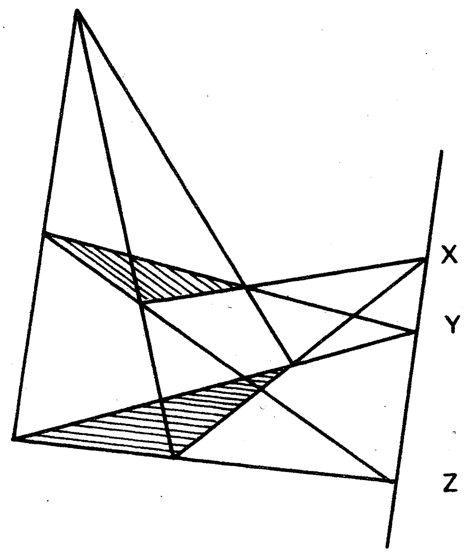


Figure 7.

The theorem states that the points X,Y,Z at the intersections of corresponding sides lie in a line. This is obvious if the triangles are in space - the line is the intersection of the planes containing them. The theorem in the plane is subtly but fundamentally different, and requires a separate proof, as Desargues realised. In fact, Desargues' theorem was shown to play a key role in the foundations of projective geometry by Hilbert (1899).

The invariance of the cross ratio answers a natural question first raised by Alberti : since length and angle are not preserved by projection, what is ? No property of three points on a line can be invariant because it is possible to project any three points on a line to any three others. At least four points are therefore needed, and the cross ratio is in fact a projective invariant of four points. The cross ratio $(ABCD)$ of points A, B, C, D on a line (in that order) is $\frac{CA}{CB} / \frac{DA}{DB}$. Its invariance is most simply seen by re-expressing it in terms of angles using Figure 8.

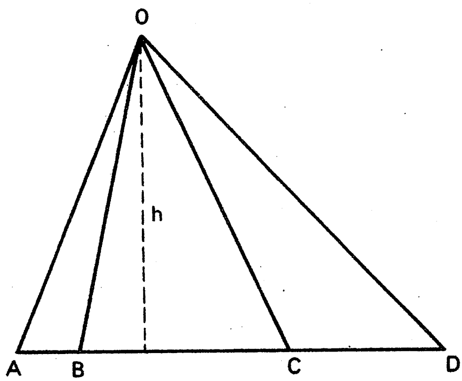


Figure 8.

Let O be any point outside the line and consider the areas of the triangles OCA, OCB, ODA, ODB. First use bases on the given line and height h , then recompute using OA and OB as bases, and heights expressed in terms of the sines of angles at O ;

$$\tfrac{1}{2}h \cdot CA = \text{area } OCA = \tfrac{1}{2} OA \cdot OC \sin \angle COA$$

$$\tfrac{1}{2}h \cdot CB = \text{area } OCB = \tfrac{1}{2} OB \cdot OC \sin \angle COB$$

$$\tfrac{1}{2}h \cdot DA = \text{area } ODA = \tfrac{1}{2} OA \cdot OD \sin \angle DOA$$

$$\tfrac{1}{2}h \cdot DB = \text{area } ODB = \tfrac{1}{2} OB \cdot OD \sin \angle DOB \; .$$

Substituting the values of CA, CB, DA, DB from these equations we find the cross ratio in terms of angles at O :

$$\frac{CA}{CB} \Big/ \frac{DA}{DB} = \frac{\sin \angle COA}{\sin \angle COB} \Big/ \frac{\sin \angle DOA}{\sin \angle DOB} \; .$$

Any four points A', B', C', D' in perspective with A, B, C, D from a point O have the same angles (Figure 9), hence they will have the same cross ratio .

