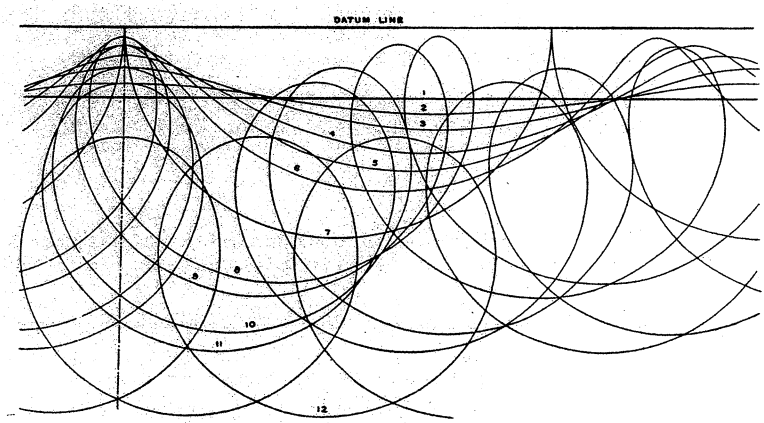
ISSN 0313 - 6825

FUNCTION

Volume 9 Part 1 February 1985



A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Reg. by Aust. Post Publ. No VBH0171

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
Function,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

The magazine is published five times a year, appearing in February, April, June, August, October. Price for five issues (including postage): \$8.00*; single issues \$1.80. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

*\$4.00 for *bona fide* secondary or tertiary students.

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Registered for posting as a periodical - "Category B"

Function's readership changes considerably each year, although we do have and value faithful friends who have been with us for a long time. To our new readers we say welcome, to our old friends welcome back.

This issue of *Function* contains articles on various aspects of Mathematics and its applications. There are also the regular *Perdix* column concerned with Mathematics competitions and the Olympiads, and our own problem section as well. Often, normally in fact, we also run a letters page, but none were to hand this time at the time of our going to press.

It is a pleasure to acknowledge receipt of a grant from the recently established Monash Mathematics Education Centre, set up with the financial backing of CRA Pty Ltd. (See the inside back cover.) Recently, *Function*, for reasons we needn't go into, has had production difficulties, particularly in relation to the art work and the layout. We hope that this assistance will boost this vital area.

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THE FRONT COVER

The curves shown on the front cover are theoretical flight paths for gliders. They were first calculated by the British engineer F.W. Lanchester, one of the pioneers in the theory of aerodynamics. Twelve such are shown varying from level flight (1), through increasing degrees of vertical instability (2 - 6), to a critically unstable one (7) beyond which the glider loops (8 - 12).

These paths or *Phugoids*, as Lanchester called them, were calculated on the assumption that the glider had negligible size and that the drag could be neglected. In this case, we have the forces shown in Figure 1. There are two of these: the lift and the weight. The weight, given by mg acts vertically downwards, while the lift is perpendicular to the flight path and is proportional to the square of v, the velocity. (m is the mass, and g the acceleration due to gravity.)

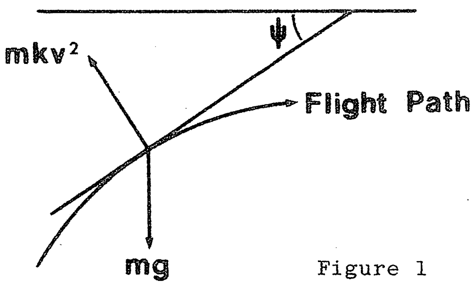

Figure 1

The glider responds to these two forces by accelerating. There are two components to the acceleration. First the glider is being slowed down, as is evident from Figure 1. Second, its path is curved and so another acceleration, toward the centre of curvature, is involved. The situation is as shown in Figure 2. a is the (backward) acceleration, the rate of change of v. ρ is the distance to the centre of curvature.

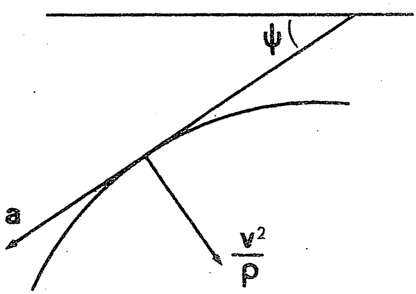

Figure 2

If we now use Newton's second law of motion, we may write down two equations, one referring to the first component of acceleration, the other to the second.

$$ma = mg \sin \psi$$

$$\frac{mv^2}{\rho} = mg \cos \psi - mk\, v^2 .$$

Lanchester wrote $v^2 = 2gy$ (an equation derived from that for free fall) where y is the distance from a given (datum) line, and was able to show that the equations of motion lead to the result

$$\frac{1}{\rho} = -\frac{2k}{3} + Ay^{-3/2} ,$$

where A is a constant. This last equation allows the curves to be plotted. Different values of A give different curves as shown on the cover.

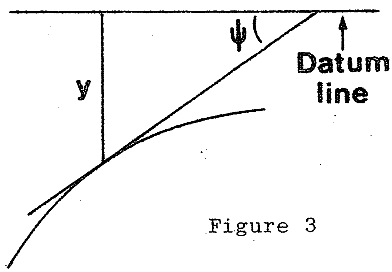

Figure 3

The illustration is taken (slightly modified) from his book *Aerial Flight, Vol.* II of 1908.

Paper aeroplanes follow approximately phugoidal paths, so readers have a ready way to explore this matter further.

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ADVICE FROM DR FRANKENSTEIN

I at once gave up my former occupations, set down natural history and all its progeny as a deformed and abortive creation and entertained the greatest disdain for a would-be science, which could never even step within the threshold of real knowledge. In this mood of mind I betook myself to the mathematics, and the branches of study appertaining to that science, as being built upon secure foundations and so worthy of my consideration.

Mary Shelley, *Frankenstein*, 1818.

If your wish is to become really a man of science, and not merely a petty experimentalist, I should advise you to apply to every branch of natural philosophy, including mathematics.

Ibid.

N.B. It is widely, and incorrectly, believed that Frankenstein was the monster in Mary Shelley's story. This is not so. Dr Frankenstein was a scientist who, in the course of his experiments, made a synthetic human being (the monster) who then turned to a life of crime to the destruction of his creator.

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THE THREE POSTCARD TRICK

John Stillwell, Monash University

There are five so-called regular polehedra - the tetrahedron, cube, octahedron, dodecahedron and icosahedron (Fig.1). Of these, the icosahedron is the most complicated and subtle.

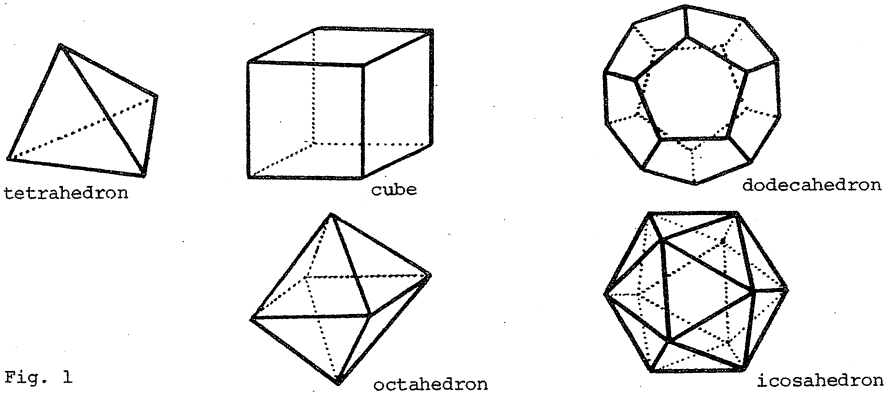

Fig. 1

It is a configuration of 20 equilateral triangle faces which makes important appearances in many parts of mathematics, from the solution of 5th degree equations to the theory of knots. While these sophisticated aspects of the icosahedron may be unfamiliar, the polyhedron itself probably is not. There are a number of children's toys which permit the building of icosahedra, either in skeleton form from rods or as surfaces made from cardboard or plastic triangles.

In view of this, it may seem silly to ask: does the icosahedron really exist? Let me put the question this way: can 20 equilateral triangles be fitted together *exactly* to form a closed surface? The plastic toys show that such a surface can certainly be formed *approximately*, but the ideal, exact, surface can only be constructed mathematically, if it exists at all.

