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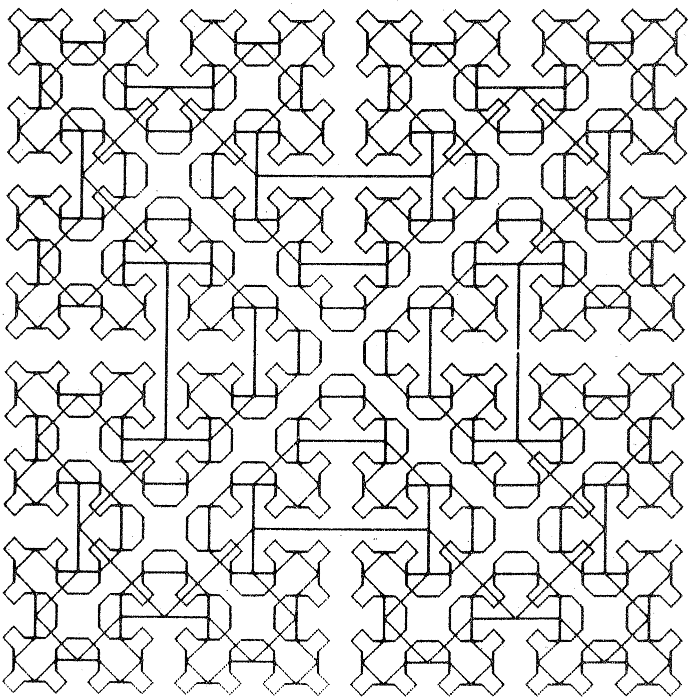

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Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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The Rubik's cube craze has given an interest in mathematics to many not normally so inclined. We thank Mr Leo Brewin for permission to print an excerpt from his forthcoming book. Because of the commercial implications, we draw attention to the fact that Mr Brewin, like all *Function* authors, holds copyright on his material.

Dr Carl Moppert writes on his eye-catching sundial, Dr L.M. Goldschlager on space-filling curves and their computer generation, and Mal Park, a mathematics graduate now studying law, looks at some wild stories in probability.

We thank Mike Morearty for the cartoon on p.14, and congratulate Richard Wilson on his bronze medal in the International Mathematical Olympiad (p.27).

THE FRONT COVER

Our front cover shows, superimposed, curves known as Sierpinski curves of orders 1-5. The background article is on pp.10-14.

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THE MONASH SUNDIAL

C.F. Moppert, Monash University

The Monash sundial is situated on the North wall of the Union building and allows us to read, on a sunny day, both the time and the date. The construction of such a sundial requires knowledge of aspects of astronomy and spherical trigonometry which are more fully explained in a longer (unpublished) paper of which this is a summary.

The Celestial Sphere

In the most favourable conditions (cloudless sky and at sea or on extensive level ground) the sky presents the appearance of a large hemisphere centred on the observer. The point of this hemisphere directly overhead is termed the *zenith*; the "rim" is the horizon. We adopt a geocentric viewpoint here, for simplicity, in which the observer is at rest and the sun and stars move relative to the earth and hence to the observer.

If we point a camera at night toward the sky and use an exposure of several minutes, the photograph will show that the stars move on circles across this hemisphere and all about a common centre. The point on the hemisphere corresponding to this centre is the *south celestial pole* (from the Latin *caelum*, the sky). The elevation of this point above the horizon is the latitude of the observer. (This angle is, by convention, counted as negative in the southern hemisphere.)

The fixed stars (as opposed to the sun, the moon and the planets) always follow the same circles (from east to west) - this is the origin of the term "fixed"; moreover they do so with constant angular velocity. Every fixed star performs a full circle in 23 hours 56 minutes and 4·091 seconds.

The apparent sphere on which these motions take place is called the *celestial sphere*. It is bisected by the *celestial equator*, a semicircle drawn on it by a star that rises due east of the observer and sets due west. Alternatively, point your outstretched left arm directly toward the south celestial pole and form a right angle between this and your right arm. Notice that you can do this in many ways - in fact, you can rotate your right arm about the left one. The right arm then lies always in a fixed plane, which is always that of the celestial equator.

Declination and Hour Angle

The sun's path through the sky is a complicated spiral, but, to an excellent approximation, we may regard its path *on any one day* as being a circle centred on the south celestial pole. Let the sun's position be denoted by P, and that of the south celestial pole by S_c. Then if O, the observer, points his left arm towards S_c and his right arm towards the sun (P) the angle so formed remains constant on any given day. It is 90° minus the constant angle between the sun and the celestial equator. (See Figure 1.) Its opposite, i.e. the *negative* angle $\angle S_cOP - 90°$ is called the *declination of the point* P in the sky. The declination is denoted by δ.

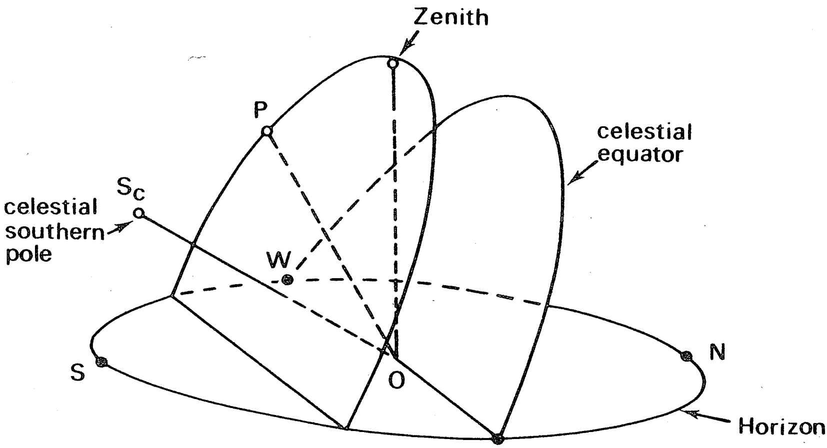


Figure 1

The declination of the sun varies over the course of a year between $-23\frac{1}{2}°$ and $+23\frac{1}{2}°$. Figure 2 shows the change of the declination of the sun during any one year. The sun's *declination* thus determines the *date*.

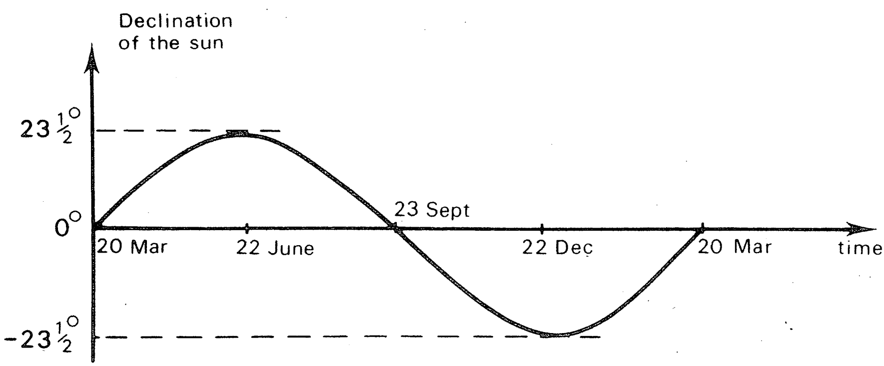


Figure 2

In order to fix the position of a point *P* in the sky, it is not enough to know the declination of its circle. We must also say where it lies on that circle.

To do this, imagine two planes. First, there is a plane determined by *O* (the observer), *Z* (his zenith), and S_c (the south celestial pole). The second plane contains *O*, S_c and *P*. These two planes form an angle τ (say), which may, however, be arrived at in another way. The first plane meets (see Figure 3) the plane of the celestial equator along a line *OZ'*, while the second meets it along a line *OP'*. τ is the angle *Z'OP'* because the lines concerned are perpendicular to OS_c, the line along which the planes intersect. δ, the declination, is the angle *POP'*, taken with a negative sign.

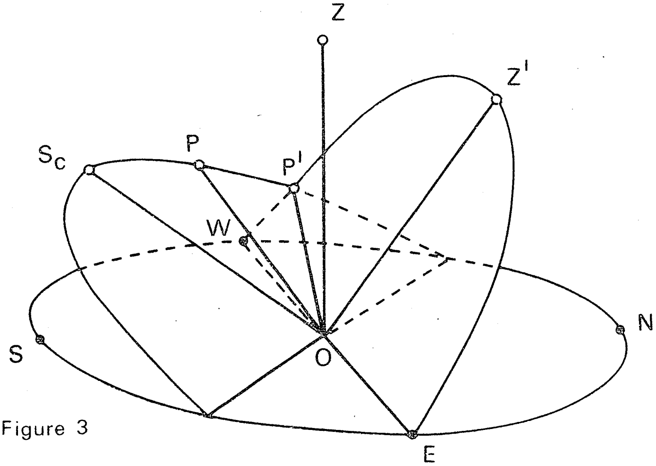


Figure 3

We shall call τ the *hour angle* of the point *P*. (Traditionally, not τ but $180° - \tau$ is the hour angle, but our definition is more appropriate to the southern hemisphere.) The *hour angle*, as its name implies, gives a measure of the *time of day*. The sun takes, on average, 24 hours to complete a full circle, but this varies slightly (up to 30 seconds between days).

