Aequationes Math. 56 (1998) 201–215
0001-9054/98/030201-15 $ 1.50+0.20/0

Aequationes Mathematicae

Research Papers

Separating sets in interpolation and geometry

BURKARD POLSTER

Summary. We introduce some basic constructions for sets of functions which solve the Lagrange interpolation problem from two or more sets of functions having the same property and sharing a common 'separating set'. We also investigate similar constructions for sets of functions which solve the Hermite interpolation problem. These constructions translate into constructions for geometries on surfaces which generalize and extend the fundamental cut and paste constructions for topological geometries on surfaces such as flat affine, projective, Möbius, Laguerre and Minkowski planes.

Mathematics Subject Classification (1991). Primary 30E05, 41A05, 41A50, 51H10.

Keywords. Non-linear interpolation, unisolvent, Chebyshev system, generalized convexity, geometries on surfaces.

Introduction

A set of continuous functions over a fixed interval solves the Lagrange interpolation problem of order n if any n points in 'general position' over the interval are interpolated by exactly one function in the set. The set of polynomials of degree at most $n-1$ is an example of such a set and sets like this are some of the most fundamental objects in classical interpolation and approximation theory. At the same time, most types of topological geometries on surfaces have interpretations in terms of such sets. Flat affine planes like the Euclidean plane, for example, correspond to special sets of interpolating functions of order two over the reals.

A large number of ways to combine different interpolating sets into new interpolating sets and other interesting sets of functions have been investigated. One of the most natural approaches seems to be to try and merge interpolating sets over two adjacent intervals into a special set over the union of both intervals. Start with two interpolating sets of order n and try to glue them together like this such that the resulting set is special in the sense that it is also an interpolating set of order n. It turns out that this is not always possible and that, in the cases where

This research was supported by the Australian Research Council.

it is possible, great care has to be taken when we pair up the functions in the two sets (see [KZ]). Similarly, the method for constructing spline spaces that are widely used in approximation theory is based on the same gluing principle (see [Sc]). As before the constructions of these special kinds of sets involve a lot of 'glue', that is, are rather complicated.

In this paper we deal with gluing constructions which require no glue. These constructions are based on separating sets of functions. The most basic kind of a separating set in an interpolating set cuts the interpolating set into an 'upper' and a 'lower' part. Given two interpolating sets which share the same separating set, the union of the separating set, the lower part of the first and the upper part of the second interpolating set is also an interpolating set.

These gluing constructions translate into gluing constructions of the geometries on surfaces associated with interpolating sets and generalize some of the constructions considered in [PS1] and [PS2]. These gluing constructions are some of the most powerful tools for constructing geometries on surfaces and are especially important when it comes to constructing rigid geometries, that is, geometries which do not admit any automorphisms.

This paper is organized as follows. In Section 1 we collect some results about sets of functions which solve the Lagrange interpolation problem and functions which are convex with respect to such sets. In Section 2 we show that in such a set, subsets which solve that Lagrange interpolation problem of order $n-1$, as well as the subsets of functions interpolating a fixed point, are separating sets. In Section 3 we discover that the separating sets in Section 2 have counterparts in sets of functions that solve the Hermite interpolation problem. Section 4 deals with separating sets in sets of periodic and half-periodic functions which solve the Lagrange or Hermite interpolation problems. In a final section we investigate some more complicated separating sets in sets of functions which solve the Lagrange interpolation problem of order 3.

1. N-unisolvent sets of functions and convexity

Let $n \geq 1$ be an integer, let $I \subset \mathbb{R}$ be an interval, and let F be a set of continuous functions $I \to \mathbb{R}$. Then F is called an *n-unisolvent set (of functions on I)* if for any set of distinct points $x_1, x_2, \ldots, x_n \in I$ and any set of n real numbers $y_1, y_2, \ldots, y_n$ there is a uniquely determined $f \in F$ such that

$$f(x_i) = y_i, \ i = 1, \ldots, n.$$

Note that two distinct elements of F cannot be equal at more than $n-1$ distinct points. We will also say that f *interpolates* the points $(x_i, y_i) \in I \times \mathbb{R}, i = 1, \ldots, n$. For a set of functions to be an n-unisolvent set just means that it solves the Lagrange interpolation problem of order n.

Following Moldovan [Mo1] and Kemperman [Ke], the above unique function f will be denoted as

$$F(x_1, x_2, \dots, x_n; y_1, y_2, \dots, y_n|x).$$

We further write

$$F(x_1, x_2, \dots, x_n; g|x) = F(x_1, x_2, \dots, x_n, g(x_1), g(x_2), \dots, g(x_n)|x),$$

where g is any function $I \to \mathbb{R}$.