When one tries to do this by calculating the distances and angles which arise as the triangles are fitted together one by one, there are horrible algebraic problems with surds. A large part of Euclid's Elements (300 BC) is devoted to developing the theory of surds and, ultimately, to a successful construction of the icosahedron at the end of the book. This is perhaps the oldest example of something which is now all too common in mathematics - a proof which everyone believes but does not have the patience to check, because it is so involved. Nowadays we are inclined to replace patience by the computer. If the surds are too hard to handle, let the computer do them to 8 decimal places. We might only confirm the existence of the icosahedron up to 8 decimal places, but this is a pragmatic world, right? Near enough is good enough.

There is good reason to reject this anti-mathematical thinking. The only satisfactory substitute for a long proof is one which is shorter, but still a proof. As well as being more comprehensible, a short proof usually throws new light on the problem.

In the case of the icosahedron, a short construction was found nearly 500 years ago by Luca Pacioli, a friend of Leonardo da Vinci. The proof is easy (particularly with the help of modern algebraic notation), and it reveals a beautiful relationship between the icosahedron and another classical geometric figure, the *golden rectangle*.

The golden rectangle has sides of length 1 and $\frac{1+\sqrt{5}}{2}$, where the so-called golden ratio $\tau = \frac{1+\sqrt{5}}{2}$ satisfies the equation $\tau^2 = 1 + \tau$. Pacioli takes three golden rectangles, each with a cut lengthwise down the middle of length 1 (Fig.2)[†].

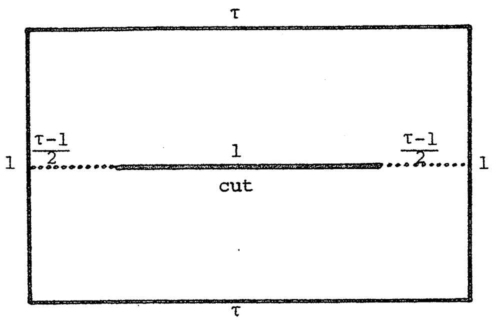

Fig. 2

The three golden rectangles can then be slotted together, perpendicular to each other, like this (Fig.3).

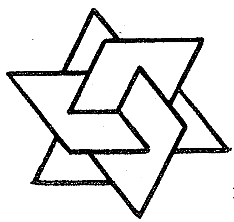

Fig. 3

† The golden rectangle is in turn related to the regular pentagon. See *Function, Vol.4, Part 1*.

Pacioli claimed that the 12 corners of the rectangles are the vertices of an icosahedron. The 20 triangles are found by joining each corner to its 5 nearest neighbours (Fig.4), so it only remains to check that each triangle is equilateral.

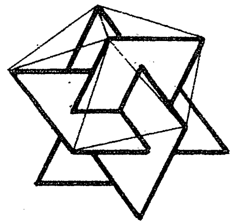

Fig. 4

We look at a typical triangle, ABC, the base AB of which we already know to be 1 (Fig.5). With D the mid-

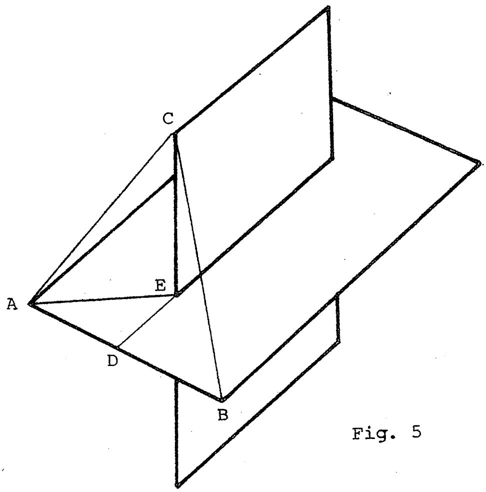

Fig. 5

point of AB, we have $AD = \frac{1}{2}$, and $ED = \dfrac{\tau - 1}{2}$ from Fig.2. Hence

$$\begin{aligned} AE^2 &= AD^2 + ED^2 \text{ (by Pythagoras' theorem)} \\ &= (\tfrac{1}{2})^2 + \left(\frac{\tau - 1}{2}\right)^2 . \end{aligned}$$

We also have $EC = \dfrac{\tau}{2}$; hence

$$\begin{aligned} AC^2 &= EC^2 + AE^2 \text{ (by Pythagoras again)} \\ &= \left(\frac{\tau}{2}\right)^2 + (\tfrac{1}{2})^2 + \left(\frac{\tau - 1}{2}\right)^2 \\ &= \frac{\tau^2 - \tau + 1}{2} \\ &= 1, \text{ since } \tau^2 = 1 + \tau . \end{aligned}$$

Thus $AC = 1$. Similarly $CB = 1$, and so the triangle ABC is indeed equilateral.

Pacioli's construction of the icosahedron was the climax of a book he published in 1509 on the properties of the golden ratio τ. He described it as "the twelfth, almost incomprehensible effect", following eleven other properties described as "singular", "ineffable", "supreme" and "most excellent", among other glowing terms. The book was written at the suggestion of the artist Piero della Francesca, with illustrations by Leonardo. Many Renaissance artists were scientists as well, and there was great interest in the golden rectangle as a theoretical basis for harmonious shapes in art and architecture. Nowadays the golden rectangle ekes out a more humble living as the shape of the ideal postcard, which brings me to another do-it-yourself construction of the icosahedron, and the title of this article.

Take three postcards, and slit them in the middle as shown in Fig.2. If the slit in the third card is continued to one end, then the three cards can be slotted together to form a model of Fig.3. Thus the corners of the postcards are the vertices of an icosahedron.

The Pacioli construction allows a very ready solution to Problem 8.2.2 in the icosahedral case. (See *Function, Vol.8, Part 2.)*

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ANOTHER LONGSTANDING CONJECTURE PROVED

In *Function, Vol.7, Part* 5 we reported the proof of the so-called Mordell conjecture. Now comes the news of the proof of another long-awaited result.

One of the longest-standing and most renowned problems in mathematics has been solved by Professor Louis de Branges of Purdue University (U.S.A.).

Felix Browder of the University of Chicago describes the 68-year-old Bieberbach conjecture as "perhaps the most important single conjecture in classical analysis - one that has stood as a challenge to mathematicians for a very long time". News that the conjecture had been verified spread quickly through the mathematical world last spring and summer as de Branges lectured in Europe and as preprints and informal communications circulated. At the August 1984 Joint Meetings of the MAA and the AMS (American Mathematical Society) in Eugene, Oregon, talk of de Branges' achievement was heard everywhere.

The technical details of the conjecture are rather too complicated for *Function*. It concerns functions of the form

$$w = f(z)$$

where both w, z are complex numbers. The branch of mathematics (complex analysis) dealing with such functions was mostly elaborated in the nineteenth century, but some problems have remained.

Continued on page 32.

WHAT IS THE BEST BASE?

M.A.B. Deakin, Monash University

If we write numbers in the form of a string of symbols and if we use the useful convention of *place value*, we must agree on a *base*. That is, the string

$$a_n a_{n-1} \ldots a_2 a_1 a_0$$

means

$$a_0 + a_1 b + a_2 b^2 + \ldots + a_{n-1} b^{n-1} + a_n b^n .$$

Here b is an integer (called the base) and $a_0, a_1, \ldots$ are other integers, satisfying

$$0 \leqslant a_i \leqslant b - 1.$$

Thus each of the a_i is chosen from a set of exactly b symbols and we may represent arbitrarily large numbers by (sufficiently long) strings of these symbols.

Virtually every society that has developed a reasonably advanced arithmetic has used base ten. The reason usually given is that we have ten fingers, and there is some anthropological evidence in support of this - many cultures do count on their fingers, as do we at times (although we may joke about it). Base 20 (Mayan) and base 60 (Babylonian) have been found (though the Babylonian system is perhaps better described as a mixed base ten and base six system), but these are really the only exceptions to the sway of base ten. Claims have been made for two, five and even quite bizarre numbers like 47, but if these are investigated, they are all found to depend on a less precise notion of what is meant by "base".