The sundial aims to measure declination and hour angle and to translate these into readings of date and time.

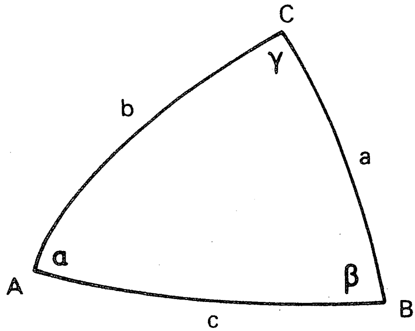
Spherical Trigonometry

There is, of course, no such thing as the celestial sphere. It is merely a convenient fiction. Nor need we, when we imagine it, place ourselves inside it. We can perfectly well imagine instead a scale model (such as is to be found in some museums) which we view from the *outside*. This has the advantage of placing us in a more familiar context, for we live on the outside of a sphere (the earth) and are all familiar with the globes used as scale models for this.

Now let A and B be two points on the sphere which are not diametrically opposite. The shortest distance between them on the sphere is the *great circle arc* joining them. On a globe this may be determined by stretching a string between A and B. The great circle arc always lies in a plane passing through the centre of the sphere. Planes and ships navigate along great circle arcs to economise on fuel[†].

Let us now fix three points A,B,C on the sphere and connect each pair by a great circle arc. The result is a *spherical triangle* (Figure 4). *Spherical trigonometry* studies spherical triangles much as ordinary (plane) trigonometry studies the more familiar plane triangles. One major difference lies in the units by which length is measured. Naturally distances along great circle arcs can be measured in linear units such as centimeters or kilometers, but it turns out to be more convenient to measure them in angular units.

Figure 4



Let A,B be two points on the sphere, and let O be its centre. Then $\angle AOB$ determines exactly the distance from A to B and we take the length AB to mean precisely this angle, i.e. the radius of the hemisphere is used as the unit of measurement. Thus in Figure 4, a,b,c are measured as angles at the centre O (not shown) of the sphere.

In what follows, we will use (without proof) some of the known formulae from spherical trigonometry.

The Conventional Horizontal Sundial

We begin by analysing the conventional horizontal sundial as found in parks and such places. Figure 5 shows the configuration. OG is the pointer or *gnomon* and it is directed toward the south celestial pole (i.e. it is parallel to the earth's axis). P is the shadow of the point G at the end of the gnomon and T is a point due south of O. Let the distance OG be taken as one (in some suitable units). Then make $OT = 1$ also and $OH = 1$ where H lies on the line OP. Then GTH is a spherical triangle (see Figure 6).

[†]See the article "Great Circle Navigation" in *Function*, Volume 4, Part 1.

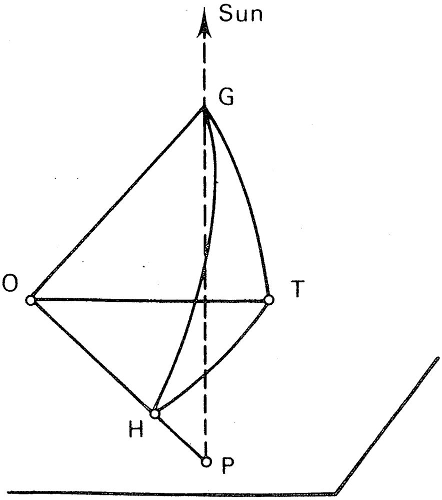


Figure 5

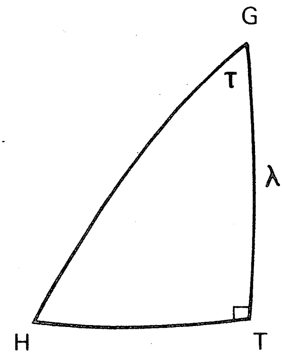


Figure 6

We readily see that $\angle GTH = 90°$, the angle between the planes GTO and HTO, and that $GT = \lambda$, the latitude. A little thought also shows that $\angle HGT = \tau$, the hour angle of the sun. Then two formulae from spherical trigonometry give the orientation of the shadow OP. We have

$$\tan HG = \tan \lambda / \cos \tau$$

$$\tan HT = \sin \lambda \tan \tau .$$

The length of the shadow is determined by plane trigonometry (see Figure 7). In the plane triangle OGP, $\angle OGP = 90° + \delta$, by the definition of δ, the declination, and by the sine rule and a few minor manipulations

$$OP = \cos \delta / \cos(\delta - HG).$$

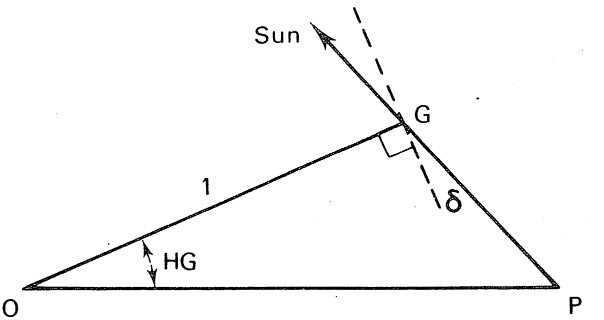


Figure 7

These three equations are, in principle, sufficient to give the position of the point *P*. In practice, more work is required: the declination δ must be related to the date (via Figure 2) and the hour angle to the standard time (via astronomical tables). (This latter correction is complicated and involves a term referred to as the "equation of time" which is responsible for the looped shape of the more "vertical" parts of the scale.) The calculations are not really difficult, however, and a programmable calculator makes them relatively easy.

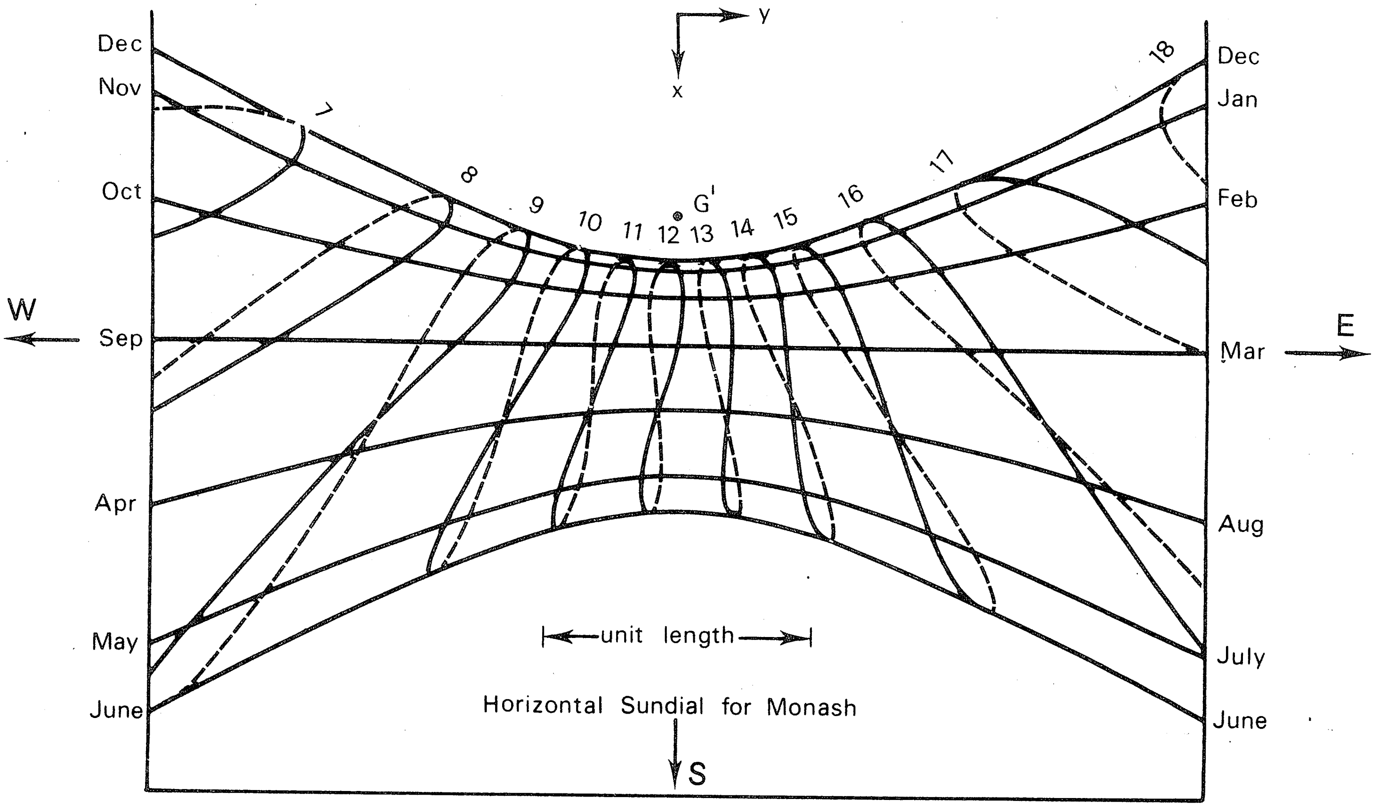
Figure 8 gives a scale model for such a sundial at Monash. The tip *G* of the gnomon is vertically above the point *G'* at a height of 0·6146 units. The plot of positions of the shadow allows us to read off time and date. Date is read by interpolating between the month curves. The shadow lies on the month curve on dates between the 20th or 23rd of the month (compare Figure 2). Time is read off by interpolating between the loops, whose fully drawn parts are used between June 22 and December 22 in any year, and whose partly-drawn parts are used the rest of the year.