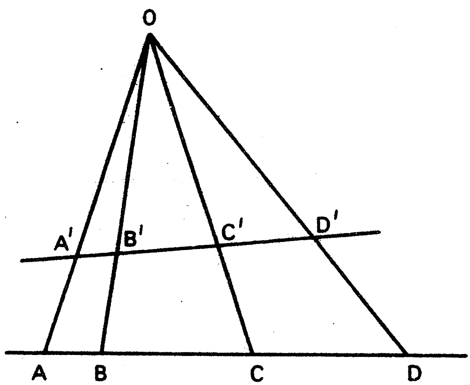


Figure 9

But then so will any four points A'', B'', C'', D'' projectively related to A, B, C, D since a projectivity is, by definition, the product of a sequence of perspectives.

THE FOLDING CHAIR†

In the market-place in Groningen there's a doughnut stand. The other day I let my doughnuts get cold, immersed as I was in the construction of the folding chairs. Look at the photograph on p.24 and the drawing on p.27. (Incidentally, a small simplication has been made in the drawing but only pettifoggers would take note of that.)

There are six turning points and no sliding mechanisms, as in many other types of folding chairs. The basic form is a quadrilateral $ABCD$ which is more or less a parallelogram. Of course - otherwise you could never fold the chair flat.

The striking feature is the linking piece AQ. What is its function? Why isn't the back leg simply connected to A? The jam had already melted when I understood that then the chair could not be folded up. For then the triangle APD would be a rigid figure, and hence the quadrilateral $ABCD$ would be rigid too.

But how do you then get the length of AQ? And why does the chair remain standing exactly in the position drawn, even when you sit down on it?

The answer is reached from the picture on p.24 and the other photographs. During unfolding, the chair angle D becomes smaller and you can imagine how the weight of the "sitter" on the chair will try to make that angle as small as possible; his or her weight pushes the parallelogram $ABCD$ into its lowest position. Now take the back-rest AD as a fixed line. Then DP and AQ become rotating rays with D,A respectively as central points.

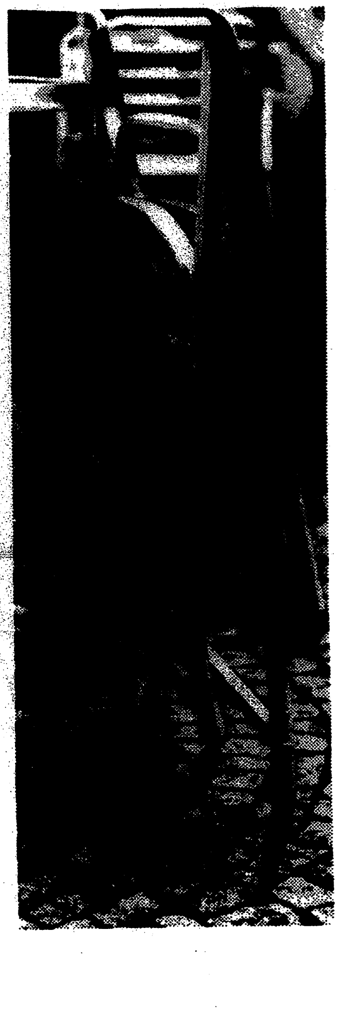
So the question is: in which position is angle D minimal? To see that we look at triangle ADP, of which DA and DP are the hinging sides. Angle D is minimal if the opposite side AP (which you'll just have to draw in your mind) is as small as possible. That occurs (and now look at triangle AQP) if angle Q is minimal. Well, angle Q is of course minimally zero, that is, if A is on PQ. So the chair remains in that position if you sit down on it!