From time to time, there are calls to reform the number system by going to a base other than ten. Most would-be reformers sing the praises of the number twelve. The New Zealand born mathematician (and calculating prodigy) A.C. Aitken devoted much of his time to this cause, and there is a Duodecimal Society of America who also promote it.

Other bases sometimes suggested are 8 and 16, and we shall look at these later. I am told, though I have no details, that eleven has also been put forward, but I find it hard to see why.

Now of course this will never happen. We are too set in our ways and the widespread use of ten suggests that there may be sound human reasons for this. This article will also show that ten is quite a good base. To judge such questions, let us investigate what is involved mathematically.

First, as mentioned, if the base is b, there are exactly b separate symbols. The smallest possible base is two and this gives the well-known binary arithmetic using the symbols 0, 1. Binary arithmetic is very easy. Its addition table reads:

$$0 + 0 = 0,\ 1 + 0 = 0 + 1 = 1,\ 1 + 1 = 10,$$

where only the last entry is non-trivial. Its multiplication table reads:

$$0 \times 0 = 0,\ 1 \times 0 = 0 \times 1 = 0,\ 1 \times 1 = 1,$$

which is completely trivial.

The trouble with binary is the length of the strings. Thus 73, which is a relatively small number, is 64 + 8 + 1 and so is represented as

$$1001001.$$

In general, if b is the base and n is a natural number. the string has length

$$1 + [\log_b n]\ ,$$

where the square brackets mean "the largest integer not greater than".

Thus in base ten, $\log_{10}73 = 1{\cdot}863\ldots$, and so $[\log_{10}73] = 1$ and the length of the string is 2. In base 2, we have

$$\begin{aligned}\log_2 73 &= (\log_{10}73)/(\log_{10}2),\\ &= 1{\cdot}863\ldots/0{\cdot}301\ldots\\ &= 6{\cdot}189\ldots\end{aligned}$$

and so the string is 7 symbols long. For large numbers, the string in base two will be about 3·3 times as long as that for base ten.

The size of the multiplication table may also be calculated. If we ignore as trivial $a \times 0 = 0$ and $a \times 1 = 1 \times a = a$, and remember that $a_1a_2 = a_2a_1$ we find that in base b, there are $\frac{1}{2}(b - 1)(b - 2)$ basic multiplication facts to be learned. (Of the b digits $0,1,2,\ldots,b-1$, omit 0,1, leaving $b - 2$. Of the $(b - 2)^2$ pairs of these, $b - 2$ have the form (a,a). The others occur twice. Thus there are $\frac{1}{2}[(b - 2)^2 - (b - 2)] + (b - 2)$ pairs, i.e. $\frac{1}{2}(b - 1)(b - 2)$.) Our system gives a multiplication table of 36 entries. Base twelve gives 55 entries, base 16 has 105, while in the other direction base 8 has 21.

If we think about the difficulty schoolchildren have in mastering the multiplication table, we may well decide that base 16 is socially quite impractical.

Another argument easily appreciated is one to do with packaging. If we are to pack b elements (as we might want to do), we might wish for a compact result. This argument favours twelve or eight: $2 \times 2 \times 3$, $2 \times 2 \times 2$ respectively. Ten results in a long and relatively narrow package: $2 \times 5 \times 1$. Still this doesn't seem to bother most people, and when we pack 12(!) eggs, we use $2 \times 6 \times 1$, not $2 \times 2 \times 3$, or $3 \times 4 \times 1$.

We may also say on psychological grounds that b should be even. We like to divide things in half. Even bases also enable us to check divisibility by two merely by examining the final digit, and this is useful. In base 3, 73 is represented as 2201, but 75 becomes 2210, and both numbers are odd, but this fact is not immediately apparent.

There are other more subtle rules for determining divisibility. In base ten, we determine divisibility by three through the use of a rule I will call A1. We add the digits in blocks of one. 73 gives $7 + 3 = 10$ and $1 + 0 = 1$, which is the remainder when 73 is divided by 3. (In base 3, of course, we determine this merely by looking at the last digit.)

To determine divisibility by 5, we merely examine the last digit, as 5 divides ten.

Seven is harder. The rule we need to use I call S3, subtraction in blocks of 3. (This is because 7 divides $10^3 + 1$.) For example we might want to know if 7 divides

$$1\ 031\ 426\ 859\ 314.$$

We write

$$1 - 031 + 426 - 859 + 314$$

which equals -149 and is not divisible by 7 (since -149 has remainder -2 or 5, when divided by 7, and 5 is indeed the remainder when the original number is so divided).

Eleven is decided similarly. The simplest rule here is S1, and thirteen is decided by S3.

In base eight, divisibility by two is readily determined by examining the final digit. For 3, since $3^2 = 8 + 1$, the rule is S1 and 5 may be tested as a divisor by S2, since 5 divides $8^2 + 1$. 7 may be tested by A1, as $7 = 8 - 1$. Eleven, however, gives trouble, although thirteen divides $8^2 + 1$ and so is decided by S2.

Systematising and generalising these observations we see that the ideal base has† b, $b - 1$, $b + 1$, $b^2 + 1$, $b^3 - 1$, $b^3 + 1$ rich in small prime divisors. For base ten this yields the

---

†We omit $b^2 - 1$ as this is $(b - 1)(b + 1)$.

following list:

2, 3, 5, 7, 11, 13, 37, 101.

For base eight we have:

2, 3, 5, 7, 13, 19, 73.

For base nine:

2, 3, 5, 7, 13, 41, 73.

For base eleven:

2, 3, 5, 7, 11, 19, 27, 61.

For base twelve:

2, 3, 5, 7, 11, 13, 19, 29.

Finally, base sixteen gives:

2, 3, 5, 7, 13, 17, 241, 257.

So from this point of view the choice must lie between ten and twelve, with twelve yielding rather simpler rules.

The other matter affected by the base is the b - mal† expansion of fractions. Thus base ten gives for $\frac{1}{n}$ the following expressions:

$\frac{1}{2} = 0 \cdot 5$ $\quad\frac{1}{3} = 0 \cdot 333\ldots$ $\quad\frac{1}{4} = 0 \cdot 25$

$\frac{1}{5} = 0 \cdot 2$ $\quad\frac{1}{6} = 0 \cdot 166\ldots$ $\quad\frac{1}{7} = 0 \cdot 142857142\ldots$

$\frac{1}{8} = 0 \cdot 125$ $\quad\frac{1}{9} = 0 \cdot 111\ldots$ $\quad\frac{1}{10} = 0 \cdot 1$

$\frac{1}{11} = 0 \cdot 0909\ldots$ $\quad\frac{1}{12} = 0 \cdot 08333\ldots$ $\quad\frac{1}{13} = 0 \cdot 076923076\ldots$

and so on, two of those listed being rather cumbersome.

If n is prime to b, the expansion of $\frac{1}{n}$ cannot terminate, but must have a repeating element whose length ℓ divides $n - 1$. Thus $\frac{1}{7}$ is as complicated as it could possibly be, but we are somewhat fortunate with $\frac{1}{13}$.

In base 12, we have (I follow the Duodecimal Society of America in writing X for ten and & for eleven), writing the fractions in base ten and their expansions in base twelve:

†b-mal is the base b equivalent of the word "decimal".

$\frac{1}{2} = 0{\cdot}6$ $\frac{1}{3} = 0{\cdot}4$ $\frac{1}{4} = 0{\cdot}3$

$\frac{1}{5} = 0{\cdot}249724\ldots$ $\frac{1}{6} = 0{\cdot}2$ $\frac{1}{7} = 0{\cdot}186X35186\ldots$

$\frac{1}{8} = 0{\cdot}16$ $\frac{1}{9} = 0{\cdot}14$ $\frac{1}{10} = 0{\cdot}124972497\ldots$

$\frac{1}{11} = 0{\cdot}1111\ldots$ $\frac{1}{12} = 0{\cdot}1$ $\frac{1}{13} = 0{\cdot}0\&0\&0\&\ldots$

which some may think simpler than ours, but I don't find it convincingly so. Base eight is not bad, but has a ten digit period in the expansion of $\frac{1}{11}$. Base sixteen is very good. There is a five-digit repetend in the 16-mal expansion of $\frac{1}{11}$ and nothing very dreadful after that till we find a 9-digit repetend in the expansion of $\frac{1}{19}$.