The Monash Sundial

In a sundial such as we have been considering, it is the shadow of *G* that is important. The gnomon itself may be omitted, if *G* is in its correct place. Such a sundial works well; an article on this type of sundial appeared in *Scientific American*, December, 1980.

The Monash sundial, however, not only omits the gnomon but has an added complexity. It is mounted on a vertical wall, whose orientation must be accounted for.

To adjust for this, we may either take the pattern of Figure 8 and modify it by calculating the intersections of a plane representing the wall and the lines drawn between *G*, the tip of the gnomon, and the points making up the curves and loops, or we may calculate from different spherical and plane triangles directly for this more complicated case. The second course of action is actually the better one - the resulting pattern is that shown on the Monash sundial.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Horizontal Sundial for Monash
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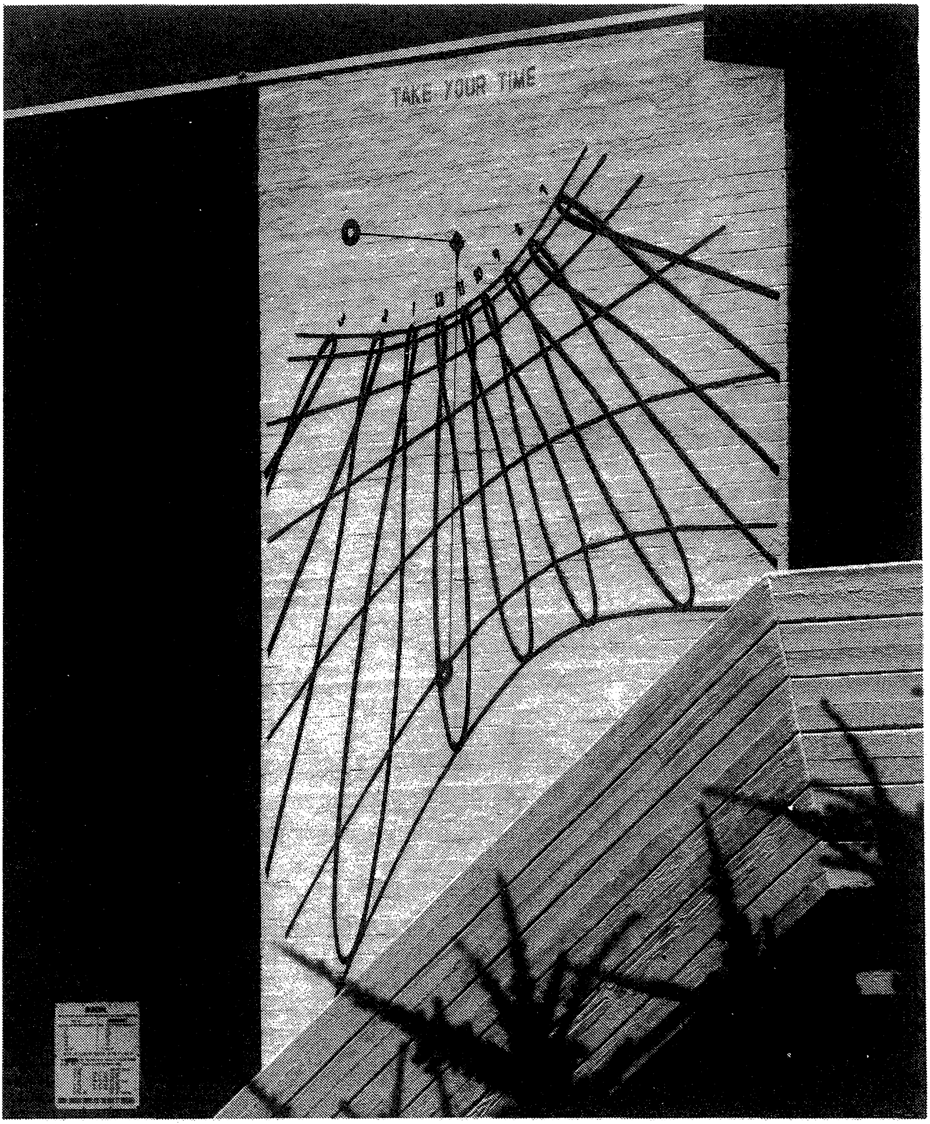
Figure 8



The Monash Sundial. This was designed by the author, Carl Moppert, with the help of Ben Laycock and Hugh Tranter. The shadow falls as shown twice a year: at 11.55 a.m. on November 23rd and at 12.25 p.m. on January 20th. The photograph is by R.L. Bryant, Physics Department, Monash University.

COMPUTER GENERATION OF SPACE FILLING CURVES

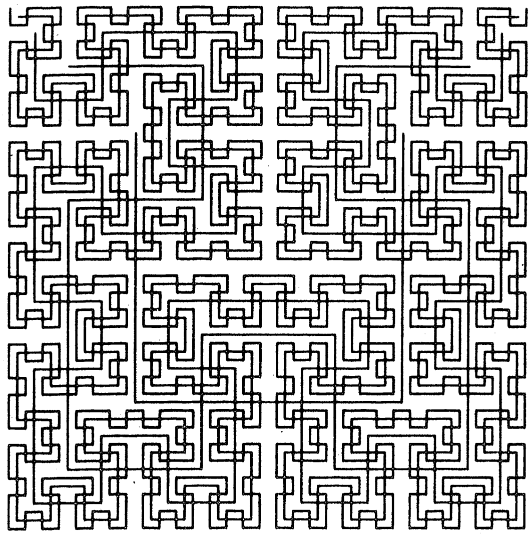
Leslie M. Goldschlager,

University of Queensland

A space-filling curve is a continuous curve, say in the unit square, which passes through every point in the square†. Examples have been given by Peano, Hilbert and Sierpinski among others.

The main interest in these curves to Computer Scientists has been recreational, as they produce pretty pictures on a plotter. Examples of some computer drawn curves appear in Figure 1 and on the cover. These figures show, respectively, Hilbert and Sierpinski curves of orders 1 to 5, superimposed. For higher orders these curves approach space-filling curves.

Figure 1



†For more on these, see *Function*, Vol.2, Part 3.

The first algorithms presented for drawing space-filling curves were two or more pages long, and they provided excellent examples of unstructured programs. More recently, Niklaus Wirth, the originator of the programming language PASCAL, has given more elegant solutions, about one page long using recursion (the ability of a procedure or subroutine to call itself). We will give algorithms of about a quarter the length and explain how these short algorithms can be used to draw these curves on a graphics screen. The algorithms will be presented in the PASCAL programming language, but a familiarity with PASCAL is not essential for generating these curves.

In order to draw the curves you can pretend that your graphics screen is a plotter whose pen can move in four directions (N,S,E,W). All that you need to do to generate the curves is to write a procedure MOVE which leaves a line on the screen as if the pen had moved. For example a call on procedure MOVE(S,W,h) causes the pen to move south-west, ending h units south and h units west of its initial position. MOVE (N,N,h) moves north a distance h. The language in which procedure MOVE is written will depend on your graphics equipment and computer. The main programs below will then generate the Hilbert and Sierpinski curves of order i.