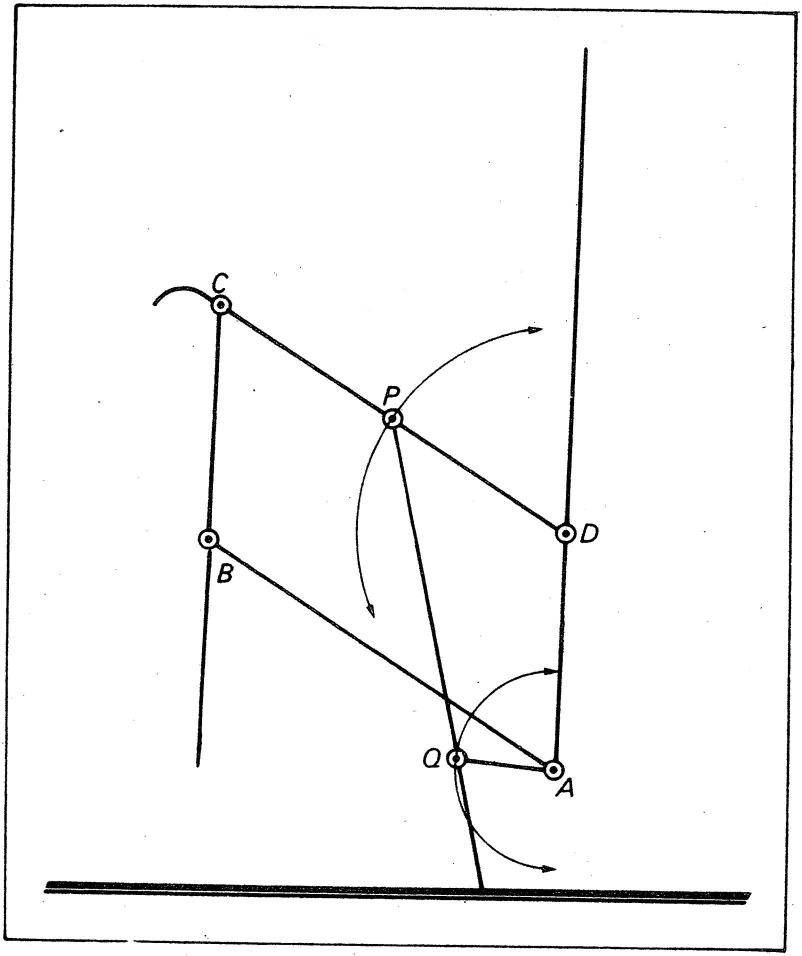
---

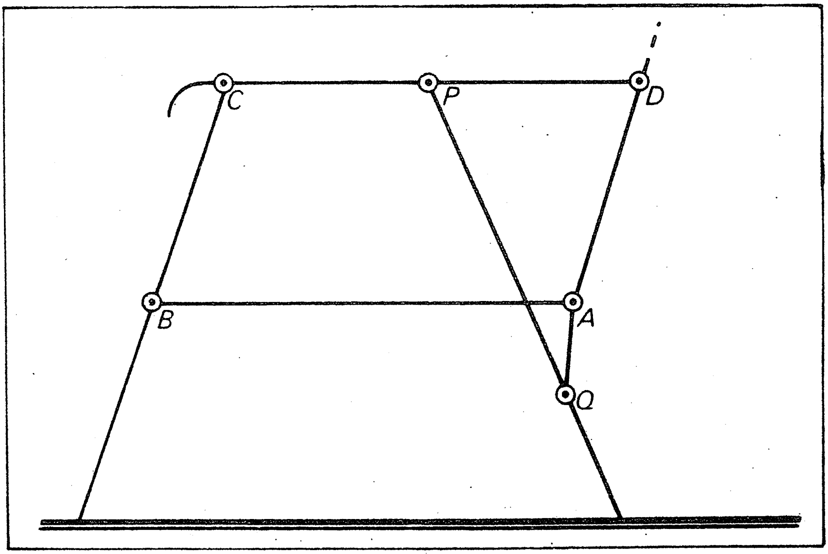
†This article is a translation from the Dutch by A.-M. Vandenberg. It first appeared in the journal *Pythagoras Vol.24, Part* 4 and is reproduced under an exchange agreement.

This doesn't happen, of course, with the real chair, because A runs into the back leg PQ. There's a reason for that, too: if A could pass the line PQ (which could be quite feasible technically) the chair would "spring" a little because you can move *through* that minimum. Something like a marble rolling to and fro in a little hollow.

After all, the lengths of DA, DP, AQ and PQ can't be chosen independently of each other. Look at the photograph of the folded chair, and you'll see that $DA + DP = AQ + PQ$.