The computer sub-culture makes some use already of bases eight and sixteen, so if we ever were to change to some other base, sixteen might be seen as a logical choice. Against it, however are its lack of a good test for divisibility by eleven, the need for a much larger multiplication table, and the need for six new symbols and four new number words.

For special purposes, other bases like 100, 1000 may be used. Indeed these relate to the earlier remarks on divisibility rules.

All in all, best stay with base ten. It's pretty good, and we're used to it.

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ANOTHER MATHEMATICAL ADVANCE

A 28-year-old mathematician at American Telephone and Telegraph's Bell Laboratories has made a startling theoretical breakthrough in the solving of systems of equations that often grow too vast and complex for the most powerful computers.

The discovery, which is to be published soon, is circulating rapidly through the mathematical world. It has also set off a deluge of inquiries from brokerage houses, oil companies and airlines, industries with millions of dollars at stake in problems known as linear programming.

These problems are fiendishly complicated systems, often with thousands of variables. They arise in a variety of commercial and government applications ranging from allocating time on a communications satellite to routing millions of telephone calls over long distances, or whenever a limited, expensive resource must be allocated most efficiently among competing users. Investment companies use them to devise portfolios with the best mix of stocks and bonds.

Continued on p.21.

THE WONDERS OF 13

D.R. Kaprekar, 311 Devlali Camp, India

409 has $4+0+9 = 13$ and $409^2 = 16\ 72\ 81$ and $16+72+81 = 169 = 13^2$.

526 has $5+2+6 = 13$ and $526^2 = 27\ 66\ 76$ and $27+66+76 = 169 = 13^2$.

607 has $6+0+7 = 13$ and $607^2 = 36\ 84\ 49$ and $36+84+49 = 169 = 13^2$.

724 has $7+2+4 = 13$ and $724^2 = 52\ 41\ 76$ and $52+41+76 = 169 = 13^2$.

823 has $8+2+3 = 13$ and $823^2 = 67\ 73\ 29$ and $67+73+29 = 169 = 13^2$.

922 has $9+2+2 = 13$ and $922^2 = 85\ 00\ 84$ and $85+00+84 = 169 = 13^2$.

1003 has $10 + 0 + 3 = 13$ and $1003^2 = 1\ 00\ 6\ 00\ 9$.

Here 13 and 169 are filled up with 00's. Also

$$1\ 0\ 3\ 0\ 5\ 0\ 3\ 0\ 1 \text{ has } 1 + 3 + 5 + 3 + 1 = 13 \text{ and}$$
$$1\ 0\ 3\ 0\ 5\ 0\ 3\ 0\ 1^2 = 01\ 06\ 19\ 36\ 45\ 36\ 19\ 06\ 01 \text{ and}$$
$$01 + 06 + 19 + 36 + 45 + 36 + 19 + 06 + 01 = 169 = 13^2.$$

1 0 3 0 5 0 3 0 1 is a palindromic number, reading backwards the same as forwards, and so is

01 06 19 36 45 36 19 06 01

palindromic in base 100.

13 is the sign of the *mathematician*, because this word has 13 letters. Also mathematical items like *circumference, approximation, perpendicular, quadrilateral, antilogarithm, parallelogram* and many others all have 13 letters.

Michael Deakin† has 13 letters and so do I if you write my name in the script of my mother language.

For wonderful books on numbers write to me and include the Pin Number†† 422401, which has $4 + 2 + 2 + 4 + 0 + 1 = 13$, and enclose ten dollars or eighty rupees.

---

†The chief editor of *Function*.

††Indian equivalent of Australia's Postcode. *Eds*.

THE PRISONER'S DILEMMA GAME

Bruce Taplin and Peter Kloeden
Murdoch University

One of the most popular games in mathematical game theory will not be found on a supermarket shelf, nor even in a specialized games shop. In fact, it is not much fun to play, because the winning strategy is obvious. Nevertheless the game continues to intrigue people on account of the insight it gives about rational decision making and the nature of optimal strategies.

The game is played by two players, each of whom secretly chooses one of two strategies "*C*" or "*D*", and who then simultaneously reveal their choices. Each player tries to rationally maximize his own score, which however depends on the other player's strategy choice as well as his own. Any malevolence or benevolence towards the other player is supposed to have already been included in the scores. These are indicated in the "bimatrix" in Figure 1, where the first element in each pair is player 1's score and the second player 2's.

		Player 2	
		C	*D*
Player 1	*C*	3,3	1,4
	D	4,1	2,2

Figure 1

The numbers here are illustrative only. The general case is indicated below.

This game is called the *prisoner's dilemma game*, because it describes the following scenario. Two criminals (the players) are suspected of committing a serious crime, and are held in separate cells for interrogation so that they cannot communicate with each other. Each knows that if neither confesses (i.e. both choose strategy *C*) then they can only be convicted on a lesser charge, whereas if both confess (play strategy *D*) they will be convicted of the serious crime, though their confessions will have a mitigating effect on their sentence. However, if only one confesses he will receive lenient treatment while the other will get the maximum sentence for the serious crime.

The actual numbers in the bimatrix here are unimportant, provided they bear the same relative orders of magnitude. The general prisoner's dilemma game has the bimatrix given in Figure 2, where outcomes *R*, *T*, *P* and *S* refer to the reward (for reciprocal cooperation), the temptation (to exploit the other player), the penalty (for failing to cooperate) and the "sucker's" punishment (for being exploited).

		Player 2		
		C	*D*	
Player 1	*C*	R, R	S, T	$T > R > P > S$
	D	T, S	P, P	$2R > S + T$

The strategies *C* and *D* here correspond to *cooperation* (with the other player) and *defection* (from possible cooperation), respectively. All of the major results discussed below apply to any game of this general class.

The prisoner's dilemma game describes many other situations too. For example, two companies (the players) are about to launch identical products on the market. Each can charge a high price (*C*) or a low price (*D*). The scores in the bimatrix then indicate the corresponding profits of the companies. Considerable attention has also been devoted to political interpretations of the prisoner's dilemma game. For example, two superpowers (the players) must decide whether to maintain existing military expenditure (*C*) or to increase it (*D*) during a period of international tension. Each would rather maintain parity in expenditure at the lower cost, but then each is also tempted to dominate the other. Alternately, in a nuclear crisis each superpower has the option of waiting until attacked first (*C*) or launching a nuclear strike first (*D*). Surely each would rather avoid a nuclear war, yet if there were to be one neither would want to wait before striking.

Each of these real world situations has obviously been highly simplified and many important factors ignored. Nevertheless the prisoner's dilemma game does capture the essence of choosing to cooperate or compete, and highlights the advantages and disadvantages of both.

A key requirement of the prisoner's dilemma game is that the players make their choices independently of one another. They may have some prior agreement, but must make their choice alone. Since both players are motivated only by the desire to maximize their own score, they may renege on any agreement, which may not be enforcable. To see what the best strategy is, suppose that the other player chooses *D*. Then if we choose *C* we would get *S* and if we choose *D* we would get *P*. Since $P > S$, it would be better for us to choose *D*. Similarly if the other player chose *C* then we would get *R* if we chose *C* and *T* if we chose *D*. Since $T > R$ it would again be better for us to choose *D*. In either case if we were a rational player trying only to maximize our score we would always play *D*. The other player, who is supposed to have the same objective, would use the same reasoning and also choose *D*. That is, both players would play strategy *D* and obtain the score *P*. Such an outcome is called an *equilibrium* because no player can do better if he alone changes his strategy from the equilibrium one.

It is ironical that "rational" maximizing players should finish with such a poor joint outcome, which in fact minimizes their average score. A frequently used criterion to judge the desirability of an outcome is that it be *Pareto optimal**, that is when there is no other outcome which would increase one player's score without decreasing the other player's. In the prisoner's dilemma game the only outcome which is not Pareto optimal is the one described above, where both play strategy *D*. In fact this outcome is Pareto-minimal as every other outcome has at least one player scoring more. Perversely, two "rational" minimizing players would both choose *C* and both score more than two "rational" maximizing players! It would seem that both players would do better if they were less maximizing (or minimizing), that is less "rational". But surely the reason that both players take the time and trouble to "rationally" maximize their scores is that they believe this would increase their expected scores. Indeed, how could maximizing one's score regardless of the other player's actions fail to produce a maximum score? If "rational" maximization does not produce the best score, what method would?