HILBERT CURVES

```
program Hilbert Curves (input, output);
type direction = (N,S,E,W);
var  i: integer; h: real;
     procedure Hilbert (R,D,L,U: direction; i: integer);
          begin if i > 0 then
               begin Hilbert (D,R,U,L,i-1);
                        MOVE (R,R,h);
                     Hilbert (R,D,L,U,i-1);
                        MOVE (D,D,h);
                     Hilbert (R,D,L,U,i-1);
                        MOVE (L,L,h);
                     Hilbert (U,L,D,R,i-1);
               end
          end; (*Hilbert*)
begin (*main program*)
     (*initialize  i  and  h *)
          ...
     Hilbert (E,S,W,N,i)
end.
```

SIERPINSKI CURVES

```
program Sierpinski Curves (input, output);
type direction = (N,S,E,W);
var  i: integer; h: real;
     procedure Sierpinski (R,D,L,U: direction; i: integer);
          begin if i > 0 then
               begin Sierpinski (R,D,L,U,i-1);
                           MOVE (D,R,h);
                     Sierpinski (D,L,U,R,i-1);
                           MOVE (R,R,2*h);
                     Sierpinski (U,R,D,L,i-1);
                           MOVE (U,R,h);
                     Sierpinski (R,D,L,U,i-1)
               end
          end; (*Sierpinski*)
begin (*main program*)
     (*initialize  i  and  h *)
          ...
     Sierpinski (E,S,W,N,i); MOVE (S,E,h);
     Sierpinski (S,W,N,E,i); MOVE (S,W,h);
     Sierpinski (W,N,E,S,i); MOVE (N,W,h);
     Sierpinski (N,E,S,W,i); MOVE (N,E,h)
end.
```

These programs assume that the curve will be drawn starting from the top left-hand corner of the screen.

How do the algorithms work?

In order to illustrate how these algorithms work, let us consider the generation of the Sierpinski curves. Observe that if procedure Sierpinski is called with the parameter $i = 0$ then nothing happens. What happens when Sierpinski (R,D,L,U,i) is called with $i = 1$? The only statements within the body of procedure Sierpinski that are executed are the MOVE instructions because the recursive calls to procedure Sierpinski are made with $i - 1 = 0$. Execution of the MOVE instructions will generate Figure 2.

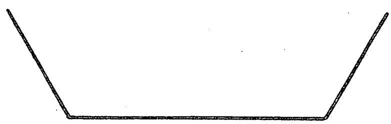


Figure 2

Let us consider the case i = 2. Now the recursive calls to Sierpinski will be made with i - 1 = 1 so that lines will be generated on the screen. The MOVE instructions will generate the dotted lines of Figure 3 and the recursive calls to Sierpinski will generate curves in the circled areas.

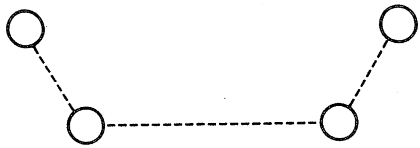


Figure 3

The first statement executed is Sierpinski (R,D,L,U,i-1) with i - 1 = 1. This is just the case described above and Figure 2 will be generated. The next statement executed is MOVE (D,R,h) and the dotted line of Figure 4 is drawn. The next call is to Sierpinski (D,L,U,R,i-1) with i - 1 = 1. The curve in Figure 2 will be generated but with a different orientation. After execution of this statement the curve of Figure 4 will be drawn and the curve after execution of the whole procedure is given in Figure 5.

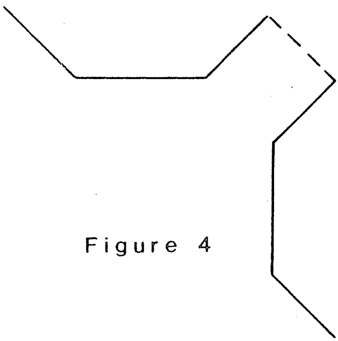


Figure 4

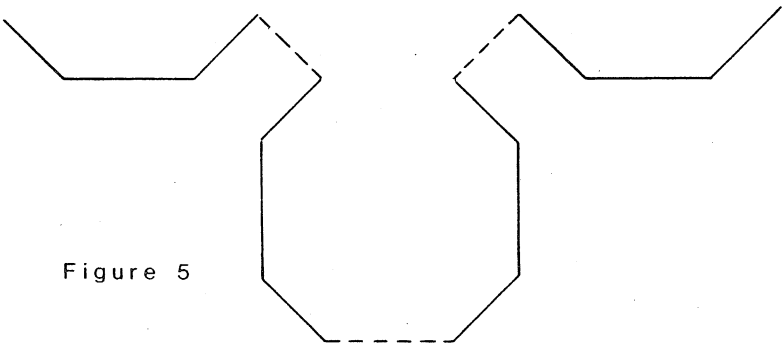


Figure 5

The main program of Sierpinski Curves with $i = 2$ will generate 4 copies of Figure 5 at different orientations and joined by the lines drawn by the 4 MOVE instructions. The cases $i > 2$ can be investigated in a similar fashion.

A challenge to BASIC programmers

If you prefer programming in the BASIC programming language, you might like to write code in BASIC for generating these space-filling curves. However, because BASIC lacks parameters and recursion an elegant solution such as the one given in this article will not be possible. In fact you might find that your BASIC program is long and unwieldy and this should cause you to reflect on the desirable properties of a high level programming language.

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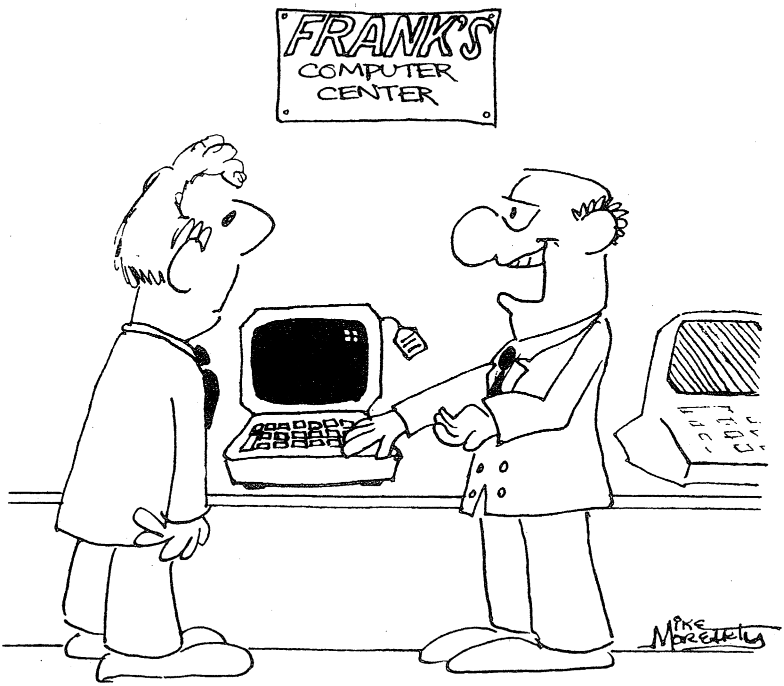

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AN ALGORITHM FOR RUBIK'S CUBE†

Leo Brewin
Monash University

Introduction and Notation

In this article I give an approach to solving the Rubik cube. To those readers who have already solved the cube this article will have little to offer, except perhaps in a comparison of different techniques. For all other readers the following algorithm will return your cube to its original glory, possibly for the first time since you purchased it!

The algorithm, as given here, is not as quick as it could be. There are some parts of the algorithm which could be replaced by more efficient techniques. However for the purposes of clarity and simplicity of presentation the number of different operations has been kept to a workable minimum.

Once you have mastered this solution you may like to modify it to include extra operations so as to streamline the solution. The fun you can have with the cube does not stop when you have the solution!

As we progress through the solution the operations will generally increase in complexity. There will come a point when it becomes quite unpractical to describe these operations in ordinary words. We w ll need to use a shorthand notation to convey the meaning of each operation.

A simple notation can be obtained by writing down the sequence of rotations of each individual face. For example: left + right would mean a quarter-rotation of the left face followed by a quarter-rotation of the right face. But in which direction should you rotate each face? By convention the rotation is in the clockwise sense to a person looking onto that face. Thus to indicate a clockwise quarter-rotation of the left face write: +left. While for a counter-clockwise quarter-rotation write: -left. We can simplify the notation even further: merely use the first letter of the name of the face. Thus for the faces UP, DOWN, LEFT, RIGHT, FRONT and BACK the shorthand notation for the rotations is U, D, L, R, F and B. On some occasions we may encounter the move U + U; this can simply be abbreviated as 2U. (Notice also that -U-U is equivalent to 2U.)

Typically then an operation on the cube will be represented by a string of letters, for example L-R+2F+R-L+2U (try this move now, and take note of what it does to each piece). These strings of letters are often referred to as 'words'.

It is most important to remember to read the string of operations from left to right and to also hold the cube in a fixed orientation throughout a move. If either of these conditions is violated then you will probably make a mess of the cube.

So we now have a technique for describing groups of rotations. Equally important is the description of the state of the cube, that is, where each piece is.

In any disordered state of the cube it is quite easy to see where each piece should go. This is because the centre pieces never move. So if we are prepared to hold the cube in a fixed orientation then we may describe the position of each piece relative to these unmoving centre pieces. For example if the piece at the top right-hand corner is adjacent to the UP, RIGHT and FRONT centre pieces, then this piece is said to be at URF. Now the three colours on this piece tell you where this piece should actually be, say UFL.