C
P
D
B
Q
A



PERDIX

Australians are good at competitive sport and the sport of mathematical olympiad contests is proving no exception. 42 countries sent teams to take part in the 1985 International Mathematical Olympiad in Finland. The Australian team was placed 11th. Team members were:

Shane Booth,	Shepparton, Victoria
John Graham,	Sydney, N.S.W.
Alisdair Grant,	Melbourne, Victoria
Andrew Hassell,	Perth, W.A.
David Hogan,	Sydney, N.S.W.
Catherine Playout,	Sydney, N.S.W.

Andrew Hassell was placed 7th out of the 209 competitors and was awarded a gold medal (14 competitors were awarded gold medals). John Graham and Alisdair Grant were awarded silver medals (35 silver medals were awarded). Shane Booth was awarded a bronze medal (52 bronze medals were awarded).

Congratulations to the Australian team!

Continued on p.30.

PROBLEM SECTION

Problems 9.1.1, 9.1.3 are still outstanding, and we leave them that way for this issue, as they are interesting and we would like to give readers a further chance to do them.

We have solutions to 9.2.1, 9.2.2 which we print below, and have also received correspondence on problems posed in Volume 9, Part 3; this will be reported on in the next issue.

SOLUTION TO PROBLEM 9.2.1

The problem, from Hall and Knight's *Higher Algebra*, read as follows.

"There are three Dutchmen of my acquaintance to see me, being lately married; they brought their wives with them. The men's names were Hendriek, Claas, and Cornelius; the women's Geertruij, Catriin, and Anna; but I forgot the name of each man's wife. They told me they had been at market to buy hogs; each person bought as many hogs as they give shillings for one hog; Hendriek bought 23 hogs more than Catriin; and Claas bought 11 more than Geertruij; likewise, each man laid out 3 guineas more than his wife. I desire to know the name of each man's wife." (Note: 1 guinea = 21 shillings.)

David Shaw of Geelong West Technical School, who submitted the problem, also sent us his solution.

Let h = number of hogs bought by each man and w = number of hogs bought by each wife. Then "each person bought as many hogs as they gave shillings for one hog" and "each man laid out 3 guineas more than his wife" leads to the equation

$$h^2 - w^2 = 63 \qquad h,w \in J^+ .$$

The factor pairs of 63 are (63,1), (21,3), (9,7).

$$(h + w)(h - w) = 63$$
$$h + w = 63$$
$$h - w = 1$$

gives $h = 32$, $w = 31$.

(21, 3) gives $h = 12$, $w = 9$,

(9, 7) gives $h = 8$, $w = 1$.

"Hendrick bought 23 hogs more than Catriin" indicates that Hendrick bought 32 hogs and Catriin 9.

"Claas bought 11 more than Geertruij" indicates that Clees bought 12 and Geertruij 1.

So, Catriin is Claas' wife
Geertruij is Cornelius' wife and
Anna is Hendrick's wife.

SOLUTION TO PROBLEM 9.2.2

We asked for a proof that no (positive) integers exist such that

$$x^4 - y^4 = z^2 .$$

Devon Cook of Urrbrae Agricultural High School (Netterby, S.A.) solved this problem. He writes:

I will use the method of infinite descent, first used I believe, by Fermat. The equation can be written as $(x^2)^2 - (y^2)^2 = z^2$, the Pythagorean identity which has the well known solutions

$$x^2 = n^2 + m^2 , \; y^2 = 2nm , \; z = n^2 - m^2 .$$

It remains to show that since the two variables x^2 and y^2 are perfect squares, there can be no solution. We have $y^2 = 2nm$ and thus there must be two integers p, q such that $n = p^2$, $2m = q^2$, as we may choose n, m to be co-prime. But $n^2 = x^2 - m^2$ which is itself Pythagorean and thus $n = s^2 - t^2$ and $m = 2st$ (or vice versa, when the analysis is similar). Thus $2m = 4st = q^2$ and following on again, there must be integers x' and y' such that $s = x'^2$, $t = y'^2$. Therefore

$$x'^4 - y'^4 = s^2 - t^2 = p^2 \quad \text{where} \quad p < z .$$

Thus there is a set of integers which satisfy the equation but using a value of z which is smaller than the original, viz: p. Since this process can be repeated ad infinitum, there can be no such integers. The problem is solved!

