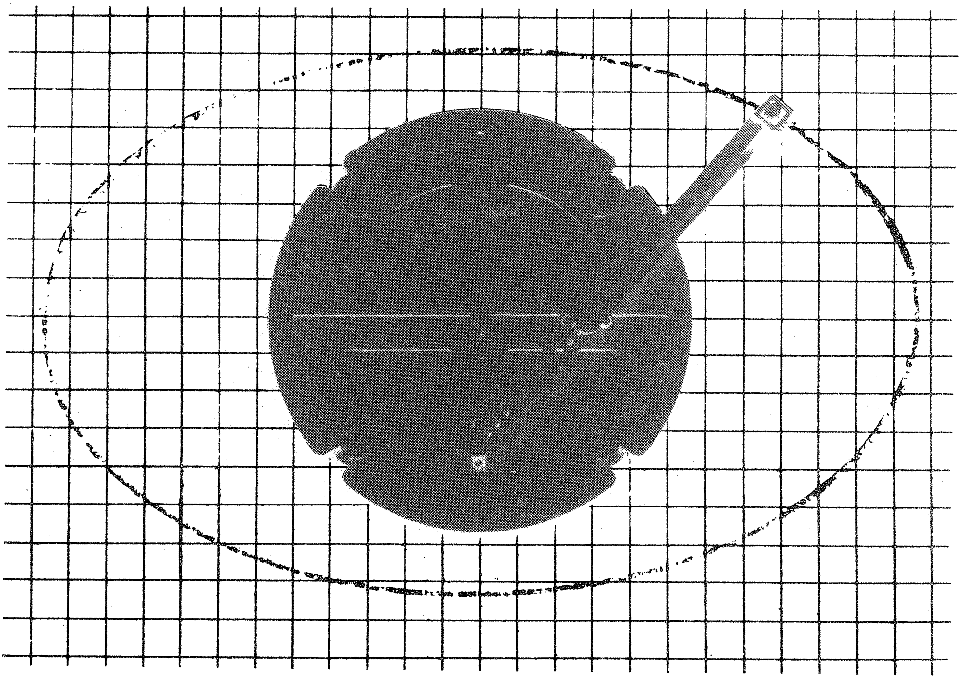
ISSN 0313 - 6825

FUNCTION

Volume 7 Part 4

August 1983



A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Reg. by Aust. Post Publ. No VBH0171

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
Function,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

The magazine is published five times a year, appearing in February, April, June, August, October. Price for five issues (including postage): $8.00*; single issues $1.80. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

*$4.00 for *bona fide* secondary or tertiary students

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Registered for posting as a periodical - "Category B"

The world of robotics is one which is fast coming upon us. The advent of the cheap computer and its marriage with other aspects of modern gadgetry are now making possible the automation of many once exclusively human tasks. In our major article for this issue, Robyn Owens, a mechanical engineer, writes on this - in particular with relation to her work on mechanical sheep shearing. Undoubtedly such devices will become, as her article indicates, more common in the decades to come. The more complex question of how society will adapt to them seems as far from solution as ever.

CONTENTS

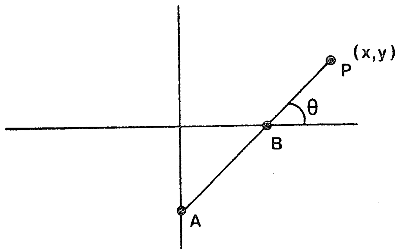
THE FRONT COVER

M.A.B. Deakin, Monash University

Last issue we illustrated a device for drawing ellipses. This time we show another - a practical device for drawing them on blackboards (as in this illustration - but we have printed the photograph in negative). The device, manufactured by the German company Leybold-Heraeus, attaches to the board by four suction cups at the back.

The points A, B slide in grooves and, as the arm thus moves, the point P traces out an ellipse. To see why this should be so, put $AP = a$, $BP = b$. Then, by elementary trigonometry, the co-ordinates (x,y) of P are given by

$$x = a\cos\theta$$
$$y = b\sin\theta.$$



Eliminating θ now gives

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

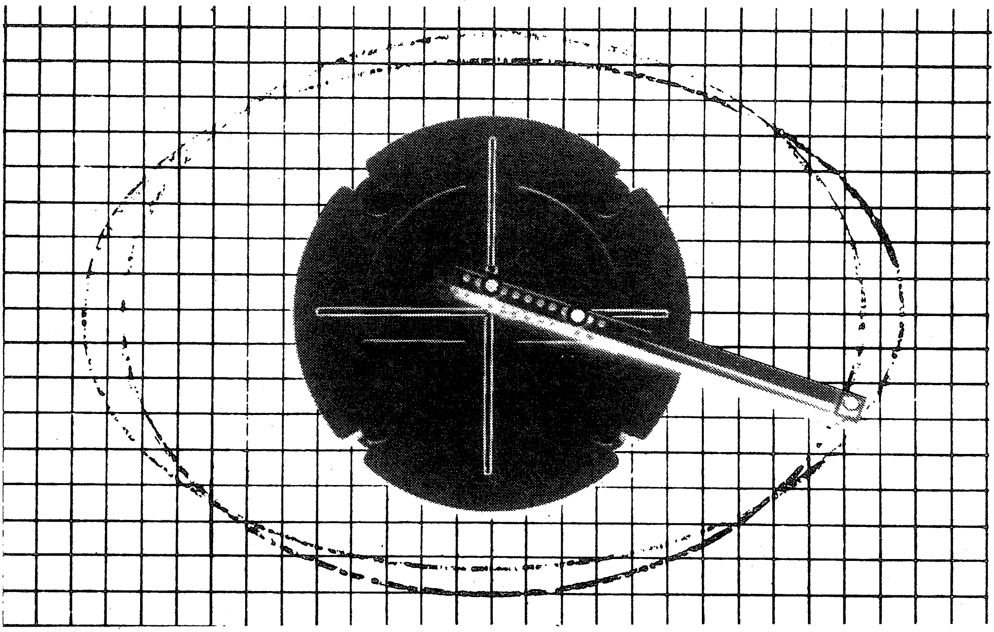
the standard equation of the ellipse.

The instrument allows a, b to be adjusted to give ellipses of different sizes and shapes. Here we illustrate two different cases. (See opposite.)

The instrument demonstrates the connection between the ellipse and another curve - the astroid. The astroid was the cover subject for Volume 2, Part 4. The easiest way to picture an astroid is to imagine a ladder up against a wall - then visualise the ladder beginning to slide. As an exercise, draw successive positions of the ladder as it slides in such a way that its top point maintains contact with the wall and its bottom point with the ground. Successive positions of the ladder are tangent to the astroid, which is referred to as their "envelope". (On page 4, we reproduce our Vol.2, Part 4 cover - a picture of an astroid generated by a computer drawing of the sliding ladder.)

The section AB of the elliptical compass is the analogue of the sliding ladder - its successive positions are all tangent to an astroid, whose equation is