This is the heart of the dilemma. Both players jointly prefer to both play *C* rather than both play *D*. Yet each player prefers to play *D* regardless of what the other does. Both of these statements appear true, and at the same time contradictory! What the dilemma highlights is the difference between *individual* and *collective* gains, that is between individual rationality and collective rationality. It is better for each individual player to choose *D* because it increases his score relative to the alternative *C*. The problem is that playing *D* rather than *C* decreases the other player's score. If both players choose *D* then each of their scores is decreased by the other player's choice more than it is increased by their own choice. Thus both players score less if both play *D* rather than *C*. It is collectively rational for both players to prefer both to play *C*, but individually rational for each player to prefer to play *D* rather than *C*.

Since both players have to make their choices independently, they are unable to coordinate their choices to their mutual advantage. If they were able to coordinate their choices, they would obviously both play *C*. Moreover if one contemplated swapping back to *D* the other would threaten to do this too. This threat is believable because the second player would gain in the circumstances whereas the first would lose. The outcome when both players choose *C* is thus fairly stable as well as desirable when the players can choose their strategies together. When the players are not allowed to coordinate their choices the existence of a governing body, the government say, that can restrict matters

*Named after the Italian economist and sociologist V. Pareto who, during the last century, investigated how individuals coordinate their choices with each other.

in some way may be beneficial to both players. For example, if the value of T were reduced so that $T < R$, then there would be an incentive to play C if the other player plays C. Both playing C is then an equilibrium, but so too is D. If P were also reduced so that $P < S$, then there would be an incentive to play C if the other player played D. Then both playing C would be the only equilibrium. In this way the government could make the individually rational outcome coincide with the collectively rational outcome, that is it could increase the actual scores obtained by the players by reducing the scores for certain outcomes. Ironically this is achieved by reducing the scores available to each player.

Caution is of course needed in applying these results to the real world. The prisoner's dilemma game is rarely a complete description of a real situation, although many of its characteristics can be observed in practice. In the example of two competing companies we would expect the companies to set a low price if they could not coordinate their prices and a high price if they could. However the Trade Practices Act prohibits companies from jointly determining prices. In this way the law encourages the optimal outcome for society of a low price. It does this not by prohibiting a high price, but, more subtly, by creating conditions where individual self interest can be relied on to create the desired result. In contrast, various attempts have been made to get the superpowers to make the collectively rational choice. The role of the United Nations as a global peacemaker was designed with this in mind. Unfortunately, according to our theory it cannot fulfil this function properly because it does not have sufficient power to limit the superpowers' strategy choices. The situation is however not as depressing as our theory predicts. Since nations interact more or less continually the model of just one play of the prisoner's dilemma game is not appropriate. Instead we should consider a sequence of games played one after the other. Here there is time for trust to develop as a player can now make his choice dependent on what the other player chose in the previous game or games, and could reward (choose C) or punish (choose D) the other player as thought appropriate. Here each player must also consider the long term consequences of playing D as well as the short term advantages. There is thus considerable incentive for mutual cooperation once the players realize the consequences of their actions.

A game theoretic analysis of prisoner's dilemma situations is very much a theoretical analysis of a highly simplified problem. Nevertheless, it gives considerable insight into the dilemma of competition versus cooperation, of individual gain versus collective gain.

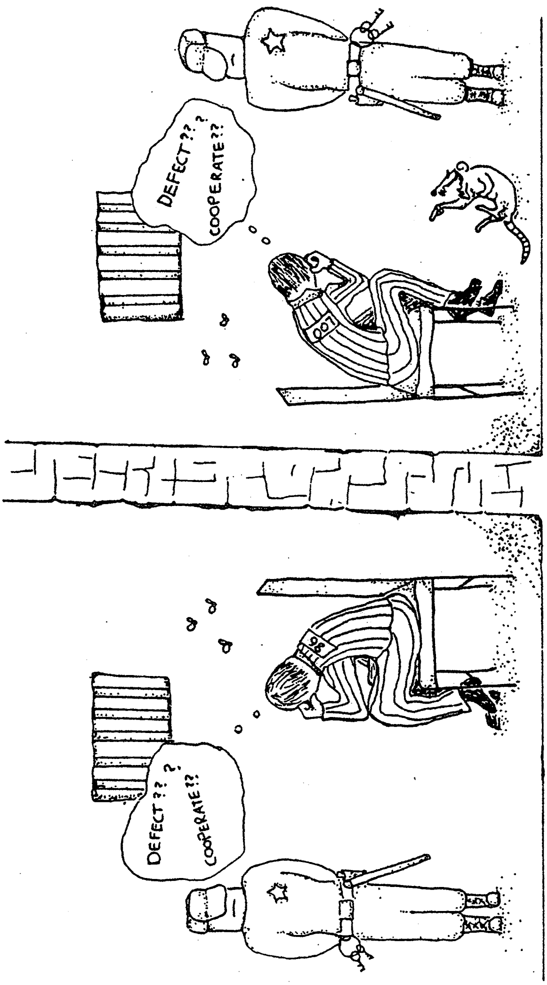
Additional reading

1. Bennett, P.G. and Dando, M.R., *The arms race: is it just a mistake?*, in New Scientist, London, Vol.97, 17 Feb.1983, pp.432-435.

2. Hofstadter, D.R., *Metamagical Themas: The calculus of cooperation is tested through a lottery* in Scientific American, New York, Vol.248, June 1983, No.6 pp.14-18.

3. Hofstadter, D.R., *Metamagical Themas: Computer tournaments of the Prisoners' Dilemma suggest how cooperation evolved*, in Scientific American, New York, Vol.248, May 1983, No.5, pp.14-20.

4. Luce, R.D. and Raiffa, H., *Games and Decisions*, John Wiley and Sons, New York, 1957.



Prisoner's Dilemma

Submitted by Tania Vincent, Year 11, Tucson, Arizona.

QUALITY AND STATISTICS

Neil S. Barnett

Footscray Institute of Technology

Most of us, at one time or another, have purchased a particular item in good faith and discovered later that it was of poor quality. Because we are often in danger of being taken advantage of, purchasing (especially expensive items) can be a rather traumatic experience. It seems often that one has to be belligerent in order to obtain a fair deal or a reasonable service for money paid.

Some 'shoddy deals', perhaps many, are committed with intent, others occur by default through genuine carelessness, accident or through plain ignorance. These latter situations are often remedied under warranty or guaranty conditions, laid down at the time of purchase.

Top management may have no desire to market poor quality goods but production workers may have little pride in their own work. Management may care little for quality and a lot for making a quick sale. Unknown to management the raw materials of manufacture may be sub-standard. Perhaps everyone in the company is striving for the best but equipment is old and worn and thus not suitable for producing the quality necessary.

It is generally acknowledged, by those who have made it their concern to study quality, that if a company is to produce quality items it must have leadership from the top. Once good quality is the prime objective of top management how then can good quality be obtained?

When I was a boy (n years ago, where $n << k$) a 'Made in Japan' label was almost a guaranty of poor quality - the expression, 'Japanese Junk' was widely used. Today, however, people seek out items made in Japan as a means of obtaining good quality. This is of course a generalization but many sectors of Japanese industry are renowned for producing top quality, reliable products. Just how has this turn-about been achieved?

The answer to this question is urgently being studied by western governments, industrialists and business men at all levels as they see their traditional overseas markets being lost to the Japanese.

The Japanese attribute their success to the implementation of a number of factors: quality must become the number one concern of the managing director, every worker within an organisation must feel responsible for quality and finally, the factor I wish to emphasise, the introduction of statistical quality control into industry in a big way.

The adoption of these principles has meant a massive educational program, with many workers being given at least a rudimentary knowledge of statistical techniques. Statistical methods used in quality control are not, by and large, new, exceptionally profound, nor Japanese in origin! Why then has the western world been 'upstaged' by the Japanese? This is arguable but whatever the philosophical issues one thing is very clear, industry is going to have to become much more concerned with quality if it is to survive the Japanese 'onslaught'. To survive, 'Made in Australia' must become synonymous with good quality and to achieve this, these steps, so vital to the Japanese success, must be taken by us in Australia - this includes a widespread use of statistical methods applied to business, industry and specifically to the area of quality control.