There does still remain the possibility

$$x^2 = n^2 + m^2 , \; y^2 = n^2 - m^2 , \; z = 2nm .$$

This, however, can be dealt with along similar lines and we leave the details to the reader.

The following fun problem came from Garnet A. Greenbury of Brisbane.

PROBLEM 9.4.1

A sequence of integers has been divided by the same number giving these remainders: 2, 4, 8, 5.

(a) What number comes next?

(b) What is the original sequence of numbers?

(c) What is the divisor?

(d) Write down the next five remainders following the 5. Call them a, b, c, d, e and show that $a + 1$, $b + 2$, $c + 4$, $d + 8$, $e + 5$ are all equal. Can you explain this?

(e) Show that the sequence formed by the first ten remainders is the same as the pattern for the next ten remainders. Can you explain this?

Other problems occur in the *Perdix* section of each issue of *Function*. Send solutions of those to *Perdix*. Solutions to problems in this section should be sent to the Editor.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

PERDIX (CONT.)

The competition took place on July 4 and July 5 with three problems to solve each day in a session of $4\frac{1}{2}$ hours. The two problem papers are reproduced below.

Problem 1 proved the easiest with 103 competitors getting the full 7 points, the average number of points obtained being 4.1. Problem 2 was a close runner up with 92 competitors getting 7 points and with an average of 3.7. In increasing order of difficulty were problem 4 (average 2.4), problem 6 (average 2.0), problem 5 (average 1.9), and problem 3 (average 0.8). Only 12 competitors got the full 7 points for problem 3 (John Graham got 6), while 153 scored 0 (including Andrew Hassell, our gold medallist, who made up for this by scoring the full 7 points for each of the other problems).

Try the questions yourself. Do you have the same assessment of their difficulty as the competitors' scores suggest?

FIRST DAY

Joutsa July 4, 1985

1. A circle has centre on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

2. Let n and k be given relatively prime natural numbers, $0 < k < n$. Each number in the set $M = \{1, 2, \ldots, n-1\}$ is coloured either blue or white. It is given that

   (i) for each $i \in M$, both i and $n-i$ have the same colour, and

   (ii) for each $i \in M$, $i \neq k$, both i and $|i - k|$ have the same colour.

   Prove that all numbers in M must have the same colour.

3. For any polynomial $P(x) = a_0 + a_1x + \ldots + a_kx^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0, 1, 2, \ldots$ let $Q_i(x) = (1 + x)^i$. Prove that if $i_1, i_2, \ldots, i_n$ are integers such that $0 \leq i_1 < i_2 < \ldots < i_n$, then

$$w(Q_{i_1} + Q_{i_2} + \ldots + Q_{i_n}) \geq w(Q_{i_1}) .$$

Time allowed: 4½ hours
Each problem is worth 7 points.

SECOND DAY

Joutsa July 5, 1985

4. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.

5. A circle with centre O passes through the vertices A and C of triangle ABC, and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangles ABC and KBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.

6. For every real number x_1, construct the sequence $x_1, x_2, \ldots$ by setting

$$x_{n+1} = x_n \cdot (x_n + \frac{1}{n})$$

for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which $0 < x_n < x_{n+1} < 1$ for every n.

Time allowed: 4½ hours
Each problem is worth 7 points.

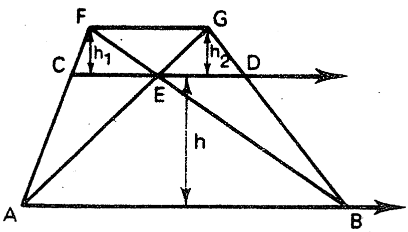
Send me your solutions. Perhaps you will find new methods of solution. For example, I found 3 solutions to Problem 1 that are different. There may be better methods.