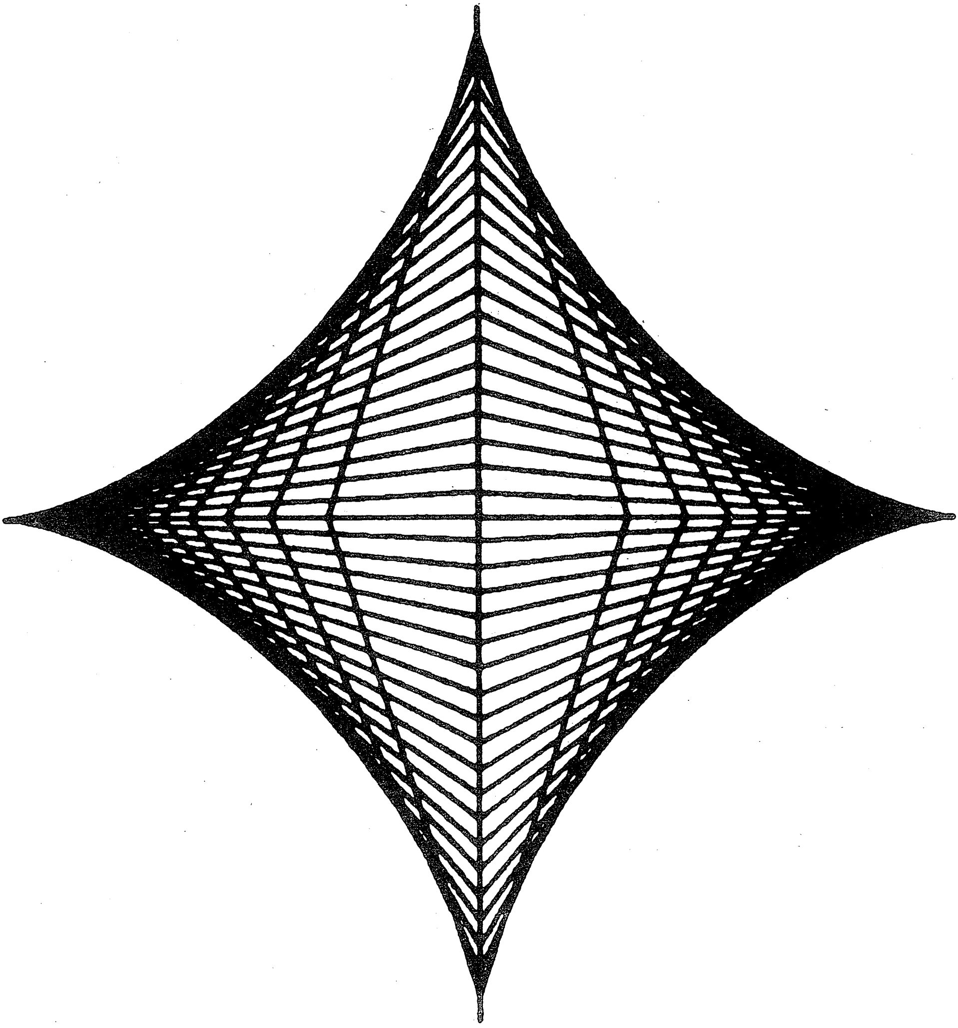
$$x^{2/3} + y^{2/3} = (a - b)^{2/3}.$$



The connections between ellipse and astroid are extensive and deep.

Each small section of arc that goes to make up the ellipse may be approximated by a circular arc. As we move along the ellipse, this circle will vary. The centre, in particular, will take up a new position. The path traced out by this centre is referred to as the *evolute* of the ellipse. This turns out to be the astroid referred to above.

Conversely, the ellipse is referred to as the *involute* of the astroid. Involutes can be realised geometrically, by wrapping a thread around the curve (here the astroid - the thread wraps from the inside) and then unwinding it. The end-point of the thread then traces out the involute. So if a pendulum were suspended from the top point of the astroid over-leaf and its length were suitably chosen, the bob would trace out an elliptical arc.



A THEOREM ON SERIES OR THE ART OF ADDING ARBITRARILY MANY SMALL NUMBERS†

Our intention is straightforward: take a sequence of real numbers

$$u_1, u_2, \ldots, u_n, \ldots$$

all less than one, all strictly positive, and form partial sums like this:

$$s_1 = u_1$$
$$s_2 = u_1 + u_2$$
$$\cdots$$
$$s_n = u_1 + u_2 + \ldots + u_n .$$

The series, denoted by $u_1 + u_2 + \ldots + u_n + \ldots$, may be viewed as a sequence of partial sums. If a limit exists for this sequence, we say that the series converges. (Otherwise, it diverges.)

Let us suppose that a series with general term u_n is convergent, let us say to s . We can write

$$\lim_{n\to\infty} s_n = s \quad \text{and} \quad \lim_{n\to\infty} s_{n-1} = s$$

and since $s_n = u_n + s_{n-1}$, we deduce immediately that

$$\lim_{n\to\infty} u_n = 0 .$$

So this last condition is necessary for the convergence of the series. But it is not sufficient ... .

---

† This article is a translation from the French. It first appeared in *Math-Jeunes* No.16. *Math-Jeunes* is a Belgian counterpart of *Function*. The article appears here under an exchange agreement between the two journals.

The display on the page opposite deals with an example which "works": the case of the series formed from the geometric sequence. The rest of this article is more concerned with the harmonic series and several of its variants.

The harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ certainly satisfies the condition $\lim_{n\to\infty} u_n = 0$, but nonetheless it diverges.

Demonstrated for the first time in 1650 by the Italian MENGOLI, the divergence of this series rests on the elementary arithmetic result

$$\frac{1}{3n-1} + \frac{1}{3n} + \frac{1}{3n+1} > \frac{1}{n}\,, \text{ where } n \text{ is a natural number.}$$

By reducing the fractions to a common denominator, this result is easily proved. Then

$$s = 1+\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\left(\frac{1}{8}+\frac{1}{9}+\frac{1}{10}\right)+\left(\frac{1}{11}+\frac{1}{12}+\frac{1}{13}\right)+\left(\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)+\ldots$$

so that

$$s > 1+ \quad \frac{1}{1} \quad + \left(\frac{1}{2} \quad + \quad \frac{1}{3} \quad + \quad \frac{1}{4} \right) \quad + \left(\quad \frac{1}{5} \quad + \ldots\right.$$

and, regrouping,

$$s > \quad 2 \quad + \quad 1 \quad + \ldots$$

and so $s > k$ (k being any natural number).

Another technique consists of grouping the terms according to the number of digits in their denominators.

$$s = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

$$s = \left(1 + \frac{1}{2} + \ldots + \frac{1}{9}\right) + \left(\frac{1}{10} + \ldots + \frac{1}{99}\right) + \left(\frac{1}{100} + \ldots + \frac{1}{999}\right) + \ldots$$

$$s > \left(\frac{1}{10} + \ldots + \frac{1}{10}\right) + \left(\frac{1}{100} + \ldots + \frac{1}{100}\right) + \left(\frac{1}{1000} + \ldots + \frac{1}{1000}\right) + \ldots$$

$$s > \frac{9}{10} + \frac{90}{100} + \frac{900}{1000} + \ldots$$

$$s > \frac{9}{10} + \frac{9}{10} + \frac{9}{10} + \ldots$$

which clearly proves divergence.

For a shorter approach, try this.

$$s = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots$$

$$s > \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \ldots$$

$$s > \quad 1 \quad + \quad \frac{1}{2} \quad + \quad \frac{1}{3} + \quad \ldots$$

$$s > s.$$

THE GEOMETRIC SEQUENCE

A *geometric sequence* (or, formerly, *progression*) is a sequence of numbers, such that each is equal to the product of its predecessor and a constant called the *constant ratio*.

We can readily show that if u_1 is the first term and q the common ratio, then u_n is given by

$$u_n = u_1 q^{n-1} .$$

Let us now calculate the partial sums s_n:

$$\begin{array}{rl} s_n = & u_1 + u_2 + u_3 + \dots + u_n \\ qs_n = & \quad u_1 q + u_2 q + \dots + u_{n-1} q + u_n q \\ \hline \end{array}$$

$$qs_n - s_n = u_n q - u_1 = u_1(q^n - 1)$$

and

$$s_n = \frac{q^n - 1}{q - 1} u_1 .$$

We can use this to show that if $-1 < q < 1$, the geometric series converges to the value

$$u_1/(1 - q) .$$

E.g. $\quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 2 .$

Intuitively, the harmonic series, although it does contain terms that get smaller and smaller, always sums them in such a way as to *exceed* any bound we try to impose. Note that the increase is very very slow: after 250 million terms the sum is still less than 20 ... .

And if we remove some terms from the series, what happens then?

A First, Easy, Example

If S is the sum of those terms remaining in the harmonic series when we remove all the terms with even denominators, then the series converges.†

$$S = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \cdots$$

$$S = \left(1 + \cdots + \frac{1}{9}\right) + \left(\frac{1}{11} + \cdots + \frac{1}{99}\right) + \left(\frac{1}{111} + \cdots + \frac{1}{999}\right) + \cdots .$$

In each bracket there are five times as many terms as in the preceding one since in a bracket each denominator appears in the next bracket preceded by each of the digits 1, 3, 5, 7, and 9.

$$S < 2 + \left(25 \times \frac{1}{11}\right) + \left(125 \times \frac{1}{111}\right) + \left(625 \times \frac{1}{1111}\right) + \cdots$$

$$S < 2 + \left(25 \times \frac{1}{10}\right) + \left(125 \times \frac{1}{100}\right) + \left(625 \times \frac{1}{1000}\right) + \cdots$$

$$S < 2 + \frac{25}{10}\left(1 + \frac{5}{10} + \left(\frac{5}{10}\right)^2 + \cdots\right) .$$

We recognise in the bracket on the right the geometric series with common ratio $\frac{1}{2}$, whose sum works out to be 2.

$$S < 2 + \left(\frac{25}{10} \times 2\right) = 7 .$$

In this first example, the series converges.