At the recent World Quality Congress in Brighton, England, attended by over 1000 delegates from 45 countries, many of the points I have raised here were discussed in greater detail.

The British Government has recently launched a nationwide quality awareness campaign. The Bradford University Management Centre has been given a government grant to study the use of statistical methods in quality control throughout the British manufacturing industry. Preliminary studies show a shockingly low use of statistical methods in quality control despite the Japanese experience and the fact that Britain has been the home of many outstanding statisticians. Generally, it has been found that where statistical methods are being wisely used, together with the other factors mentioned, quality is improved and when quality is improved a company becomes more competitive. A need has been established for a widespread expansion in broad based statistical education and for trained statisticians to apply their skills in industry.

So where does all this leave you? Emphasis on the intelligent use of statistical methods is going to increase in society in general but particularly in business and industry. So don't be in a hurry to regard your school statistics as so much academic game playing - the 'old' normal distribution forms the basis of many of the current applications of statistical quality control. Means, variances, range, histograms, cumulative plots are all useful tools and are again fundamental to many applications of statistics to quality control.

For example, production processes are often monitored by use of so-called mean and range control charts. Suppose that a machine should be producing parts of a prescribed diameter;

fluctuations about this target value can be plotted and under standard operating conditions observed to have a normal distribution. Actual diameters of the parts produced will also have a normal distribution. When values are distributed normally then the means of random samples can also be shown to have a normal distribution. Remarkably, in most cases, provided that the sample sizes are not too small, sample means are very nearly normally distributed, even when the values themselves do not have a normal distribution. This being the case, the performance of a machine mass producing parts to a target diameter can be monitored by sampling regularly parts produced, calculating the mean and standard deviation (or range) of sample values and plotting these on a chart. Because of the normality property alluded to above, extreme, average and/or range values can be shown to correspond to high probabilities of the machine starting to deviate from its standard operating mode. Corrective action can then be taken. The monitoring of *both* mean and standard deviation (or range) is necessary since a change in either will affect the probabilities of having parts produced with a diameter beyond prescribed limits. More will be said of such techniques in a future article. To understand how the technique operates requires only a knowledge of the normal distribution, what a sample mean, standard deviation and range are and an understanding of the implications of changes of these on the probability of producing defective parts (parts with a diameter unacceptably far from the target value).

So build on these fundamental tools and you just might have a part in helping to ensure that 'Made in Australia' means made well.

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ANOTHER MATHEMATICAL ADVANCE (CONT.)

The Bell Labs mathematician, Dr Narendra Karmarkar, has devised a radical procedure that may speed the routine handling of such problems by businesses and government agencies and also make it possible to tackle problems that are now far out of reach.

Because problems in linear programming can have billions or more possible answers even high-speed computers cannot check every one. So computers must use a special procedure, an algorithm, to examine as few answers as possible before finding the best one - typically the one that minimises cost or maximises efficiency.

A procedure devised in 1947, the simplex method, is now used for such problems, usually in the form of highly refined computer codes, sold by the International Business Machines Corporation, among them.

The Karmarkar approach exists so far only in rougher computer code. Its full value will be impossible to judge until is has been tested experimentally on a wide range of problems. But those who have tested the early versions at Bell Labs say that it already appears many times faster than the simplex method, and the advantage grows rapidly with more complicated problems.

New York Times 19.11.84

PROBLEM SECTION

Function has two separate sections dealing with problems and problem solving. There is the *Problem Section* in which various problems of very different levels of difficulty appear. Readers are encouraged to submit problems and solutions to this section. Many readers find that this is the most rewarding way to interact with *Function*.

Perdix is the name of a famous problem solver in classical myth, whose uncle *Daedalus* envied his prowess and tried to have him killed. He was, however, rescued by the goddess Minerva, and lived to write for us. *Perdix* concerns himself mainly with questions from the International Mathematical Olympiads and with the techniques of problem solving. Readers are also encouraged to write to *Perdix*, c/- The Editor.

We begin by giving the solution of some outstanding problems, beginning with

SOLUTION TO PROBLEM 8.4.2.

This problem, due to Lewis Carroll, and submitted by S.J. Newton, read:

A room has a light switch at each corner. It is not possible by examining the switch to tell if it is on or off. The light will, however, be off unless all switches are in the "on" position. A person comes into the room and finds the light off, then presses each switch in turn with no result. He then presses again in order the first, second and third switches, still with no result. How should he proceed if he is to turn the light on?

Label the switches A, B, C, D. Switch A may be in state a or state $\bar{a}$, one of which is off and one on but we do not know which is which. Similarly for the other switches.

Suppose we have initially $abcd$. The first circuit of the room produces in order:

$$\overline{a}bcd,\ \overline{ab}cd,\ \overline{abc}d,\ \overline{abcd}.$$

The next gives:

$$a\overline{bcd},\ ab\overline{cd},\ abc\overline{d}.$$

At this point no success has been reached and eight out of a possible 16 configurations have been tested. The person is now at C, and going on to D will merely re-establish the initial set up.

Nonetheless, this is not a bad way to proceed. Our nocturnal explorer could go to D and so re-establish $abcd$, the initial state. One way to proceed now is to go to B, and establish $a\bar{b}cd$, then back to D and so reach $a\bar{b}c\bar{d}$.

After this, he may proceed around the room as before going to A,B,C,D,A,B and setting up the six outstanding cases: $\bar{a}\bar{b}c\bar{d}$, $\bar{a}bc\bar{d}$, $\bar{a}b\bar{c}\bar{d}$, $\bar{a}b\bar{c}d$, $ab\bar{c}d$, $a\bar{b}\bar{c}d$.

This solution involves only one repetition of a state. There seem to be no solutions that involve no state's being repeated at all.

SOLUTION TO PROBLEM 8.4.4.

This was a four-part problem on Pythagorean triples.

a,b,c are integers satisfying $a^2 + b^2 = c^2$. Prove:

1. Of these integers, one is divisible by 3, another (possibly the same one) by 4, and one (again possibly the same one) by 5;

2. The product abc is divisible by 60;

3. $ab/2$ (the area of a corresponding right-angled triangle) is never a perfect square;

4. The radius of the in-circle of the right-angled triangle is always integral.

In order to solve these, it is necessary to know that a,b,c are expressible in the form

$$a = 2kuv, \; b = k(u^2 - v^2), \; c = k(u^2 + v^2)$$

where u,v,k are integers. (See *Function, Vol.6, Part* 3.)

1. Clearly a is divisible by 2. If u, or v is divisible by 3, so is a. If not, $u = 3m \pm 1$, $v = 3n \pm 1$ (for some integers m,n) and a quick calculation shows b to be divisible by 3. If u or v is even, then a is divisible by 4. If not, $u = 2n + 1$, $v = 2m + 1$ (again, for some integers m,n) and another quick calculation shows b is divisible by 4.

   Finally, if u or v is divisible by 5, so is a. If not there are integers m,n such that $u = 5n \pm 1$ or $5n \pm 2$ and $v = 5m \pm 1$ or $5m \pm 2$. There are four possible combinations, two of which make 5 divide b and the other two of which make 5 divide c. (E.g. $(5n \pm 1)^2 + (5m \pm 2)^2$ $= 25(n^2 + m^2) \pm 10n \pm 20m + 5$.)

2. As 3,4,5 are mutually prime (have no divisors in common), this follows from Part 1.

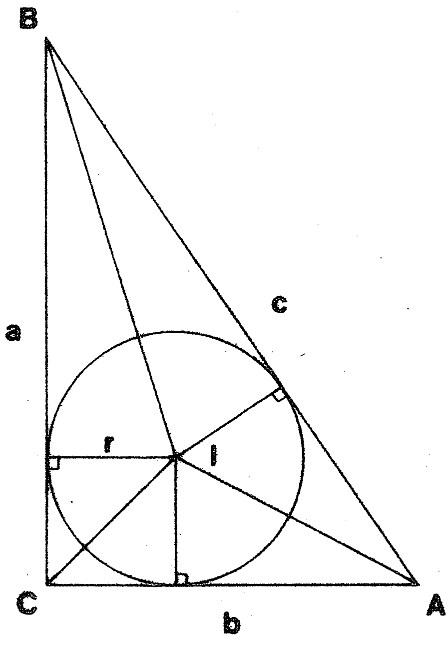
3. $ab/2 = k^2uv(u^2 - v^2)$. We show that $uv(u^2 - v^2)$ cannot be a perfect square. If u,v have divisors in common, call the largest such divisor h. Then $u = hU$, $v = hV$ (say), where U,V are integers with no common divisors. Then $uv(u^2 - v^2)$ $= h^4UV(U^2 - V^2)$ and U,V have no divisors in common and neither have U, $U^2 - V^2$, for if a divisor divided U and also

$U^2 - V^2$, it would have to divide V^2 and so contain a divisor common to U, V. Thus each divisor of $UV(U^2 - V^2)$ occurs only once and this expression cannot be a perfect square. We will complete the proof in the next issue.