* * *

I now give solutions to some of the problems I have set earlier this year. Here are solutions to Problems 7, 8, and 9, provided by Hai Tan Tran, 15 Arthur Street, Plympton Park, S.A., 5038.

PROBLEM 7. AB and CD are two parallel straight lines, E is mid-point of segment CD. Line AC meets BE at F and AE meets BD at G. Show that FG is parallel to AB.

Solution.

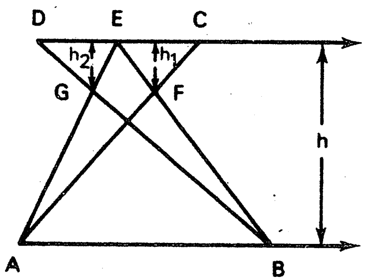


There are two variants of the figure for which slightly different arguments are required. In each figure h_1 is distance of F from CD, h_2 is the distance of G from CD and h is the distance between the parallel lines CD and AB.

In Δs CEF and ABF

$$\frac{h + h_1}{h_1} = \frac{AB}{CE}$$

$\left(\dfrac{h - h_1}{h_1} = \dfrac{AB}{EC}\right.$ in the lower figure$\Big)$;



similarly, $\dfrac{h + h_2}{h_2} = \dfrac{AB}{ED}$

$$\left(\frac{h - h_2}{h_2} = \frac{AB}{DE}\text{, in the lower figure}\right).$$

Hence, since $CE = ED$,

$$\frac{h + h_1}{h_1} = \frac{h + h_2}{h_2}$$

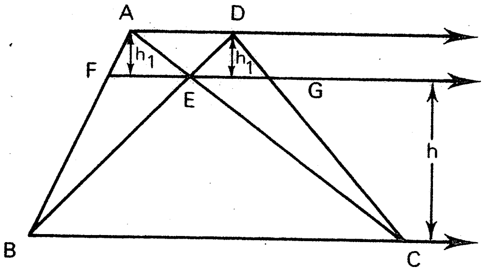
$$\left(\frac{h - h_1}{h_1} = \frac{h - h_2}{h_2}\right),$$

whence, in both cases, $h_1 = h_2$.

Thus FG is parallel to CD, and so to AB.

PROBLEM 9. Let ABC and DBC be two triangles such that AD is parallel to BC. Let BD and AC meet at E. Draw a line parallel to BC through E and let this line meet AB at F and CD at G. Show that $FE = EG$.

Solution. This problem asks for the proof of a result converse to that of problem 7.



There are again two variants of the figure. We consider merely the one shown.

Let h_1 be the distance from each of A and D to FG and let h be the distance between AD and BC. Then, since FE is parallel to BC, in Δs AFE and ABC

$$\frac{h_1}{h + h_1} = \frac{FE}{BC} .$$

Similarly,

$$\frac{h_1}{h + h_1} = \frac{EG}{BC} .$$

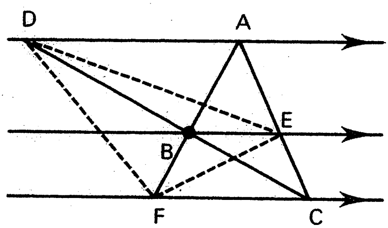
Hence

$$\frac{FE}{BC} = \frac{EG}{BC}$$

whence $$FE = EG .$$

PROBLEM 8. Let ABC be any triangle and draw parallel lines through the vertices A, B, and C to meet the opposite sides in D, E, and F, respectively. Show that the area of ΔDEF is twice the area of ΔABC.

Solution. You will see how to use the result of problem 9 to solve this result. Alternatively we may proceed as follows.



$\Delta BDE = \Delta BAE$ .. (1)

[Δs with same base and equal heights];

similarly

$\Delta BFE = \Delta BCE$ .. (2)

and $\Delta FDC = \Delta FAC$.

Subtracting ΔFBC from the latter gives $\Delta DBF = \Delta ABC$. (3)

Adding (1), (2) and (3) now gives the result.