A Second, More Surprising, Example

If we delete from the series all terms whose denominator contains the digit 9, then the series converges!

$$1 + \frac{1}{2} + \cdots + \frac{1}{8} + \frac{1}{10} + \cdots + \frac{1}{18} + \frac{1}{20} + \cdots + \frac{1}{88} + \frac{1}{100} + \cdots$$

$$+ \frac{1}{108} + \frac{1}{110} + \cdots = ?$$

---

† It follows from this that those terms with even denominators themselves form a *divergent* series, but this may also be proved quite directly from the divergence of the harmonic series itself. This is left as an exercise for the reader.

We put

$$a_1 = 1 + \frac{1}{2} + \dots + \frac{1}{8}$$

$$a_2 = \frac{1}{10} + \dots + \frac{1}{88}$$

$$a_3 = \frac{1}{100} + \dots + \frac{1}{888}$$

$$\dots$$

The initial terms of each a_i are given by $1/10^{i-1}$. The number of terms in a_i is less than or equal to 9^i. The first statement is obvious; we now justify the second†.

First note that it holds for a_1, which contains exactly 9 terms. Suppose now that the statement has been proved for $i = n$; we shall show that it remains true for $i = n + 1$.

a_{n+1} contains all the terms of the series whose denominator d lies between 10^n and 10^{n+1}. This interval can be subdivided into nine intervals

$$\alpha.10^n \leqslant d < (\alpha + 1).10^n, \quad \alpha = 1,2,\dots,9.$$

The last interval contains no terms of the series; the other eight each contain exactly as many terms as there are in the interval

$$0 < d < 10^n.$$

But, by our assumption, this number is less than or equal to

$$9^n + 9^{n-1} + \dots + 9.$$

Thus α_{n+1} contains fewer than $8(9^n + 9^{n-1} + \dots + 9)$ terms, i.e. fewer than 9^{n+1} terms. This proves the second statement.

Hence

$$a_1 + a_2 + \dots + a_n + \dots < 9 + \frac{9^2}{10} + \dots + \frac{9^n}{10^{n-1}} + \dots .$$

We recognise a geometric series with first term 9 and common ratio 9/10. So our series sums to a value less than 9/(1 - 9/10), or 90.

---

†The method of proof used here is called *mathematical induction*, a technique widely employed to prove statements concerning positive integers. Let $S(i)$ be such a statement. To prove $S(i)$ true for all i, we prove:

(a) $S(1)$ is true
(b) $S(n) \Rightarrow S(n+1)$.

Then, by (b), $S(1) \Rightarrow S(2)$ and so $S(2)$ holds. Apply (b) again: $S(2) \Rightarrow S(3)$ and so $S(3)$ holds. And so on.

THE INTELLIGENT ROBOT

Robyn Owens,

University of Western Australia

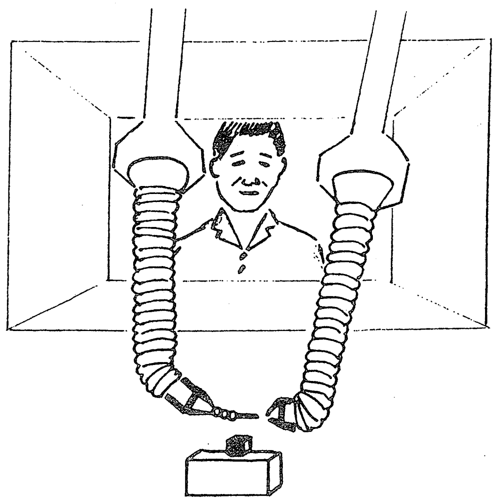
Despite the popularity of such robots as R2D2 in Star Wars, or the manically depressed Marvin in The Hitchhiker's Guide to the Galaxy, the robots of today are still at a rather primitive level and can only accomplish fairly limited tasks.

In fact, robots were only mentioned for the first time in 1921 in the play "R.U.R." or Rossum's Universal Robots by Karel Čapek. The plot concerns a huge factory manned by robots that have been designed to produce all types of goods, including other robots. However, the robots are eventually programmed to have emotions and it is then that they rebel and destroy their creators.

Today's industrial robot, on the other hand, is a long way from being equipped with emotions. Its development began during the Second World War with the *teleoperator*, a machine designed to handle radioactive materials at a distance. The teleoperator was a substitute for the operator's hand; it consisted of a pair of tongs and two handles which were connected together by linkage mechanisms. The tongs were on one side of a wall with the radioactive material, and the operator and handles were outside. By moving the handles the operator could achieve almost any position and orientation of the tongs, thus manipulating the radioactive material at will.

Figure 1.

A master-slave manipulator (teleoperator) for handling radioactive materials.



In 1947 the teleoperator was electrically powered for the first time but because the operator could no longer "feel" what was going on (there was no force feedback) the task of picking up and placing objects became very difficult. Force feedback was introduced just one year later and this made the operation of a teleoperator much simpler. Teleoperators are widely used these days in situations which are difficult or dangerous for people to operate, for example, in maintenance work on the outside of submarines, doing such delicate tasks as screwing up small screws.

After the end of the war the numerically controlled (i.e. computer controlled) machine tool was developed in answer to the need to make advanced aircraft. This allowed complicated aircraft parts to be described in terms of mathematical curves which were stored in the computer and the necessary computations were made to tell the tool where to cut the metal.

Then in the 1960's the machine which was to become Unimate's[†] first industrial robot was demonstrated. This device could be taught to perform any simple job by driving it by hand through the sequence of task positions, which were then recorded in a digital memory. The robot could replay the task exactly any number of times, for example, indefinitely spray painting car doors; but it was completely unable to adapt should the car door be moved to another position or exchanged for a side panel. Such a robot is known as a *first generation* robot; these machines have some form of memory and are programmable (through a microprocessor) but have no sensors through which they can learn about their environment.

A *second generation* robot is a first generation robot which has been given touch, sight or some other signal sensing. Touch sensors were first added to a robot in 1961 whilst by 1967 a computer could be equipped with a television camera and then identify a limited number of objects and decide where they were in space.

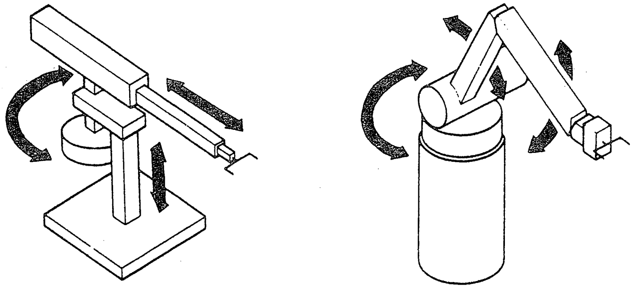
Third generation robots will be advanced robots that are mobile and have a high level of sensing and artificial intelligence. Although they appear frequently in science fiction films, their actual production is still some way off.

Most existing robots in use in industry today have the following features:

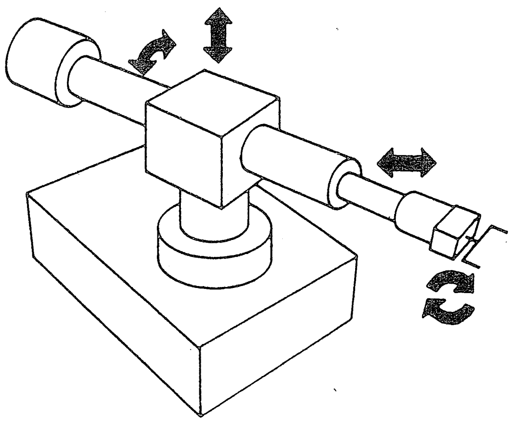
The mechanical structure. This consists of a central pedestal and the mechanical linkages and joints which allow movements in various directions. At the end of the "arm" is a "wrist" which may have a gripper or a tool attached. The number of independent motions that can be carried out by the assembly is known as the number of degrees of freedom of the system. The design of the arm is clearly very important since it determines which points in space can be reached by the wrist. See Figure 2.

[†]Unimate is a large U.S. Corporation specialising in robotics.

Figure 2



The movements of two robots with different mechanical structures.



Pick-and-place robot with four independent movements.

[Courtesy of the Australian Government Publishing Service.]

The control system. The movements of the mechanical structure can be controlled in a number of ways, from simple switches or mechanical stops to position sensors that are linked to the computer. Robots with mechanical stops and switches have only a limited number of positions and hence move from one point to another with no control over intermediate positions. Such robots are called *point-to-point* or *pick & place* robots. Robots with larger memories and the ability to control intermediate positions are called *continuous path* robots.

The power system. The power used to drive the mechanical structure of today's robots is either pneumatic, hydraulic or electrical.