Simple examples of n-unisolvent sets are the set of all polynomials of degree at most $n-1$ on any interval, the linear span of the set $\{1, \sin x, \sin 2x, \sin 3x, \dots, \sin(n-1)x\}$ on the half-open interval $[0, \pi)$, and, in general, the linear span of any Chebyshev system $\{u_1(x), u_2(x), \dots, u_n(x)\}$ of continuous functions $u_i : I \to \mathbb{R}$, $i = 1, 2, \dots, n$. A well-known example of a non-linear 2-unisolvent set on $\mathbb{R}$ is the set that contains all linear functions $ax + b$, $a, b \in \mathbb{R}$, $a \leq 0$ and the functions $e^{c+x} + d$, $c, d \in \mathbb{R}$ [To].

In everything that follows, all intervals of $\mathbb{R}$ we will be dealing with are supposed to be open (unless otherwise specified).

A function $g : I \to \mathbb{R}$ is said to be *convex (concave) relative to F* if

$$(-1)^{n-i}(g(x) - F(x_1, x_2, \dots, x_n; g|x)) \geq 0 (\leq 0) \text{ for } x_i < x < x_{i+1}, i = 0, 1, \dots, n$$

whenever the $n+2$ points $x_i \in I$ are chosen such that $x_0 < x_1 < \dots < x_{n+1}$. The function g is *strictly convex (concave) relative to F* if all the above inequalities are strict.

In the case where $n = 2$ and the 2-unisolvent system under discussion is the set of all linear functions over any interval, a function is (strictly) convex in our sense if and only if it is (strictly) convex in the usual sense. Our notion of convexity coincides with some of the generalized notions of convexity investigated in the papers by Kemperman, Moldovan, Tornheim and Umamaheswaram listed in the References.

Let J be a subinterval of I, and let g be a function defined on I. Then g_J will denote the restriction of g to J and F_J will denote the set of all restrictions of functions in F to J. Clearly, if g is (strictly) convex or concave relative to F, then g_J is (strictly) convex or concave relative to F_J (F_J is, of course, an n-unisolvent set on J.)

The following first result is an immediate consequence of the definition of convexity (see also [Mo2, Lemme 3]).

Proposition 1 (Transitivity of convexity). *Let H_1 and H_2 be be two n-unisolvent sets on I.*

If all elements of H_1 are convex with respect to H_2 and $f : I \to \mathbb{R}$ is a continuous function that is (strictly) convex with respect to H_1, then f is (strictly) convex with respect to H_2.

If all elements of H_1 are concave with respect to H_2 and $f : I \to \mathbb{R}$ is a continuous function that is (strictly) concave with respect to H_1, then f is (strictly) concave with respect to H_2.

In the following we make frequent use of the following lemmas which are due to Tornheim [To, Theorems 3 and 5] (see also [Ke, 3.1, 3.2]).

Lemma 1. *Let F be an n-unisolvent set on the interval I and let $f, g \in F$ be distinct. If f and g are equal in $n-1$ distinct points of I, then the function $f - g$ has exactly $n-1$ zeros and changes signs at every single one of these zeros.*

Lemma 2. *Let F be an n-unisolvent set. Then $F(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n|x)$ is jointly continuous in the $2n+1$ variables $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, x$.*

Here the domain of $F(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n|x)$, considered as a function in $2n+1$ variables, may be defined by $x_i \in I$, $x_1 < x_2 < \ldots < x_n$, and $y_i \in \mathbb{R}$. Using these lemmas it is no problem to prove

Proposition 2. *Let F be an n-unisolvent set, $n \geq 2$, on the interval I and let H be an $(n-1)$-unisolvent subset of F. Then every function $g \in F \setminus H$ is either strictly convex or strictly concave relative to H.*

We have to remark that this proposition is also an easy consequence of results in [Mo1].

Proof. As a consequence of Lemma 2, the function

$$g(x) - H(x_1, x_2, \ldots, x_{n-1}; g|x)$$

is jointly continuous in the n variables $x_1, x_2, \ldots, x_{n-1}, x$. Futhermore, by Lemma 1, as a function of x alone this function has exactly $n-1$ zeros (the x_i's) and changes signs at every single one of these zeros. Clearly, as we continuously vary the points $x_1, x_2, \ldots, x_{n-1}, x_n$ such that at all times we have $x_1 < x_2 < \ldots < x_{n-1} < x_n$ the sign of

$$g(x_n) - H(x_1, x_2, \ldots, x_{n-1}; g|x_n)$$

will not change. Hence, as a consequence of Lemma 1, if this sign is positive (negative), then g is strictly convex (concave) with respect to H. □

In the following, whenever we are dealing with a situation like this let $F^+(H)$ ($F^-(H)$) denote the set of all strictly convex (concave) functions relative to H in $F \setminus H$.

Corollary 1. *Let F be an n-unisolvent set, $n \geq 2$, on the interval I and let H be an $(n-1)$-unisolvent subset of F. Then F is the disjoint union of the sets H, $F^+(H)$, and $F^-(H)$.*

We say that two n-unisolvent sets F_1, F_2 defined on intervals I_1, I_2, respectively, are *topologically equivalent* if there exists a homeomorphism $I_1 \times \mathbb{R} \to I_2 \times \mathbb{R}$ that maps the set of vertical lines in $I_1 \times \mathbb{R}$ to the set of vertical lines in $I_2 \times \mathbb{R}$ and that maps the graphs of the functions in F_1 to the graphs of the functions in F_2. It is easy to prove that topological equivalence is an equivalence relation. Clearly, two 1-unisolvent sets F_1 and F_2 are topologically equivalent since I_1 and I_2 are both supposed to be open. We remark that our example of a non-linear 2-unisolvent set on $\mathbb{R}$ (above) is not topologically equivalent to the set of linear functions on any interval.