So we know that the piece presently at URF should really be at UFL. Using this notation we can describe the state of the cube. It is important to notice that URF is not the same as RFU. Even though the piece is still in the same location of the cube it does not have the same orientation. It has been twisted in its place.

We can apply a similar notation to the edge pieces - those pieces which have only two adjacent centre pieces. Thus an edge piece may be located at UR but should be positioned at UF, and so on.

When a piece is in its correct location of the cube, we will refer to the piece as being correctly located, if it is also correctly oriented then we shall say that it is correctly positioned. Thus the word 'location' will refer to a physical place in the cube whereas 'position' will also imply that the orientation is significant.

On occasion you will encounter the phrase 'the top face of the piece lies in the X face'. This is meant to imply that the side of the piece that would normally appear in the top face does in fact (presently) appear in the X face.

Throughout the solution we will treat the edge and corner pieces separately. Typically we will correctly position each edge piece by first locating and then orienting it. Once this is complete we will then position the corner pieces. The positioning of the edge and corner pieces will proceed in a layer by layer process.

1. *The First Layer.*

We start by reconstructing the top layer.

1.1. *The edge pieces.*

Choose any one of the (incorrectly positioned) edge pieces. This piece will be in either of the bottom or middle layers (if this is not the case then rotate the face that contains it).

First orient the whole cube such that the chosen piece is in the front face.

1.1.1. *The piece is in the bottom layer.*

(a) If the top face of the piece is on the bottom rotate the bottom layer so that a face of the piece matches one of the centre spots. Now turn this layer upside down. This should correctly position this piece.

(b) If however the top face is in the front rotate the bottom layer until the piece is directly below its correct position. By rotating this vertical layer the piece can be placed into the middle layer.

1.1.2. *The piece is in the middle layer.*

Rotate the whole cube so that the top face of the piece is not in the front face. Now ensure that the top face of the piece is on the side of the front layer. Now locate the correct position for this piece and rotate the top layer so as to bring this position to the front. You should now be able to rotate the front layer so as to place the piece into its correct place in the top layer. Finally return the top layer to its original position. This ensures that no other edge pieces will be displaced.

1.2. *The corner pieces.*

Choose any of the incorrectly located corner pieces. This piece will be in either of the top or bottom layers. Should the piece be in the top layer then we will first shift it to the bottom layer.

1.2.1. *The piece is in the top layer.*

Rotate the whole cube so as to place the chosen piece into the top right-hand position of the front layer. By rotating the right-hand layer place the piece into the bottom right position (i.e. directly below its original position). Now rotate the bottom layer one quarter of a turn in either direction. Finally rotate the right-hand layer back into its original position.

The chosen piece should now be in the bottom layer.

1.2.2. *The piece is in the bottom layer.*

Rotate the whole cube so as to place the correct position for the piece at the top right position of the front layer. Now by rotating the bottom layer you should be able to place the piece directly below this correct position.

If you are unlucky the top face of the piece will lie on the bottom. Should this occur you will need to shift this top face to one of the sides (i.e. twist the piece).

(a) If the top face of the piece is on the bottom then, by rotating the right-hand layer, place the piece into the back right-hand corner of the bottom layer. Now make one half turn of the bottom layer (i.e. turn it upside down). You can now return the right-hand layer to its original position. Finally rotate the bottom layer so as to place the cube directly below its correct position (i.e. its starting position).

The top face of the piece should now be on one of the sides.

(b) Take a look at the piece and determine in which of the layers the top face lies (the front or right layers). Rotate this vertical layer one quarter turn, keeping the piece in the bottom layer. Now rotate the bottom layer so as to place the piece back in its starting position; the piece should now match with one of its edge pieces. Now return the vertical layer to its correct position.

The chosen corner piece should now be correctly positioned. This procedure can now be applied to all the remaining corner pieces to complete the first layer.

2. *The Second Layer.*

Throughout this section we will always leave the first layer in the left-hand position. Although some of our moves may appear to destroy the regularity of this layer do not get worried. As each second layer piece is correctly positioned the first layer will return to its correct structure.

The construction of the second layer is the simplest part of the whole process, there being only four edge pieces to position.

Choose a piece which you wish to correctly position in the second layer. We will assume that this piece is in the right-hand layer. Unfortunately in some cases this will not be possible, the pieces will already be in the second layer (but incorrectly located). When this occurs you can flush out one of these pieces by shifting a third layer piece in its place as if it were a true second layer piece. Thus we may safely assume that the piece to be inserted is in the third layer.

Rotate the right-hand layer so that one face of the piece matches with one of the centre spots. Now rotate the whole cube so as to make this face the top face of the cube. The chosen piece, now at UR, needs to be shifted to either UF or UB.

2.1. *The UR to UF transition.*

(a) To displace the piece at UB use: 2U+2R+U+2R+2U.

(b) To preserve the piece at UB use: B+2U+R+U-R+2U-B.

2.2. *The UR to UB transition.*

(a) To displace the piece at UF use: 2U+2R-U+2R+2U.

(b) To preserve the piece at UF use: -F+2U-R-U+R+2U+F.

In each of the above transitions you could choose to use the type (b) move, however there are two advantages in using the (a) move. First, it requires only five primitive operations as opposed to seven and, second we can use this move to flush out some of the second layer pieces which are incorrectly positioned. This may save some time in the completion of this layer.

3. *The Third Layer.*

This layer is certainly the most laborious to reconstruct since each group of operations must not affect the previous two layers.

The technique which we are about to employ is similar to that used for the first layer. We will firstly orient and locate the edge pieces, this will then be followed by locating and orienting the corner pieces.

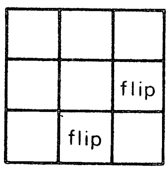
To start, hold the cube so that the third layer is pointing upwards.

3.1. *Flipping the edge pieces.*

At this point there should be zero, two or four edge-pieces with their top faces pointing upward (i.e. in the top face). If this is not the case then you have either made a mistake or your cube has been tampered with!

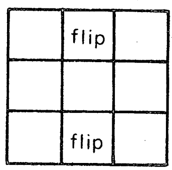
If all four edge pieces need flipping then first apply: F-R-F+R+U+R-U-R. This should flip two pieces. We are now in a position where only two pieces need flipping. There are two possibilities as indicated in Figure 3.1a and 3.2b. Now apply the operation associated with the figure that matches your situation.

All four edge pieces should now have their upper face pointing upwards.



F-R-F+R+U+R-U-R

Fig. 3.1a



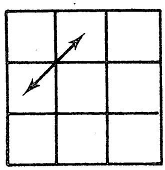
R+U-R-U-R+F+R-F

Fig. 3.1b

3.2. *Shunting the edge pieces.*

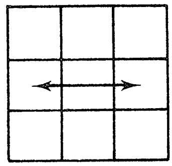
In this section we will cycle the edge pieces into their correct locations. By rotating the top layer you should be able to correctly position two of the four edge pieces. For the remaining two pieces refer to figures 3.2a and 3.2b to obtain the required operations.

The cross on the third layer should now be complete.



F+U-F+U+F+2U-F+U

Fig. 3.2a



F+U-F+U+F+2U-F-U+F+U-F+U+F
+2U-F

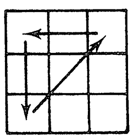
Fig. 3.2b

3.3. *Locating the corner pieces.*

In all that follows we will only need to use two distinct operators together with their inverses. These operations and their effect on the cube is indicated in the figures 3.3a and 3.3b. Since we will be making constant use of these moves let us introduce two new symbols: X and Y to represent them. With these two operators (and their inverses) we can cycle the corner cubes as in figure 3.3a and 3.3b. However this may not cover all possibilities.

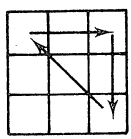
The two remaining cases are shown in figure 3.3c and 3.3d. The new symbol Q represents a quarter rotation of the whole cube, which looks as if the top face was rotated clockwise one quarter of a turn. This must be one of the simplest moves possible since all you need do is hold the cube differently!

With this collection of operators you should be able to locate correctly all of the corner pieces.



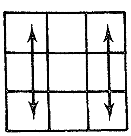
X=L+F-R-F-L+F+R-F

Fig. 3.3a



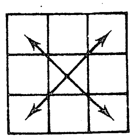
Y=-R-F+L+F+R-F-L+F

Fig. 3.3b



X-Q+X

Fig. 3.3c



X+Q+X

Fig. 3.3d

3.4. *Twisting the corner pieces.*

In this part of the solution we need only one basic operation: X+Q+Y-Q. Its effect on the cube is to twist the two rear pieces in opposite directions. The result is to shift what was the back face of each piece onto the top face of the cube. When making this move we will be attempting to orient correctly one or both of the corner pieces. However on occasions this will not succeed; we will need to follow this with exactly the same move on the same pair of pieces. In fact this double move can be accomplished in one move when applied to the same pair of pieces but from the other layer in which the pair lie. If you like you may try this move on the bottom layer, for there it is easy to see the effect the move has on the cube.