4. We first need a formula for the radius of the inscribed circle. From the figure, the area $\frac{ab}{2}$ of the triangle ABC is the sum of the areas of the triangles IBC, ICA, IAB, i.e. $\frac{ra}{2} + \frac{rb}{2} + \frac{rc}{2}$. Thus $ab = r(a + b + c)$. So

$$\begin{aligned} r &= \frac{ab}{(a + b + c)} \\ &= \frac{ab(a + b - c)}{(a + b + c)(a + b - c)} \\ &= \frac{ab(a + b - c)}{(a + b)^2 - c^2} \\ &= \frac{ab(a + b - c)}{2ab} \\ &= \tfrac{1}{2}(a + b - c). \end{aligned}$$

Now $a + b - c$ is even (as c is even if both a, b are, and c is odd if a or b is odd). Hence the result.



Those were quite difficult problems. Still outstanding are three rather easier ones, repeated here for new readers.

PROBLEM 8.5.1.

Let r, x, y and z be real or complex variables. Show that:

(i) $y \propto x$ if y changes by a factor of r whenever x changes by a factor r. [Assume y is known to be a function of x.]

(ii) $z \propto xy$ if $z \propto x$ for each fixed y and $z \propto y$ for each fixed x. [Assume z is known to be a function of both x and y, i.e., to each suitable x and y, there corresponds just one z.]

PROBLEM 8.5.2.

Let $A = \sqrt{5} + \sqrt{22 + 2\sqrt{5}}$ and

$B = \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}}$. Prove that $A = B$.

PROBLEM 8.5.3.

A particle is projected vertically into the air; it ascends to a certain height and then descends to the point of projection, all in the same straight line. Taking air-resistance into account, show that the initial (projection) speed is greater than the final (impact) speed and that the ascent time (the time to reach maximum height) is less than the descent time.

And now some new problems.

PROBLEM 9.1.1 (from Mathematical Spectrum).

A boy took a calculator with him when he went to the store. He bought four items and calculated correctly that their total cost was \$7.11. The only trouble was that he had used the multiply key to reach this answer.

What did the items cost?

PROBLEM 9.1.2 (from Mathematical Digest).

Which is larger:

$$\frac{1}{1984}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1984}\right)$$

or

$$\frac{1}{1985}\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{1985}\right) ?$$

PROBLEM 9.1.3.

The decimal representations of $n/13$, $n = 1,2,\dots,12$ each have a period of 6 and belong to one of two families (each of which has 6 members). If $i/13$ belongs to one family, then $(13 - i)/13$ also belongs to that family. Why is this so? (Cf. pp.9 - 13.)

PERDIX

Welcome to new readers of *Function*! Perdix will continue in 1985 to write about mathematics competitions, both Australian and international, and present you with competition problems and discuss their solutions.

Preparations for selection of the 1985 Australian Olympiad team are well under way. The Australian Mathematical Olympiad (AMO) Competition takes place in mid-March. The best ten, or so, in this competition will then start training, and will go to the IBM financed mathematics camp in May. From these will be selected the final team of six, to represent Australia in Finland in July, in the 27th International Mathematical Olympiad (IMO).

The AMO is competed in by those who do well enough in the AMO Interstate Finals. These take place in October. In 1984, nine participants in the Interstate Finals were awarded certificates of excellence, four from New South Wales, two from Victoria, two from South Australia, one from Western Australia. Another 13 participants were awarded certificates of achievement. Well done, all of you!

How can you enter the Interstate Finals, the first step on the way to selection for the Australian Olympiad team? You must join one of the various training groups in your State. In Victoria, in 1984, there were training programs at Monash University, at Melbourne University, at Melbourne Grammar School, and by a correspondence scheme very well run by David Easdown at La Trobe University.

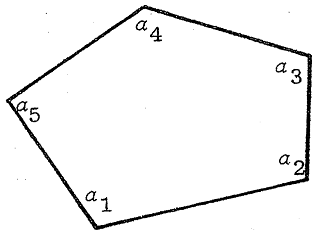
If you wish to be involved in the next round ask your school teachers to tell you how to join one of the training groups. A strong recommendation from your school teacher to the State organizer will probably get you an invitation to join a group. Others will receive invitations to join because they have performed well in some competition such as the Australian Westpac Mathematics Competition.

Problems

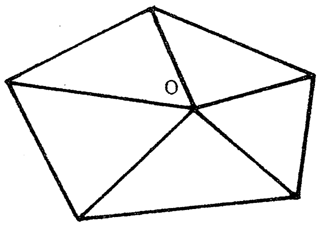
This year I shall begin with geometry problems. Geometry has always played a large part in Olympiad problems. In many years up to half the problems set in the IMO have been entirely or principally geometry problems. I shall begin with some simple problems and gradually build up to those of Olympiad standard.

REMEMBER always to try to do a problem before reading its solution. Also try to find different solutions, if you can, from those I give.

PROBLEM 1. Show that the sum of the measures of the angles marked in the diagram is 540°.



Solution. Take a point O in the interior and connect it to each vertex. We then have 5 triangles. The interior angles of each triangle have total measure 180°. Subtract the 360° measure of the angles at O to get the required sum of (5 × 180 − 360)° = 540°.

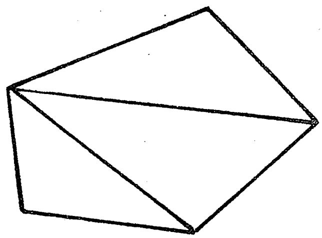


REMARK 1.

The only basic result used in this solution is the important fact that the sum of the measures of the interior angles of a triangle is 180°. The method we have used would also apply for a polygon with any number n of sides, instead of 5. We then get that the sum of the interior angles is $(n \times 180 - 360)°$.

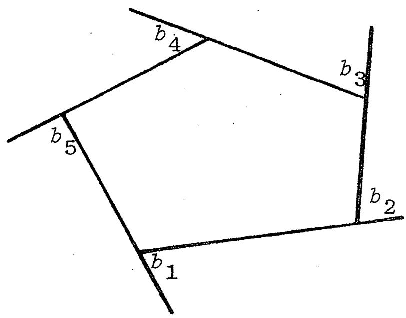
REMARK 2.

Another solution is obtained by dividing the pentagon up into triangles thus:



The sum of the measures of the interior angles of the triangles is 3 × 180°, and this is the required sum, for this time there are no extra angles involved in the sum.

PROBLEM 2. Find the sum of the measures of the angles marked in the diagram. Generalize your answer to a polygon with n sides.



Solution. You probably noticed immediately that you can deduce the answer from the result of Problem 1. For the sum of the measures of a_1 and b_1 is 180°, as also is that for a_2 and b_2, and so on. Thus, subtracting the sum of the measures of $a_1, a_2, \ldots, a_5$, i.e. 540°, from $5 \times 180°$ we get the sum of the measures of $b_1, b_2, \ldots, b_5$. Thus the required answer is 360°.

For an n-sided polygon it is also just 360°.

Alternative Solution. Start with the side connecting the angles b_1 and b_2. Move this side so as to be parallel to the side connecting the angles b_2 and b_3 by rotating it anticlockwise through an angle equal to b_2. Similarly rotate it through an angle equal to b_3 to make it parallel to the side connecting b_3 and b_4; and so on. The side returns to its original position (or parallel to it) when it has been rotated through 360°; but this is just a rotation, as we have seen, through an angle of measure equal to the sum of the measures of $b_1, b_2, \ldots,$ and b_5.