A robot is thus a piece of equipment which uses ideas from mechanical engineering, electrical and electronic engineering and computer science. To simplify and solve many of the problems that occur in each of these areas we need a large number of mathematical tools, ranging from calculus and algebra to numerical analysis and statistics. The mathematical areas that one encounters in robotics arise from the need to have:

(i) Mathematical descriptions of one or many objects, of the robot, and of the relative positions and orientations between them. These mathematical descriptions are stored in the computer memory and the computer makes calculations which tell the power system how to operate the mechanical structure so as to manipulate the objects.

(ii) Mathematical descriptions of the forces and trajectories necessary to ensure that the robot accomplishes its task.

(iii) An analysis of sensor information; that is, analysing electrical signals, television pictures, sound waves, etc.

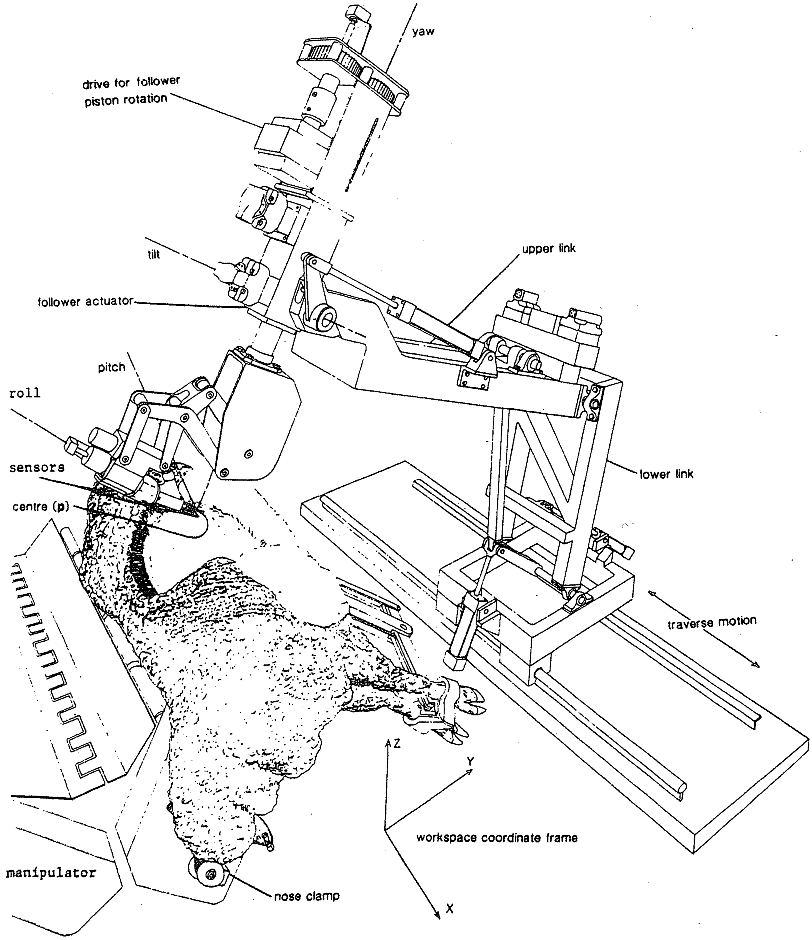
(iv) An anlysis of errors in the robot's knowledge of the world so that it can adapt its behaviour and learn about its environment.

(v) Computer programs that allow the robot to calculate its position and forces quickly enough so that it can work at a reasonable speed.

To illustrate some of these mathematical concepts let us consider one robot in particular, the experimental sheep shearing robot which has been developed at the University of Western Australia. A simplified diagram of the robot is given in Figure 3.

To shear a sheep, the robot needs to have a description of the sheep's surface, called a "software sheep". This description is mathematical and must be in a form that the computer can handle easily and quickly. To make the task easier, the sheep's surface is divided up into a number of patches, namely the back, side, belly, neck and legs, and each of these patches is described separately. Once we have a good description of each patch, we can then match all the patches up together to obtain the whole sheep.

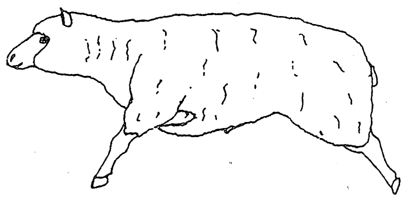
So let's begin by concentrating on the side patch. One of the simplest things to do is to consider this patch being made



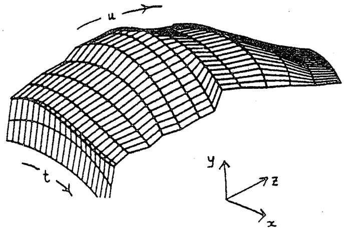
Simplified sketch of sheep-shearing rig at University of Western Australia
Source: Department of Mechanical Engineering, University of Western Australia

Figure 3.

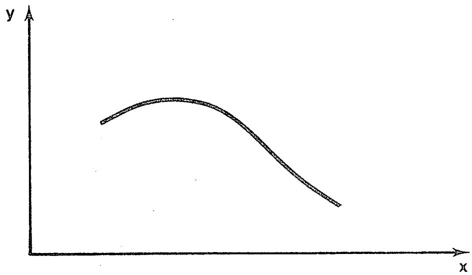
up of a number of parallel plane curves in equally spaced section planes in 3-dimensional space. The curves are then just objects in 2-dimensions (within each plane) and can be given a simple description.



(a) The sheep



(b) Side and rear leg patches



(c) A side section curve

Figure 4.

We describe the sheep's 2-dimensional (u,t)- surface in 3-dimensional space by saying that a point r on the surface is given by

$$r(u_i,t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a + bt + ct^2 + dt^2 \\ e + ft + gt^2 + ht^3 \\ ku_i \end{bmatrix}$$

where $0 \leqslant t \leqslant 1$ and $0 \leqslant u \leqslant 1$. The u_i are u values for each section curve.

The variables u and t are allowed to range between 0 and 1 and are called *parameters*. Notice that each section curve is given by describing the x-variable and the y-variable as a function of t (rather than just making the y-variable a function of x). In such a case we say that the curve is described *parametrically* with parameter t. The adjacent curves are joined up together by joining points of equal t-value as in Figure 4(b). The advantage of a parametric description is that it allows the curve to turn back or loop over itself. (Why is this so?) This could not occur if y was written simply as a function of x.

The functions describing x and y are called cubic polynomials and the constants a, b, c, d, e, f, g and h are the coefficients of these polynomials. These coefficients will be different for each of the section curves and by changing these coefficients we can change the shape of the surface. The constant k is just a scaling constant which tells us how far apart the section curves are and we can move from one section curve to the next by changing the value of u.

Although this gives us a general mathematical description of a sheep's side patch that we can program into a computer, there still remains the problem that individual sheep have different shapes, some being bigger than others, some fatter, some older, or they may be of different sexes. To cope with this problem we need to be able to *predict* the shape of an individual sheep's surface underneath the wool before we shear it. And this is where we use some statistical theory.

The idea is to notice that there should be some *relationship* between various *physical measurements* that we can make on a sheep (for example, weight, length, width across its shoulders, leg length, etc.) and the *coefficients* that we use to describe the surface. Since a big fat sheep will have different coefficients from a small skinny sheep, our aim is to predict the coefficients by using the physical measurements. This can be done using a technique known as *linear regression*; this allows us to write each coefficient as a linear function of the physical measurements. (A linear function is just one in which the physical measurements are multiplied by constants and then added together.) So we have

$$\text{coefficient} = (\text{constant } 1) \times \text{weight} + (\text{constant } 2) \times \text{length} + \ldots$$

In this way we can obtain an approximate predicted surface of the sheep before we shear.

To cope with the errors that arise from predicting the sheep's surface (since no prediction can be exact and the sheep moves!) the robot is equipped with electrical sensors that can feel where the sheep's skin is when it comes close to it. So the software sheet (or surface map) is used to guide the shearing cutter up close to the sheep's skin and then the sensors will make sure that the sheep is not cut by "feeling" where the skin is and always remaining a few millimeters away from it.

In this way the robot's behaviour is modified by the information it receives through its sensors.

The robot can also "learn" how to shear sheep more accurately by accumulating the surface data it finds on shearing one sheep, and using this data to get a better prediction of the next sheep it will shear. The more and more sheep it shears, the better the predictions it will be able to make and the better will be the quality of the shearing.