To return the bottom layer to its original state simply apply the move a total of three times.

The decisions we must make involve the choice of the pair of pieces to twist. We will need to distinguish three different cases depending on the number of correctly oriented pieces, this being zero, one or two.

In the instance when two pieces need twisting and they are diagonally opposite it will simplify the discussion if we disregard the fact that one of the other pieces is correctly located. This places this situation in the same case as when one piece is correctly positioned.

3.4.1. *Zero pieces correctly oriented.*

By inspecting the four corner pieces you should be able to locate a pair of pieces for which the operation X+Q+Y-Q (or twice X+Q+Y-Q) will twist them into their correct orientations. This will leave only two pieces which require twisting, so apply X+Q+Y-Q once again (or perhaps twice).

If you were careful in your choice for the first pair of pieces you should only need to apply X+Q+Y-Q once for each pair of corner pieces. You can make this choice by finding a pair of pieces which require their face at the back of the piece to be shifted to the top face.

3.4.2. *One piece correctly oriented.*

Our strategy here is to twist a pair of pieces so as to orient correctly one of the pieces adjacent to that piece already correctly oriented.

Select a pair of pieces not containing the correct piece. Twist these pieces by applying X+Q+Y-Q. If this does not correctly orient the adjacent piece, then apply X+Q+Y-Q once more. This should result in two pieces being correctly oriented. Then finally apply X+Q+Y-Q (once or twice) to the two remaining pieces.

3.4.3. *Two adjacent pieces require twisting.*

In this case you should only need to apply the basic move (X+Q+Y-Q) once to orient the two pieces. To do so you may need to redefine the top layer of the cube (i.e. rotate the whole cube in your hand so that the top faces of the two pieces are not visible from the top layer).

If all has gone well then you should now have a completely re-assembled cube. Now all you need to do is mess it up and start all over again!

RARE EVENTS AND SUBJECTIVE PROBABILITY

Mal Park, Monash University

People can assess probabilities in strange ways, often, but not always, at some loss to themselves.

Example 1. In June, 1950, an unidentified man walked up to a dice table at Las Vegas and won an amazing twenty-eight consecutive times at the game of craps. As the probability of winning once at this dice game is $\frac{976}{1980} = \cdot 49\dot{2}\dot{9}$, the probability of winning 28 times consecutively is $(\cdot 49\dot{2}\dot{9})^{28} = 2\cdot 5 \times 10^{-9}$. The probability of winning 28 times consecutively and then losing is $2\cdot 5 \times 10^{-9} \times (1 - \cdot 49\dot{2}\dot{9}) = 1\cdot 3 \times 10^{-9}$ as in the "geometric" distribution. Unfortunately for the man, he did not believe his luck, winning only $750 with modest bets. "Other gamblers, packed four deep around the board, were less conservative. Witnesses said Zeppo Marx raked in $25,000. Gus Greenbaum, owner of a rival club, walked out $48,000 to the good."

[From the New York Herald Tribune report, discussed in the book "Lady Luck, The Theory of Probability" by Warren Weaver.]

Example 2. In law, it has been suggested that some characteristic known to be possessed by a criminal should not be used as strong evidence against an accused person who also has that characteristic, unless it is "much more rare than a frequency of one in a thousand". Given this desired rarity, consider the following report from the Melbourne *Sun* (16 December, 1980) concerning a coroner's inquest at Alice Springs:

"According to the alleged statement, Mrs Chamberlain said: 'So there is blood on the rock with the same blood grouping as me.'

Sergeant Charlewood: 'A little more than just the same group as you.'

Mrs Chamberlain: 'Identical group.'

Sergeant Charlewood: 'Not identical, but coming from 14 per cent of the population.'"

Example 3. For another example consider a lottery recently conducted in Victoria designed to provide a total of \$1.5 m. in prize money including a \$1 m. first prize. Before the lottery can be drawn 100 000 tickets at \$25 each must be sold. Thus a total of \$2.5 m. was paid in by purchasers in return for a total payout of \$1.5 m. It would be more expedient, with less administrative overheads, to simply pay out a prize of \$15 to each purchaser of a \$25 ticket. In this way each purchaser would receive exactly the same amount as his "expectation" would be, in the randomly drawn lottery.

As far as the organizers of the lottery are concerned the result would be no different from that following from the lottery as it is presently conducted. The ticket purchasers, however, are hoping to gain more than their expectation. The first prize winner of \$1 m. has certainly done just that, but the great majority will be luckless participants whose tickets fail to secure a prize of any size. There is a one in 100 000 chance of being the lucky first prize winner.

Example 4. An experiment was reported by the psychologists Kahneman and Tversky in the journal *Science* in 1974. Given a population made up of 70 engineers and 30 lawyers, Kahneman and Tversky's subjects were required to estimate the probability that a person drawn from the population of 100 was an engineer rather than a lawyer. Where there was no further information provided the subjects assigned a 0·70 probability value to the proposition. The authors found nothing remarkable in this assessment which was nothing more than an application of the Principle of Indifference: there being no apparent reason why any one of the 100 persons should be more likely to be drawn than another, the subjects assigned a probability of 0·01 to each person with the conclusion that there was a 0·7 probability that the drawn person was an engineer.

Kahneman and Tversky went further. They provided a personality profile of each member of the group of 100 and, unexpectedly, the subjects tested returned probability assessments of the drawn member's vocation of about 0·5 for each, that is, approximate equal probabilities that the member was an engineer or that the member was a lawyer. That this was unexpected is explained by the fact that the personality profile was deliberately drawn so as to be uninformative and thus the subjects tested should be no more able to assess the probability values than they were in the initial tests when they returned assessments of 0·7 favouring engineering as the member's vocation.

Example 5. Fighter pilots in the Pacific during World War Two encountered situations requiring incendiary shells about 1/3 of the time and armour-piercing shells about 2/3 of the time. There was no general procedure for predicting on every mission which type of shell would be required. It was observed, however, that when left to their own devices, pilots armed themselves with incendiary and armour-piercing shells in the proportion of 1 to 2. Thus, the experienced fighter pilots acted much like the naive subjects in the psychological laboratory, even though their own lives were at stake.

Unfortunately further information is missing, but on a purely probabilistic analysis the pilots appear to be wrong. If a pilot was always armed with armour-piercing shells, he would be well prepared for 2/3 = 6/9 of the missions he flew, while a pilot who randomly chose his ammunition, subject only to the constraint that he arm himself with armour-piercing shells twice as often as incendiary shells, would be well prepared for only $\frac{1}{3} \times \frac{1}{3} + \frac{2}{3} \times \frac{2}{3} = \frac{5}{9}$ of his missions. Another way of evaluating these figures would be to suggest that, given that it was fatal for a pilot to find himself in a situation requiring ammunition different from that with which he was armed, sensible pilots who always took armour-piercing shells would average a 'life expectancy' of 3 missions, that is they would expect to fail to return from their third mission. The pilots who randomly decided on their ammunition would average a 'life expectancy' of only two and one-quarter missions.

Example 6. In *Function*, Volume 5, Part 3, there is an article by Doug Campbell on the reliability of a witness.

In that situation, a witness can be 80% reliable in general, but when he claims to have seen a rare event occurring, then with a proper use of Bayes' Theorem from Probability Theory his reliability turns out to be only 41% on that occasion.

We can formulate the problem in a different setting. Consider five urns, four of which contain 85 blue marbles and 15 green marbles while the fifth urn contains 85 green marbles and 15 blue marbles. Choosing an urn at random and from that urn, again at random, choosing a marble the probability of ultimately drawing a blue marble is higher than that of drawing a green marble: 0·71 as opposed to 0·29.

$$Pr(B) = \frac{4}{5} \times \frac{85}{100} + \frac{1}{5} \times \frac{15}{100} = 0{\cdot}71.$$

The problem can now be reversed. Given that we have chosen an urn at random and from that urn we have drawn, at random, a marble and found it to be green, what is the probability that the green marble so drawn came from one of the four urns containing only 15 green marbles as opposed to the probability that it was drawn from the fifth urn which contained 85 green marbles? Using probability theory it can be shown that there is a higher probability that the drawn green marble came from the fifth urn (0·586) and not from any of the first four urns (0·414).

If a witness said that the marble came from one of the first four urns, we might believe him with probability 0·414. However, in this context we could use hindsight. That is, after drawing a green marble we can tip out the remaining contends of the particular urn and count them. We would expect to count 14 green and 85 blue if the urn was one of the first four while if the urn was the fifth, we would count out 84 green and 15 blue marbles. Valuable as it is, hindsight is not available to the jury in the taxi accident case discussed by Doug Campbell, it not being possible to tip out the contents of the witness and, by counting his marbles, determine the reliability of his testimony!