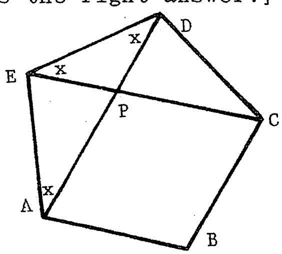
PROBLEM 3. Use the result of Problem 2 to deduce the result in Problem 1.

DEFINITION 1.

A *regular* pentagon is a pentagon with all its sides equal and all its interior angles equal. A *regular* n-sided polygon is defined similarly: thus a regular triangle is equilateral, a regular quadrilateral is a square.

PROBLEM 4. Show that each angle of a regular pentagon has measure 108°. Generalize: show that each angle of a regular n-sided polygon has measure equal to $\left(180 - \dfrac{360}{n}\right)^{\circ}$. [Check for $n = 3$, 4 and 5, that this formula gives the right answer.]

PROBLEM 5. Show that in the regular pentagon in the diagram the length of the segment AP equals the length of each side of the polygon.



Solution. Since the lengths of DE and EA are the same, ΔDEA (read this as "triangle DEA") is isosceles. Its angle DEA is an interior angle of the regular pentagon, so $\angle DEA = 180°$ (Problem 4). So the other two angles of the triangle are both 36° (36 + 36 + 108 = 180). Similarly the other angle marked x on the diagram is of measure 36°. Hence

$$\begin{aligned}\angle PEA &= \angle DEA - \angle DEP \\ &= (108 - 36)° = 72°\end{aligned}$$

and

$$\angle APE = 180° - \angle PEA - \angle EAP$$
$$= 180° - 72° - 36°$$
$$= 72°.$$

Thus, $\angle PEA = \angle APE$; so ΔPAE is isosceles. Thus AP has length equal to that of the side EA of the pentagon.

REMARK 3.

We have used another geometrical fact in this solution: if two angles of a triangle are equal, then the sides opposite those angles are equal.

Here are some more facts about triangles.

DEFINITION 2. Two triangles are said to be *congruent* when one can be obtained from the other by moving it, without distortion or stretching, into the position of the other.

Remember the following results: they give tests for congruency of triangles.

RESULT 1. Two triangles are congruent if the lengths of the three sides of one are the same as the lengths of the three sides of the other. [3 sides]

RESULT 2. Two triangles are congruent if two sides of one triangle are of the same lengths as two sides of the other and also the angle between these sides is of the same measure in each triangle. [2 sides and an included angle]

RESULT 3. Two triangles are congruent if the angles of one are the same (i.e. have the same measures) as the angles of the other and in addition two equal angles, one in each triangle, have sides of equal length opposite them. [3 angles and a corresponding side]

To use the test for congruency in Result 3 you only have to check that 2 angles in one triangle have the same measure as 2 angles in the other. For then the remaining third angles are necessarily equal. [2 angles and a corresponding side]

RESULT 4. Two right angled triangles (i.e. triangles in which one of the angles is a right angle) are congruent if their hypotenuses are the same length and there is another side of each of the same length. [right angle, hypotenuse and side]

The comments inside [ ] at the end of each of these results gives the way that I shall refer to these results in future.

Each of these results will be used again and again in later issues, to establish other results, and to solve problems.

ROBERVAL'S BALANCE

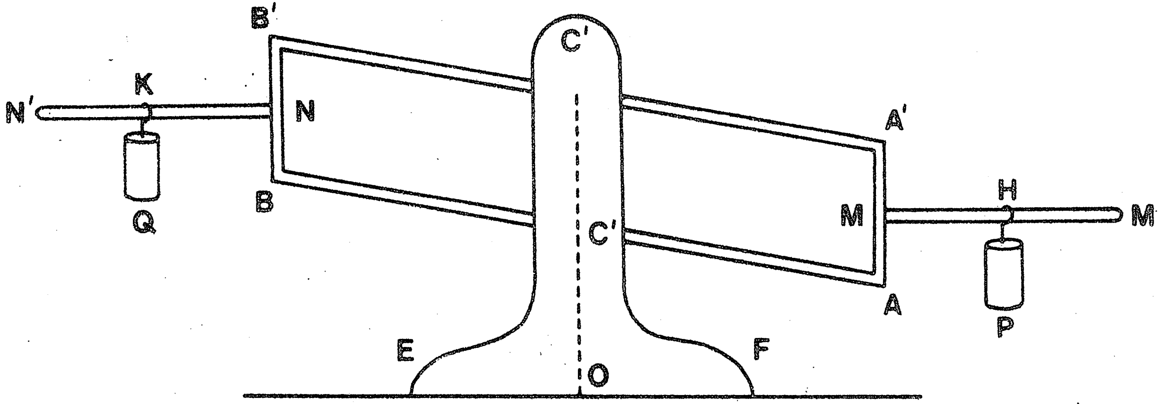
The diagram opposite depicts Roberval's balance, whose theory once formed part of school mathematics courses. Our account here is based on that in Volume 1 of E.J. Routh's *A Treatise on Analytical Statics* (Cambridge University Press, second edition, 1896).

In this balance the four rods AA', $A'B'$, $B'B$, BA are hinged at their extremities and form a parallelogram. The sides AB, $A'B'$ are also hinged at the points C, C' to a fixed vertical rod OCC'. The line CC' must be parallel to AA' and BB', but need not necessarily be equidistant from them. Two more rods MM', NN' are rigidly attached to AA', BB' so as to be at right angles to them. These support the weights P and Q suspended in scale-pans from any two points H and K. As the combination turns smoothly round the supports C, C', the rods AA', BB' remain always vertical, and MM', NN' are always horizontal.

The peculiarity of the machine is that, if the weights P, Q balance in any one position, the equilibrium is not disturbed by moving either of the weights along the supporting rods MM', NN'. It may also be remarked that, if the machine is turned round its two supports C, C' so that one of the rods MM', NN' descends and the other ascends, the two weights continue to balance each other.

If the balance is so constructed that the weights P, Q are equal, when in equilibrium, we can detect whether any difference in weight exists between two given bodies by simply attaching them to any points of the supporting rods. The advantage of the balance is that no special care is necessary to place them at equal distances from the fulcrum.

This should be compared with the simple beam balance, which, like a see-saw, depends critically on the placement of the weights on either side. We have never heard of a see-saw constructed on the principles of the Roberval balance, but such a thing could be quite interesting and make a useful addition to your friendly neighbourhood adventure playground!



Roberval's Balance

ANOTHER LONGSTANDING CONJECTURE PROVED (CONT.)

The Bieberbach conjecture was a refinement of an earlier result due to the nineteenth century mathematician Riemann and it has applications to problems of optimization.

Earlier results by Bieberbach himself (1916), Loewner (1923), Littlewood (1923) and others since produced only partial results, so de Branges has made a major advance.

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1985 APPLIED MATHEMATICS CONFERENCE

Every year in Australia, as also in other countries, Mathematices conferences are held. Here delegates relate, listen to and discuss advances in Mathematics. Such conferences contribute greatly to the quality of research in the areas of mathematics under discussion.

In February, the 21st Applied Mathematics Conference (of the Australian Mathematical Society's Division of Applied Mathematics) was held in Launceston, Tasmania (on the campus of the Australian Maritime College).

The strong prevailing theme was the use of mathematics in the biological sciences. Among the questions posed in this area and to which mathematics has been found to be useful are those concerned with biochemical reactions, ecological questions, nervous conduction, electrophysiology and cellular differentiation.

Other papers to attract attention because of their potential utility concerned the effect of cyclone wire fences on halting the advance of bushfires and the effect of soil cracking on water runoff. The potential economic significance of such studies in an arid country such as Australia should not need to be stressed.

Computing played a large part at the conference, as might be expected. One paper concerned the design of silicon chips, while others dealt with aspects of software or numerical analysis.

A disappointment was the low level of support from industry. Of the more than 100 delegates present, only five came from industry. Australia, it seems, employs nearly all its mathematicians in Universities, CAEs and the CSIRO. A panel discussion on the need for closer cooperation between mathematicians and industry did more to show the need for such cooperation than to suggest ways to bring this about.

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ACKNOWLEDGEMENT

We are pleased to acknowledge financial assistance given by CRA Ltd through the Monash Mathematics Education Centre. This support has been used specifically to pay for art work and so to enhance the quality of exposition and understanding.