This robot has now sheared over 200 sheep, but it is still a very complicated machine and many problems remain unsolved. By combining the tools of mathematics with the skills of engineering the problems will eventually be overcome and what was initially an experimental machine will have grown into an intelligent, helpful and reliable robot.

By now the world has seen the development and growth of many "intelligent" robots. "Intelligence" usually means that the robot has the ability to *recognize* various situations and is able to *take decisions* depending on what it finds. In conclusion, two of these "intelligent" robots are:

> The Automatic Chocolate Decorating Robot. This robot consists essentially of an arm with a chocolate nozzle at one end. The arm is attached to a television camera that watches various shapes of chocolate go past on a conveyor belt. The robot has to *recognize* what shape the chocolate is and then *decide* where to squirt on the pattern of chocolate topping that corresponds to that particular shape. When deciding on the shape, it is not enough to say "square", "round" or "rectangular"; the robot also has to calculate *where* the chocolate is and in which direction it is pointing.
>
> The Ohio State University Hexapod. This is a robot with six feet and is designed to walk like an insect. In fact, cockroaches were studied especially for deciding how the hexapod robot could turn around corners. The research is being sponsored by and done for the U.S. Army and is eventually seen as producing a 2-ton vehicle which will replace pack mules and helicopters. Such a robot could also be used extensively on a construction site, serving as an apprentice to a craftsman in constructing brick walls, lifting and setting window and door frames, painting, pouring concrete and many other such tasks.

By the 1990's we expect to have household robots that can carry out such jobs as vacuum cleaning, dusting, setting tables, scrubbing bathrooms and almost any other domestic task. Such robots will need to be equipped with very sophisticated sensors and be able to take quite complicated decisions. This will certainly be the age of the third generation robot.

Ref. "Robots" - A report to the Prime Minister by ASTEC, prepared by the Technological Change Committee (1982). [Available from the Australian Government Printing Office.]

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

TESTS FOR DIVISIBILITY ON LARGE NUMBERS

Bruce Henry,
Victoria College, Rusden Campus

The number B = 123 456 789 101 112 131 415 161 718 192 021 222 328 has 39 digits and is regarded by most people as a BIG number. But you can tell instantly that it is divisible by 2 and not by 5. (We will use the term "divisible by" to mean "exactly divisible by".)

We write $2 \mid B$ and $5 \nmid B$.

It is easy to tell that $4 \mid B$, since $B = 100K + 28$, where K is a 37-digit number and $4 \mid 100$ and $4 \mid 28$, so $4 \mid B$. Similarly $8 \mid B$, since $B = 1000L + 328$ and $8 \mid 1000$ and $8 \mid 328$.

The most common reason for wishing to test for divisibility by a number is to find out if a number is a prime or not; thus we wish to find tests for divisibility by primes, and tests which a computer can handle are useful. You probably know a test for divisibility by 3. A number is divisible by 3 if the sum of its digits is divisible by 3. You certainly know a test for divisibility by 5 and you may know one for 11. But other primes have difficult tests, all different, and therefore hard to program on a computer. Furthermore, few computers will store a 36-digit number as a *number* (most will store it only as a string of characters). We must look for tests which are similar for any prime and which can be used on the computer in that they only use the last few digits of the number. Here is a test for 7, 11 or 13.

1. Call the number to be tested N_0.
2. Note the number q formed by the last 3 digits of N_0.
3. Strike out the last 3 digits of N_0 and subtract q from the number left. Call this new number N_1.
4. Repeat steps 1, 2, 3 with N_1 instead of N_0, obtaining N_2, and so on, until N_a is reached, with 6 or fewer digits.
5. Complete the test on N_a by division. N_0 divides by 7, 11 or 13 if and only if N_a divides by 7, 11 or 13.

Example: $N_0 = 1234567893$.

$q = 893$.

$N_1 = 1234567 - 893$

$= 1233674$.

$N_2 = 1233 - 674$

$= 559$.

$7 \nmid N_2,\ 11 \nmid N_2,\ 13 \mid N_2$.

So $7 \nmid N_0,\ 11 \nmid N_0,\ 13 \mid N_0$,

i.e. 13 is a factor of 1 234 567 893 but 7 and 11 are not factors.

The test relies on three facts:

1. $1001 = 7 \times 11 \times 13$.

2. $p \mid N$ if and only if $p \mid (N - kp)$ where k is integral and p is prime.

3. If p is a prime, $p \neq 2$, $p \neq 5$, then $p \mid 1000N$ if and only if $p \mid N$.

The test subtracts suitable multiples of 1001 from N_0 so that the result ends in 000. Then N_1 is this number divided by 1000. In the above example $N_1 = \dfrac{N_0 - 893 \times 1001}{1000}$.

$p \mid N_0$ if and only if $p \mid N_1$ where $p = 7$ or 11 or 13.

Now this test is a good one in that it only operates on the right hand digits of the number and that it shortens the number by 3 digits at each application (iteration). Unfortunately it cannot be generalized to tests for divisibility by other primes. Nonetheless, the principle is a sound one. To test N for divisibility by prime p, we will try to subtract a multiple of p from N so that the result ends in zero.

Let us try to find a test for divisibility by 17. Fifty-one is a multiple of 17. If we subtract suitable multiples of 51 we can easily get a zero in the last place of the number being tested.

E.g. let $N_0 = 1606463297$.

Since the last digit is 7, we must subtract a multiple of 17 which ends in 7. 51×7 will do

$$1606463297 - 357 = 1606462940.$$

Let $N_1 = 1606462940 \div 10$

$= 160646294$.

This time, subtract $4 \times 51 = 204$.

$N_1 - 204 = 160646090$.

Let $N_2 = 16064609$.

To speed up our test, call the last digit q; then

$$N_{n+1} = \frac{N_n - q \times 51}{10} .$$

So $N_3 = \dfrac{N_2 - 9 \times 51}{10} = 1606415.$

$N_4 = (1606415 - 5 \times 51)/10 = 160616.$

$N_5 = (160616 - 6 \times 51)/10 = 16031.$

$N_6 = (16031 - 51)/10 = 1598.$

$N_7 = (1598 - 408)/10 = 119.$

Since $17 \mid 119$, $17 \mid N_0$.

We can speed up the test a little more by using only the 5 from the 51:

"To test if N_0 divides exactly by 17, note the last digit q of N_0. Strike out q and subtract $5q$ from the number left. Call this new number N_1. Repeat with N_1, and so on, obtaining N_2, etc. till N_a is reached with 3 or fewer digits; $17 \mid N_0$ if and only if $17 \mid N_a$."

For example $N_0 = 1234567893$

$q = 3$

$N_1 = 123456774$

$N_2 = 12345657$

$N_3 = 1234530$

$N_4 = 123453$

$N_5 = 12330$

$N_6 = 1233$

$N_7 = 108.$

Since $17 \nmid 108$, $17 \nmid N_0$.

This test has the desired properties; it reduces the number under test and only changes a few digits at the right hand end. It is slower to give a result than the (7,11,13) test in that it takes more steps - it only removes one digit at each step where the (7,11,13) test removes 3 digits at each step. We can devise a test to eliminate 2 digits at once if we can find a multiple of 17 which ends in 01. 901 is such a number $(901 = 53 \times 17)$. Or use 6001 $(= 353 \times 17)$ to remove 3 digits at once.

For example: $N_0 = 18606463297$

$N_1 = 18606463 - (6 \times 297) = 18604681$

$N_2 = 18604 - 6 \times 681 = 14518$.

Since $17 \mid N_2$, $17 \mid N_0$.

This example uses a multiple m_p, by which the last d digits of the number under test must be multiplied.

Thus we can establish numbers which are related to the primes which will be used in the above way to test for divisibility by the primes. So far we have:

Prime (p)	m_p	d=No. of digits in q
7	1	3
11	1	3
13	1	3
17	5	1
17	9	2
17	6	3

Clearly, the larger the number of digits in q the better, we so discard smaller numbers if we can find a larger one.

It is easy to find m_p if d is 1.

If p ends in 1, $m_p = \dfrac{p-1}{10}$, e.g. $p = 31$, $m_p = 3$.

If p ends in 3, $m_p = \dfrac{7p-1}{10}$, e.g. $p = 23$, $m_p = 16$.

If p ends in 7, $m_p = \dfrac{3p-1}{10}$, e.g. $p = 37$, $m_p = 11$.

If p ends in 9, $m_p = \dfrac{9p-1}{10}$, e.g. $p = 19$, $m_p = 17$.

To find formulae like these for $d = 2$, we will need to find 40 formulae as p may end in 01, 03, 07, 09, 11, 13, 17, 19, ..., 91, 93, 97, 99. We will need 400 formulae for $d = 3$ (!) We cannot store all these in a microcomputer, so we must find a way to produce them within a program, or settle for $d = 1$.