ST. MARTIN †

St Martin was a bishop of Tours in France, who lived in the fourth century. According to legend, his charitable nature led him to divide his cloak with a beggar, while he was a young soldier of the Roman Empire.

Can we proceed from this to deduce the effect of sharing all one has? St Martin will pardon us this sick joke - but let us imagine for a moment that, moved by St Martin's example, the beggar had decided in his turn to share his goods with his benefactor. St Martin would then have divided his goods in half, the beggar then divided his, St Martin further divided his ... .

Let us mathematicise the situation: let S_0 be St Martin's initial wealth and B_0 the beggar's initial wealth. We examine the state of their fortunes after each transaction.

Initial Situation:

St Martin: S_0

Beggar: B_0.

Situation after St Martin has divided his goods:

St Martin: $\frac{1}{2}S_0$

Beggar: $B_0 + \frac{1}{2}S_0$.

Situation after the beggar has divided his goods:

St Martin: $\frac{1}{2}S_0 + \frac{1}{2}(B_0 + \frac{1}{2}S_0)$

Beggar: $\frac{1}{2}(B_0 + \frac{1}{2}S_0)$

Call these last amounts S_1 and B_1, to find:

$$S_1 = \frac{3}{4}S_0 + \frac{1}{2}B_0, \quad B_1 = \frac{1}{4}S_0 + \frac{1}{2}B_0.$$

This step may be written (and calculated) in matrix notation:

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† This article is a translation from the French. It first appeared in a Belgian counterpart of *Function*, the magazine *Math-Jeunes* Vol.3 (Part 10), May-June 1981, pp.78-80. The article is based on a suggestion of Claude Delmez and is reproduced here under an exchange agreement between *Function* and *Math-Jeunes*.

$$\begin{bmatrix} S_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} S_0 \\ B_0 \end{bmatrix}$$

The sum of the elements in either column of this matrix (a transition matrix) equals one, because the first gives the split-up between the two parties of St Martin's goods in the previous swaps, the second the split-up of the beggar's.

One can easily see that the same matrix also represents the change in the state of affairs at the next stage.

$$\begin{bmatrix} S_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} S_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} S_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} \frac{11}{16} & \frac{5}{8} \\ \frac{5}{16} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} S_0 \\ B_0 \end{bmatrix}$$

If we assume that the process leads to a stable limiting distribution of possessions, we may find it by solving the equation

$$\begin{bmatrix} S_{n+1} \\ B_{n+1} \end{bmatrix} = M \begin{bmatrix} S_n \\ B_n \end{bmatrix} \qquad \text{with} \quad M = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} ,$$

where we hope that $S_{n+1} = S_n$, $B_{n+1} = B_n$,

i.e.

$$\begin{bmatrix} S_n \\ B_n \end{bmatrix} = M \begin{bmatrix} S_n \\ B_n \end{bmatrix} ,$$

or, following the rules of matrix algebra,

$$M \begin{bmatrix} S_n \\ B_n \end{bmatrix} - \begin{bmatrix} S_n \\ B_n \end{bmatrix} = 0, \text{ i.e. } (M - I) \begin{bmatrix} S_n \\ B_n \end{bmatrix} = 0.$$

Notice that we are now faced with a system of two homogeneous equations in two unknowns. But the equations are not independent because the columns of the matrix $M - I$ sum to zero. We write the system as

$$\begin{bmatrix} \frac{3}{4} - 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} - 1 \end{bmatrix} \begin{bmatrix} S_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

The system is equivalent to

$$- \frac{1}{4} S_n + \frac{1}{2} B_n = 0.$$

That is to say

$$S_n = 2B_n .$$

After n steps (n being large), St Martin will have twice as much as the beggar has. At the start, between them they owned $S_0 + B_0$. At the end, $S_n = \frac{2}{3}(S_0 + B_0)$ and $B_n = \frac{1}{3}(S_0 + B_0)$.

We can sum up this problem by remarking that *whatever the original distribution*, if the exchanges are stopped (for large n) when the first giver has just received, he will receive two thirds of the total.

The beggar didn't know the subtleties of mathematics, and St Martin was truly a saint ... and the moral remains: in all your dealings, be the first to give ... .

[*We may remark that if the problem finishes after* $(n+\frac{1}{2})$*-steps,* n *being large, i.e. after the beggar has just received, the beggar receives approximately* 2/3 *of the total and St Martin* 1/3. *Perhaps the more realistic advice is not so much to be the first to give, as to be the last to receive*! *Eds.*]

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

BRONZED AUSSIE

Function, Vol.5, Part 3 contained an account of the first team ever from Australia to enter the International Mathematical Olympiad.

Now comes the news that one of the team, Richard Wilson, has won a bronze medal. Students from 27 countries took part in two $4\frac{1}{2}$-hour sessions held on the 13th and 14th of July.

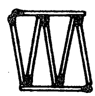
Richard Wilson is 17 years old and studies at The Kings School, Parramatta. Congratulations!

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

MATCH TRICK NO.5

At right, we show match trick no.5 from the series supplied to us by the Wilkinson Match Co. The solution is discussed on the inside back cover.

MATCH TRICK No. 5



It is easy enough to make four triangles out of seven matches as you see illustrated. But can you produce the same result in a different way?

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

PROBLEM SECTION

SOLUTION TO PROBLEM 5.2.2.

This problem read:

A farmer has 10 sheep, all of which are identical in their feeding characteristics, and 3 paddocks which are all equally good pasture in all respects. He puts 6 sheep in the first paddock, 3 in the second and one in the third. After 3 days, the 6 sheep in paddock 1 have eaten it out and he sells the 6 sheep; after 4 more days, the 3 sheep in paddock 2 have eaten it out and he sells them. When will the last sheep exhaust the pasture of paddock 3?

To solve it, let G be the amount of grass present in a paddock initially. (We measure amounts of grass in sheep-days; the amount one sheep eats in one day.) Suppose the rate of grass-growth per day is R. Then

$$G + 3R = 6 \times 3 = 18$$
$$G + 7R = 3 \times 7 = 21$$
$$G + nR = 1 \times n = n,$$

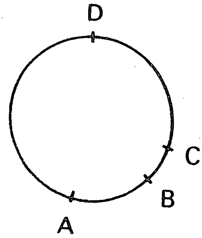
where n is the number of days the solitary sheep remains in the third paddock. Solving these equations gives $n = 63$.

This problem is based on one in Newton's *Arithmetica Universalis* first published in Latin in 1707. The English translation first appeared in 1720. The book is much more than a text of arithmetic. It contains a great deal of algebra, with the emphasis on the use of algebra in solving problems. The second half is concerned with the solution of geometrical problems by reducing them to equations.

This problem is based on Problem XI - pages 79, 80 of the 1728 English edition. Newton speaks of 'Oxen' and 'a piece of pasture' and expresses the area of the pasture in acres.

SOLUTION TO PROBLEM 5.2.3.

One circuit of a running track is 1300 metres. The track is to be marked at the least number of points which can be used as starting and/or finishing lines for races of any multiple of 100m. Where should the points be chosen?



The diagram at right shows how this is done. The distances are AB = 200m, BC = 100m, CD = 400m, DA = 600m. Then AC = 300m, BD = 500m, AD = 700m, DB = 800m, DC = 900m, CA = 1000m, BA = 1100m, CB = 1200m, the sense being anticlockwise in all cases. For distances greater than 1300m, incorporate first a sufficient number of complete laps.

SOLUTION TO PROBLEM 5.2.4.

Arrange the 52 cards of a pack into 13 tricks of 4 cards each so that:

(1) in each trick all cards belong to different suits;

(2) in each trick all cards are of different ranks;

(3) each pair of tricks has just one rank in common;

(4) given any two ranks, they occur together in just one trick;

(5) no trick contains more than one pair of cards with adjacent ranks.

This problem uses the above. Lay each of the 13 hearts (say) sequentially in the 100m intervals beginning with an ace in BC. Next lay the spades around, beginning with the 2 in BC (and ending king, ace in AB). Next take the diamonds and lay them out beginning with a 6 in BC and finally do the same with the clubs, stating with a queen in BC. It is readily seen that conditions (1), (2) and (5) are now met. That conditions (3), (4) are satisfied may be verified with work.

SOLUTION TO PROBLEM 5.3.1.

This problem concerned the so-called Armstrong Numbers.

An *Armstrong number of order* m is an m-digit number the mth powers of whose digits add up to the number itself. Thus 153 is an Armstrong number of order 3 as

$$1^3 + 5^3 + 3^3 = 153,$$

and each of the digits 1, 2, ..., 9 is an Armstrong number of order 1.

The problem was to prove:

(a) There are no Armstrong numbers of order 2;

(b) There are only finitely many Armstrong numbers.