It would be good if we could find a formula or an algorithm to compute m_p for large d. This is not easy. It is interesting that m_p is easy to find for $d = 1$ but difficult for $d > 1$. The formulae for $d = 1$ are derived from our knowledge of tables e.g. to get m_p for $d = 1$, when p ends in 3, we know that $7 \times 3 = 21$ ends in 1, and the rest is easy. But we do not know tables for odd numbers up to 99, so the appropriate formula is hard to find. Perhaps this article will stimulate someone to devise suitable formulae.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

SCIENTIFIC LAWS I

G.B. Preston, Monash University

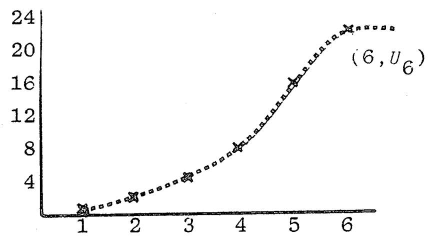
In *Function*, Volume 6, Part 5, p.7, in the item headed "The next term in the sequence" an argument was given to show that "a perfectly logical answer" to the question, "What is the next term in the sequence 1, 2, 4, 8, 16 ?" is that the next term is 31.

In Volume 7, Part 1, pp.24-25, two letters from readers commented on this item. The first from J.A. Deakin, pointed out that any number, or indeed any sequence of numbers, could follow the first five numbers 1, 2, 4, 8, 16, as logical continuations of this sequence. Deakin demonstrated this by giving a formula for u_n, the nth term of a sequence, such that $u_1 = 1$, $u_2 = 2$, $u_3 = 4$, $u_4 = 8$, $u_5 = 16$, and such that u_n, for $n > 5$, could take any sequence of values we chose, including non-integral values.

The question of "what is the next term of a sequence" is in fact perhaps the most fundamental in science, economics, sociology, or indeed any area of study in which it is possible to use numbers to measure what one is investigating.

One way of thinking about the question is to regard it as asking how to find a curve through a given set of points in a plane. For example, suppose we have a sequence $u_1, u_2, \ldots, u_5$. This sequence of 5 terms, determines the 5 points with coordinates $(1,u_1),(2,u_2),(3,u_3),(4,u_4),(5,u_5)$. In the special case when $u_1 = 1$, $u_2 = 2$, $u_3 = 4$, $u_4 = 8$, $u_5 = 16$, Figure 1 graphs the corresponding set of points, (1,1),(2,2),(3,4),(4,8),(5,16). Note that in this graph different scales are used on the horizontal and vertical axes.

Figure 1



Finding an answer to the question "what is the next term in the sequence $u_1, u_2, \ldots, u_5$" may perhaps be regarded as equivalent to finding a curve through these points and then continuing the curve to the right, so as to find a point with co-ordinates $(6, u_6)$. The dotted curve drawn gives, approximately, $u_6 = 22$.

Is there any good reason for supposing that the sequence should be 1,2,4,8,16,22,... ?

I said earlier that perhaps this question was the most fundamental in the whole of science. Why? How can this apparently trivial playing with a few numbers be basic in science?

The reason is that science is based upon measurements. In any scientific investigation we start with a set of measurements. We hope that the measurements we make will enable us to understand what is happening. We have a finite sequence of measurements $u_1, u_2, \ldots, u_n$, say, and from these we hope to be able to find some explanation of what we have measured, i.e. what we have observed. The explanation will be a good one if we can predict what the next measurement will be.

When we speak here of "the next measurement" we are not necessarily speaking in terms of time. Each measurement that is made is made in certain circumstances that are recorded as part of the scientific observation. What we want to be able to do is to predict what we shall observe when the circumstances under which the observation is made are specified arbitrarily. If we can do this correctly in all circumstances, then we have certainly understood the phenomena we are observing: we have a complete understanding of the situation.

Such a complete understanding is rare. When it occurs we have what is called a "law of nature". What is accepted as a law of nature by one generation may be overturned by later experiments or observations. Perhaps the most well-known of such changes of view is from the acceptance of Newton's explanation of motion of the planets to a preference for that of Einstein. In fact Newton's explanation is a special case of Einstein's, applying when the speeds involved are not too large. The evolution from Newton's theories to those of Einstein is required to accommodate a longer (i.e. including more) sequence of observations than was known to Newton.

What makes us prefer one explanation of a sequence of observations to another?

This is the first of a sequence of articles that will discuss various approaches to such interpretations. There are several strictly mathematical approaches, one of which is evidenced in the letter of J.A. Deakin in *Function*, Volume 7, Part 1. There are scientific approaches and statistical approaches. There are what might be described as psychological approaches. There are historical approaches, which to some degree comment on what are the acceptable psychological approaches.

A famous approach to what is acceptable as a scientific explanation is embodied in the principle of Occam's razor. An English translation of this principle is: *Entities should not be multiplied except when it is necessary.*† When applied to interpreting scientific observations the principle requires that all unnecessary assumptions be eliminated or, in more general terms, that simpler explanations be preferred to more complicated ones. Newton refers to the importance of the principle in the 3rd edition of his *Principia Mathematica.*

Let us consider an application of the principle to a simple situation. Suppose we consider the sequence 0, 16, 64, 144 and ask "What is the next term in this sequence?". Let us also suppose that these numbers represent measurements taken of some phenomenon at times $t = 0,1,2,3$, respectively. If these numbers really represent a natural phenomenon and we apply Occam's razor, then we ought to be looking for the simplest possible formula $f(t)$, such that $f(0) = 0$, $f(1) = 16$, $f(2) = 64$, $f(3) = 144$.

It is unfortunately not possible to make precise the notion of what is meant by "the simplest possible formula". For example here are two candidates for the choice of $f(t)$.

(a) $f(t) = 16t^2$.

[In fact this is the formula that results from using, for a sequence of 4 terms only, the appropriate part of the formula exhibited by J.A. Deakin in his letter in *Function* Volume 7, Part 1, already referred to.]

(b) $f(t) = -\frac{1}{6}t^4 + t^3 + \frac{85}{6}t^2 + t$.

The reader should check that for (b), as for (a), $f(0) = 0$, $f(1) = 16$, $f(2) = 64$, and $f(3) = 144$.

Now which formula is the simplest? Whick makes the least number of unnecessary assumptions? There is no formal logical answer to this question. Though perhaps, like me, you would opt for possibility (a). In (a) $f(t)$ is a monomial (i.e. a polynomial with a single term) whereas in (b) $f(t)$ is a polynomial with four terms; in (a) the coefficient of the sole term, in t^2, is an integer whereas in (b) the coefficients involve fractions that are not integers. Altogether simpler!

The principle of Occam's razor would seem to favour (a) because, for example, the formula (b) suggests that the observed measurements depend on the fourth power t^4 and also the third power t^3 of the time at which the measurement is taken; while

†The name Occam's razor for this principle was first used in 1852 by the Scottish philosopher Sir William Hamilton (1788-1856) in his book *Discussions*. The Latin original of which the above is a translation is: *Entia non sunt multiplicanda, praeter necessitatem*, a phrase apparently first used by John Ponce of Cork (Ireland) in 1639. William of Ockham, a 14th century English logician, after whom the principle is named, made very similar statements, and was perhaps the first to argue convincingly for its importance.

on the other hand formula (a) would say that we need not assume that the third and fourth powers of t are needed to explain what has been observed.

Of course, as earlier observed, there is a way of testing the usefulness of the formula we have chosen, to see whether it has the desirable feature of applying to other observations in addition to those already considered. Suppose that at time $t = 4$, the next (term of the series) observation was 252. Then formula (a), $f(t) = 16t^2$, which gives $f(4) = 256$ would not apply (unless an experimental error had been made in the observation: but this is another story) while formula (b) gives exactly $f(4) = 252$. So, unless we strongly suspect an experimental error in the measurement at $t = 4$, we adopt formula (b): the alternative formula (a) has failed.

As another example consider the motion of the planets in relation to the sun. If we take the sun as stationary and take the planets to be revolving about the sun, then we get a relatively simple mathematical description of the observed motion of the planets. If, on the contrary, we suppose that the earth is stationary, and take the sun and the other planets to be revolving round the earth, then an adequate mathematical description of the motions observed, is extremely complicated. The principle of Occam's razor would seem to demand that we take the sun as stationary and so avoid unnecessarily complicated concepts and calculations. Here however we are not dealing with two theories, one of which is wrong, while the other is right. We have two equivalent theories, or formulations of a single theory, one of which is much simpler to use than the other. Occam's razor is a most useful principle to invoke to justify our choice.

In the next issue of *Function* I shall discuss some mathematical approaches to the answer to the question "what is the next term in this sequence?". Alternative approaches will be discussed in later issues of *Function*.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

In general we mean by any concept nothing more than a set of operations; the concept is synonymous with the corresponding set of operations. If the concept is physical, as of length, the operations are actually physical operations, namely, those by which length is measured; or if the concept is mental, as of mathematical continuity, the operations are mental, namely those by which we determine whether a given aggregate of magnitudes is continuous.

P.W. Bridgman, *The Logic of Modern Physics*, 1934

Data obtained by the processes of measurement, numbers constructed by definite algorithms, are the basis of knowledge. *Pi*, as the idealization of a limiting process, is forever beyond our reach, but as the 707-place approximation obtained by Shanks, it is within our range of knowledge.

H.T. Davis, *The Theory of Linear Operators*, 1936

LETTERS TO THE EDITOR

IMPROVING A RESULT

A short note on p.28 of your last issue showed that $100! + 2, \ldots, 100! + 100$ were all composite. Thus the number $100! + 1$ is followed by 99 composite numbers. In fact this is not the smallest number with this property. $97! + 1$ is also followed by 99 composite numbers.

$$97! + 2,\ 97! + 3,\ \ldots,\ 97! + 97$$

are all composite as before. Furthermore

$$97! + 98 \qquad \text{and} \qquad 97! + 100$$

are clearly divisible by 2, while $97! + 99$ is divisible by 3. It may be that there are even smaller numbers with the same property.

Bruce Henry,
Victoria College, Rusden Campus.

LIOUVILLE'S NUMBERS

The French mathematician Joseph Liouville (1809-92) set himself the task of finding sets of integers $a_1, a_2, \ldots, a_n$ having the property

$$(a_1^3 + a_2^3 + \ldots + a_n^3) = (a_1 + a_2 + \ldots + a_n)^2 \quad \text{for some fixed integer } n.$$

The remarkable result he obtained is best first illustrated with a few examples.

(i) Consider the integer 6. It has four divisors: 1, 2, 3, 6. Let n be 4 and let a_i, $i = 1,2,3,4$, be the number of factors of each of the divisors of 6.

Hence we have the following situation:

Divisor of 6	1	2	3	6
Factors of divisor	1	1,2	1,3	1,2,3,6
a_i	1	2	2	4

Now we observe that

$$1^3 + 2^3 + 2^3 + 4^3 = 81 = 9^2 = (1 + 2 + 2 + 4)^2 .$$

(ii) Consider the integer 20 and its divisors: 1,2,4,5, 10,20. For $n = 6$ we can construct the following table:

Divisor of 20	1	2	4	5	10	20
Factors of divisor	1	1,2	1,2,4	1,5	1,2,5,10	1,2,4,5,10,20
a_i	1	2	3	2	4	6

Again we observe the result.

$$1^3 + 2^3 + 3^3 + 2^3 + 4^3 + 6^3 = 324 = (18)^2$$
$$= (1 + 2 + 3 + 2 + 4 + 6)^2 .$$

(iii) We see that the integer 2^{n-1} has n divisors $1, 2, 2^2, \ldots, 2^{n-1}$. The respective numbers of factors for each of these numbers are $1, 2, 3, \ldots, n$.

Hence using the same method as above we have the result:

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} = (1 + 2 + \ldots + n)^2 .$$

This result is in fact a special case of a more general theorem due to Liouville, which we may state as follows:

If a natural number K has n divisors then let the integers a_i, $i = 1, \ldots, n$, be the number of divisors of each of the divisors of K. Then

$$(a_1^3 + a_2^3 + \ldots + a_n^3) = (a_1 + a_2 + \ldots + a_n)^2 .$$

Garnet J. Greenbury.
Brisbane.

[*This result intrigued us and we asked Dr R.T. Worley, a number theorist, to comment. He remarked that "Liouville's result is just a neat way of generalising*

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2 \text{ "}, \qquad (*)$$

He proceeded to give a proof which is perhaps a little technical for Function, *but whose gist can be given. First suppose we take the divisors of a perfect power. Then as in Greenbury's example (iii) we merely reach the known formula (*). If we take the divisors of a product of perfect powers, we multiply the separate results. Thus Greenbury's example (ii) is expressible as*

$$(1^3 + 2^3 + 3^3)(1^3 + 2^3) = (1 + 2 + 3)^2(1 + 2)^2$$

(as 20 *has* 6 *divisors other than itself, and* 6 = 3 × 2*). Eds.*]

PROBLEM SECTION

We have had a gratifying response to our problems and give here the solutions of those set in Volume 7, Part 2.

SOLUTION TO PROBLEM 7.2.1.

$ABCDEF$ is a convex hexagon, all of whose angles are equal. The problem was to show that

$$AB - DE = EF - BC = CD - FA.$$

This problem was solved by J. Ennis, Year 11, M.C.E.G.S. He writes as follows.

Since all the interior angles are equal, they must be 120° each. Extend the sides as shown in Figure 1.

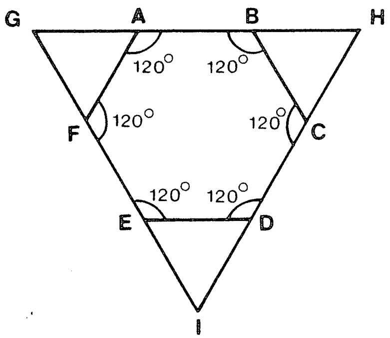


Figure 1

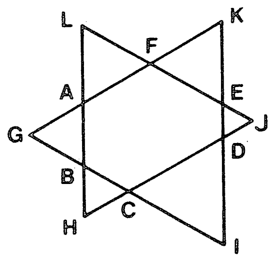


Figure 2

Each of the triangles GAF, HBC, IDE is equilateral as its interior angles are all 60°. Thus $GA = GF = AF$, $BH = HC = BC$, $EI = ID = ED$. Furthermore, since

$$\measuredangle FGA = \measuredangle CHB = \measuredangle DIE = 60°,$$

the triangle GHI is equilateral and so

$$GA + AB + BH = GF + FE + EI = HC + CD + DI.$$

So by the equalities proved earlier

$$FA + AB + BC = FA + FE + ED = BC + CD + ED.$$

Now subtract $BC + FA + ED$ from each expression to obtain the required result.

This problem first appeared in the British counterpart of *Function, Mathematical Spectrum*. They published a slightly different proof based on Figure 2. All the triangles adjoined to the original hexagon are equilateral and furthermore opposite sides of this are parallel. Then

$$AB + BC = AB + BH = KE + ED = FE + ED$$

Thus $$AB - DE = EF - BG.$$

The rest of the problem follows similarly.

SOLUTION TO PROBLEM 7.2.2.

This problem asked for a proof that if

$$(a + b + c)^3 = a^3 + b^3 + c^3$$

then

$$(a + b + c)^5 = a^5 + b^5 + c^5.$$

Clayton S. Smith of 184 Wilkilla Road, Mt Evelyn solved the problem by noting that

$$(a + b + c)^3 - a^3 - b^3 - c^3 = 3(a + b)(b + c)(c + a).$$

Thus, either $a = -b$, $b = -c$ or $c = -a$. By symmetry, choose $a = -b$. Then clearly $(a + b + c)^5 = a^5 + b^5 + c^5$.

John Barton of 1008 Drummond Street, North Carlton submitted the same argument. John Percival, Year 12, Elderslie H.S., Narellan, also solved the problem but by a different method. He wrote a cubic equation $x^3 + px^2 + qx + r = 0$ of which a, b, c are to be the roots. This yielded a relation $r = pq$. But a, b, c are also roots of $x^5 + px^4 + qx^3 + rx^2 = 0$. The relation $r = pq$ then allows us to deduce that $a^5 + b^5 + c^5 = p^5 = (a + b + c)^5$.

John Barton considered the question of whether the result generalises. Is

$$(a+b+c+d)^4 = a^4+b^4+c^4+d^4+4(a+b+c)(b+c+a)(c+d+a)(d+a+b) \ ?$$

The answer is no, because the left-hand side, if expanded, would have 4^4 terms, while the right would have only $4 + 4.3^4$ and (dividing by 4)

$$4^3 \neq 1 + 3^4 .$$

It happens that $3^2 = 1 + 2^3$ and this is what allows the cubic case to proceed.