(a) Let the digits be a,b. Then for an Armstrong number of order 2, we require

$$a^2 + b^2 = 10a + b$$

$$\text{i.e.} \quad a(10 - a) = b(b - 1).$$

Examination of the cases $a = 1,2,\ldots,9$ in turn shows that there is no solution, as the solution for b involves $\sqrt{1+4a(10-a)}$ which is non-integral for $a = 1,2,3,4,5$. The remaining four cases duplicate the cases $a = 4,3,2,1$ in order.

(b) The powers of the digits sum to give a left-hand side L_m. Clearly $L_m \leqslant m.9^m$. Let the number itself be R_m. As R_m has m digits, then $R_m \geqslant 10^{m-1}$. Thus no Armstrong numbers will exist if $10^{m-1} > m.9^m$, i.e., taking logs to base 10 of both sides and rearranging, if

$$1 + \log m < m(1 - \log m).$$

This inequality may readily be seen to hold for all m greater than 60. Thus all Armstrong numbers have orders less than 61 and, in consequence, only finitely many exist.

SOLUTION TO PROBLEM 5.3.2.

We asked if $2222^{5555} + 5555^{2222}$ was divisible by 7. It isn't. To prove this, note first that $2222 = 7 \times 317 + 3$. Thus in the expansion of $(2222)^{5555}$, all terms will be divisible by 7 except (possibly) 3^{5555}, i.e. $(243)^{1111}$. Put $243 = 7 \times 34 + 5$. So all terms in this expansion are divisible by 7, except possibly 5^{1111}, i.e. $(5^{11})^{101}$. But $5^{11} = 5 \times 25^5 = 5 \times (21 + 4)^5$. This is divisible by 7 if and only if 5×4^5 is divisible by 7, i.e. if 20×16^2 is. But 20×16^2 is divisible by 7 if and only if 20×2^2 is. This, after a little calculation, shows that we need only examine 3^{101}. But $3^{101} = 3 \times 9^{50} = 3 \times 2^{50}$ + a multiple of 7 = 12×8^6 + a multiple of 7 = 12 + a multiple of 7. Hence

$$2222^{5555} = 5 + \text{a multiple of } 7.$$

Similarly

$$5555^{2222} = 4 + \text{a multiple of } 7.$$

Thus, when $2222^{5555} + 5555^{2222}$ is divided by 7, the remainder is 2.

SOLUTION TO PROBLEM 5.3.3.

The problem was to prove that

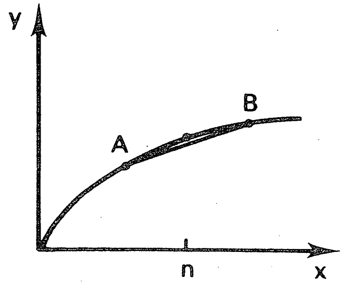
$$\sqrt[n]{n + \sqrt[n]{n}} + \sqrt[n]{n - \sqrt[n]{n}} < 2\sqrt[n]{n}.$$

The case $n = 2$ is readily solved by squaring. The proof for general n is graphical. The graph of $y = \sqrt[n]{x}$ has the general concave downwards shape shown in the diagram. The values of x given by

$$n - \sqrt[n]{n}, \quad n + \sqrt[n]{n}$$

are respectively to the left and right of the value $x = n$. Thus the average of their nth roots lies at the mid-point of the chord AB joining them. As this chord lies below the graph,

$$\tfrac{1}{2}\left(\sqrt[n]{n - \sqrt[n]{n}} + \sqrt[n]{n + \sqrt[n]{n}}\right) < \sqrt[n]{n},$$



and the result follows.

SOLUTION TO PROBLEM 5.3.4.

Let $S(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}$. The problem was to show that $S(n)$ cannot be integral if $n > 1$.

The problem is a difficult one in that the key to it is so simple that it is easily overlooked. Walter Vannini of the University of Melbourne produced this startlingly elementary proof.

To add up fractions, we express them all in terms of a common denominator, which is the lowest common multiple of all the separate denominators. If $n > 1$, this number is even and is $2^m.B$, where 2^m is the highest power of 2 less than or equal to n. (E.g. if $n = 11$, $m = 3$; if $n = 32$, $m = 5$, etc.) B is clearly odd.

Now each of the numbers $\frac{1}{k}$ $(k \leqslant n)$ will be expressed as $b_k/2^mB$, and b_k is even unless k contains a factor 2^m. There is only one such case, given by $k = 2^m$, since all other multiples of 2^m (by the definition of m) exceed n. Thus the numerators with one exception are all even, and hence their sum is odd. Thus $S(n)$ has an odd numerator and an even denominator and cannot be integral.

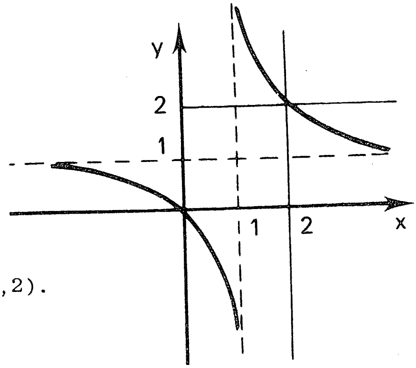
SOLUTION TO PROBLEM 5.3.5.

We asked for all integer pairs (x,y) whose sum and product are equal.

The problem may be tackled by the methods used to solve Ray Bence' football problem (Problem 3.5.1) - indeed it is rather simpler. We print, however, an alternative approach due to Ravi Sidhu of Ignatius Park College, Townsville, who attacked the problem graphically. He noted that we require

$$y = \frac{x}{x - 1},$$

and that this, when graphed, gives the shape at right. The left-hand branch lies almost entirely within a "corridor" between the axes and the asymptotes, and so cannot pass through integral values of both x, y except at (0,0), where it touches the edge of the corridor. By symmetry, the right-hand branch lies in a similar corridor, but passes through the point (2,2). These then give the required solutions.



We end, as usual, this section with a new batch of problems.

PROBLEM 5.5.1.

ABCD is a quadrilateral. Circles are drawn on each of *AB*, *BC*, *CD*, *DA* as diameter. Let *P* be any point in the interior of the quadrilateral. Show that *P* lies on or within at least one of the four circles.

5 PROBLEM 5.2.2.

To settle a point of honour, three men, *A*, *B* and *C* engage to fight a three-cornered pistol duel. *A*, a poor shot, has only a 30% chance of hitting his target; *C* is somewhat better, his chance of a hit being 50%; *B* never misses. *A* however has first shot. *B*, if he survives, fires next; then *C*; then *A* again, etc. However, if a man is shot, he takes no further part in the contest either as a marksman or a target. What should *A*'s strategy be?

[*Note: It is incorrect to speak of a "three-wheeled bicycle" as this is a contradiction in terms. It is, however, correct English to speak of a three-cornered duel, as the word "duel" is not derived from the Latin* duo *(two) but from* duellum *a variant of the Latin* bellum *(war). Eds.*]

PROBLEM 5.5.3.

Police Witness: Your worship, the defendant had to brake suddenly while travelling up a 30° slope. His skid marks measured 30m. I later tested the defendant's car on the level road outside the police station. Both roads are paved with the same material. I slammed on the brakes at 60 km/h and skidded to a halt in exactly 30m. Obviously he was travelling faster to require 30m to stop on the steep grade.

Magistrate (after consulting a pocket calculator): Case dismissed.

Explain the magistrate's reasons for his decision.

PROBLEM 5.4.3 (CLARIFIED)

This problem concerned the path of a billiard ball on a rectangular billiard table, the table being divided up into squares. Two readers misunderstood this problem. The diameter of the ball is equal to the side of each small square. This ensures that the ball travels as stated. Alternatively, forget about billiards and consider a bishop moving on a rectangular chessboard.

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GIRLS AND MATHS

In *Function*, Vol.5, Part 3, we announced a conference addressed to girls in years 7 to 12 at SCV Burwood. This was so successful that a follow-up is organised for October 24th. For more information, contact Dr Susie Groves at SCV Burwood.

We hope soon to bring our readers details of some of the sessions at these conferences.

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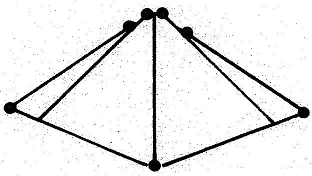
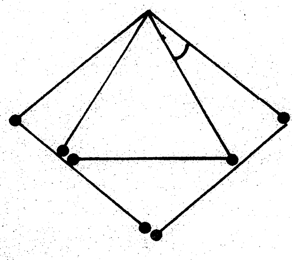
SOLUTION TO MATCH TRICK NO.5

We do not print the "official" solution, as in this one instance, it is unclear. There are very many solutions. Our top solution (at right) is due to Rainer Ignetik from the Caulfield Institute of Technology.



The remarkable one below this is due to Paul Nash of the Gippsland Institute of Advanced Education. The angle marked must be exactly 20° for this solution to work. Can you prove this?



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