SOLUTION TO PROBLEM 7.2.3.

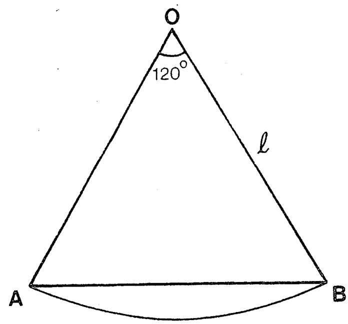
This problem read:

A new planet has been discovered. Its shape is that of a right circular cone whose flat circular face is land. This continent has a diameter of 5000 km. The curved surface is entirely covered by water and the oceanic area is three times that of the continent. The inhabitants of the planet are all keen sailors and are planning an "around-the'cone" yacht race. What is the shortest route, starting and ending at the same point on the coastline, that circumnavigates the cone?

J. Ennis, Year 11, M.C.E.G.S. solved this problem. The curved surface of the cone may be imagined to be cut by a straight line joining the vertex to the point at which the circumnavigation commences and ends. This surface may then be imagined as flattened out into a sector of a circle, as shown.

It is readily deduced that $\ell = 7500$ km and that $\measuredangle AOB = 120°$.

Now the shortest route is seen to be the straight line AB. The length of this is (by the cosine rule) $7500\sqrt{3}$ km or approximately 12990 km.



SOLUTION TO PROBLEM 7.2.4.

This problem, submitted by J. Ennis, asked for a proof that

$$e + \ln 4 > 4 .$$

Simple use of a calculator was outlawed - tables likewise.

The shortest of the proofs submitted came from the proposer himself and used integration. It proceeds

$$\begin{aligned}\int_{\ln 2}^{1} (e^x - 2)dx &= (e - 2) - (2 - 2\ln 2) \\ &= e + \ln 4 - 4 .\end{aligned}$$

But, as $e > 2$, $\ln 2 < 1$. It follows that the integral is positive, for $x > \ln 2$ implies that $e^x > 2$. Thus $e + \ln 4 > 4$.

Clayton S. Smith sent us a computationally based solution, using infinite series.

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

$$> 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = \frac{163}{60}$$

$$\ln 4 = 2 \ln 2 = 2\{1 - \frac{1}{2} + \frac{1}{3} - \ldots\}$$

$$> 2\{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}\}$$

$$= 2 \times \frac{1627}{2520} = \frac{3254}{2520} > \frac{3240}{2520} = 1 + \frac{2}{7} .$$

Thus $$e + \ln 4 > 3 + \frac{43}{60} + \frac{2}{7} > 4 .$$

Clayton notes that, with a calculator, or even without one, but with extra work, we could show that $e + \ln 4 > 4.1$.

John Barton, 1008 Drummond Street, North Carlton also submitted a series approach. It may be proved from the series for $\ln(1 + x)$ [used above with $x = 1$] that

$$\tfrac{1}{2} \ln \frac{1 + x}{1 - x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots .$$

Put $x = \frac{1}{3}$.

$$\tfrac{1}{2} \ln 2 = \frac{1}{3} + \frac{1}{81} + \frac{1}{5 \times 243} + \ldots .$$

This gives a series for $\ln 2$ that converges rather more rapidly than Clayton Smith's. Otherwise the details are much the same.

SOLUTION TO PROBLEM 7.2.5.

This problem is one of a number of related problems in which the absence or presence of information at various stages of a dialogue itself forms part of the data for a problem. A related (indeed harder) problem appeared as 1.2.6.

"I hear some youngsters playing in the garden", says Jones, a graduate student in mathematics, "Are they all yours?" "Heavens, no", exclaimed Professor Smith, the eminent number theorist. "My children are playing with friends from three other families in the neighbourhood, although our family happens to be the largest. The Browns have a smaller number of children, the Greens have a still smaller number, and the Blacks the smallest of all."

"How many children are there altogether?" asked Jones.

"Let me put it this way", said Smith. "There are fewer than 18 children, and the product of the numbers in the four families happens to be my house number which you saw when you arrived."

Jones took a notebook and pencil from his pocket and started scribbling. A moment later he looked up and said, "I need more information. Is there more than one child in the Black family?"

As soon as Smith replied, Jones smiled and correctly stated the number of children in each family.

Knowing the house number and whether the Blacks had more than one child, Jones found the problem trivial. It is a remarkable fact, however, that the number of children in each family can be determined solely on the basis of the information given above! How many children are in each family?

To solve this, first note all possibilities for the numbers of children. Let S be the number of Smiths, B the number of Browns, G the number of Greens and A the number of Blacks. Then $A < G < B < S$ and $A + G + B + S < 18$. 38 possibilities satisfy these conditions. For each of these, the product $AGBS$ is formed.

In some cases, this has a unique factorisation satisfying the conditions specified. E.g. $1 \times 3 \times 4 \times 9 = 108$ and *no other set on the list yields this product.* In other cases, an ambiguity remains. E.g. $1 \times 3 \times 4 \times 9 = 60$, $1 \times 2 \times 3 \times 10 = 60$ and $1 \times 2 \times 5 \times 6 = 60$. Presumably Jones remembered the Smith's house number, so, *had the factorisation been unique, he would have known the answer at this stage.* He did not, but needed further information.

Those possibilities of the original 38 that involve $A > 1$ all give unique products except

$$A = 2, \; G = 3, \; B = 4, \; S = 5 \qquad (*)$$

whose product is 120. The combinations 1,3,5,8 and 1,4,5,6 also multiply to give 120. Thus if the Blacks had had only one child, Jones would have required yet more information. Since he did not, (*) is the solution.

The above expands on a brief outline of the solution submitted by J. Ennis.

COMMENT ON PROBLEM 7.1.2.

Problem 7.1.2 concerned a triangle of sides 13,14,15. The question asked for the distance from the side of length 14 to the opposite vertex. Rather too late for inclusion in our last issue, a solution arrived from John Barton that differed from those we published earlier. This uses <u>*Hero*'s (or *Heron*</u>'s) formula for the area of a triangle: Area $= \sqrt{s(s-a)(s-b)(s-c)}$, where the sides have lengths a, b, c, and $s = \frac{1}{2}(a + b + c)$. In this example, the area is found to be $84 = \frac{1}{2} \times 14 \times 12$, so that the requisite distance is 12.

The response to our problems has been most gratifying lately, and we hope it will continue unabated on the new crop that follows.

PROBLEM 7.4.1.

A *palindromic number* is one which reads the same backwards or forwards. It can happen that when a number and its reverse are added, the sum is a palindrome. [E.g. 1030 + 0301 = 1331 .] In other cases, it isn't. 812 + 218 = 1030 , which is not palindromic. But in this case a further operation of reversal and addition produces, as we saw above, and palindromic result.

Does this hold true in general if the process is repeated often enough? The answer depends on the number base employed and is known only in the case of base 2.

In that base, 10110 can never produce a palindromic number in this way. Can you prove this?

PROBLEM 7.4.2. (Submitted by J. Ennis.)

Let S be a sphere with A, B, C three points interior to it. AB, AC are perpendicular to the diameter through A. There will be exactly two spheres passing through the points ABC and tangent to S. Prove that the sum of the radii of these spheres is equal to the radius of S.

PROBLEM 7.4.3. (Submitted by S. Ethier.)

A snowplow starts plowing snow at noon. By 1.00 p.m. it has travelled 10 km. By 2.00 p.m. it has travelled 15 km. When did it start snowing? (Assume constant rates of snowfall and snow displacement.)

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

NEGLIGENCE[7]

In 1928 the New York *Times* carried a cabled story from Paris, dated February 21. It reported the fact that six persons had been found guilty of the "accidental" death of a M. Desnoyelles. He had been in a sanitarium, and a medicine had been prescribed for him. But the chief physician gave M. Desnoyelles a prescription really intended for another patient named Desmalles. Second, the chief physician failed to check that the prescription, dictated to a clerk, was written as he had ordered. Third, the interne who filled the prescription confused two drugs, and introduced one of a poisonous nature. Fourth, the order for the prescription was mistakenly written on a slip used for medicines for internal use, rather than externally as the doctor intended. Fifth, the head pharmacist, who was supposed to check all prescriptions, was busy. He left the matter to his assistant, and she neglected to check. Sixth, an assistant "corrected" the error in name, and wrote on the medicine that it was intended for M. Desnoyelles. Seventh, the interne who administered the medicine disregarded the indicated dose, handed the bottle to the patient, and instructed him to "take a good big drink".

Warren Weaver, *Lady Luck*, 1963.