N-unisolvent sets, or even more general, varisolvent sets form the topological foundation for classical interpolation and approximation theory [Ri1], [Ri2]. They can also be interpreted in terms of geometries on surfaces: the point-line geometry, or *incidence geometry*, associated with a set of functions over the interval I has point set $I \times \mathbb{R}$ and a line set that consists of all graphs of functions in the set. The 2-unisolvent sets and the 3-unisolvent sets are of particular interest to geometers, as point-line geometries such as flat affine planes [Sa] and Laguerre planes [St], respectively, can be represented in terms of such sets. The point-line geometry associated with the linear functions, for example, is the Euclidean plane (minus the verticals), that is, the classical example for a flat affine plane. Many 2- and 3-unisolvent sets that are essentially non-linear, that is, not topologically equivalent to 2- and 3-unisolvent sets that arise from Chebyshev systems have been constructed and classified by topological geometers. Incidence geometries corresponding to n-unisolvent sets with $n > 3$ are investigated in [Po3] (see also [HK]).

Most of the results in this paper deal with combining two or more n-unisolvent sets into new n-unisolvent sets. All these results can also be stated in terms of combining geometries on surfaces into new such geometries.

For more information about n-unisolvent sets of functions the reader is referred to [Cu], [Mor], [Po1], [Ri1], [Ri2] and [To]. See [KS], [Sc] and [Zi] for information about Chebyshev systems.

2. Separating sets in n-unisolvent sets

Given n distinct fixed points $x_1 < x_2 < \ldots < x_n$ in the interval I, it is possible to identify an n-unisolvent set F on I with $\mathbb{R}^n$ by mapping the function $f \in F$ to the point $(f(x_1), f(x_2), \ldots, f(x_n)) \in \mathbb{R}^n$. Under this identification an $(n-1)$-unisolvent subset H of F corresponds to a closed subset of $\mathbb{R}^n$ homeomorphic to $\mathbb{R}^{n-1}$ that separates $\mathbb{R}^n$ into two open components. These two open components correspond to $F^+(H)$ and $F^-(H)$. This separating property of H goes even deeper as we can see from the following

Theorem 1. *Let F_1 and F_2 be n-unisolvent sets, $n \geq 2$, on the interval I and let H be a common $(n-1)$-unisolvent subset of F_1 and F_2. Then both $H \cup F_1^+(H) \cup F_2^-(H)$ and $H \cup F_1^-(H) \cup F_2^+(H)$ are n-unisolvent sets.*

Proof. As an immediate consequence of Lemma 1, we know that for any choice of distinct points $x_1, x_2, \ldots, x_n \in I$ with $x_1 < x_2 < \ldots < x_n$ and real numbers $y_1, y_2, \ldots, y_n$ the sign of

$$y_n - H(x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_{n-1}|x_n)$$

alone determines whether $F_i(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n|x)$ is strictly convex with respect to H, belongs to H or is strictly concave with respect to H. □

Example 1. Let $F_i = \text{span}\{1, x, x^2, \ldots, x^{n-2}, f_i(x)\}$, $i = 1, 2$ such that $f_i : \mathbb{R} \to \mathbb{R}$ is $n-1$ times differentiable and its $n-1$st derivative is a positive function. Then F_i is an n-unisolvent set on $\mathbb{R}$ and $H = \text{span}\{1, x, x^2, \ldots, x^{n-2}\}$ is an $(n-1)$-unisolvent set contained in both F_1 and F_2. Using Theorem 1, we can combine these two sets into an n-unisolvent set that is no longer linear. See [LP] for further information about the case $n = 3$.

We are interested in constructing $(n-1)$-unisolvent subsets of an n-unisolvent set F and other subsets of F that separate F in a similar way. A *separating set* like this can then be used to construct a new n-unisolvent set from two given n-unisolvent sets that share this subset, as demonstrated above.

Let F be an n-unisolvent set on I, and let $p = (s, t)$ be a point in $I \times \mathbb{R}$. Let $F(p)$, $F^+(p)$, $F^-(p)$ denote the set of all functions $f \in F$ such that $t = f(s)$, $t < f(s)$ and $t > f(s)$, respectively.

The following obvious result gives one way of finding $(n-1)$-unisolvent subsets of n-unisolvent sets.

Proposition 3. *Let F be an n-unisolvent set, $n \geq 2$, on I, let J be a proper subinterval of I, and let $p = (s, t) \in (I \setminus J) \times \mathbb{R}$. Then $F(p)_J$ is an $(n-1)$-unisolvent subset of F_J. Furthermore, if s is greater than all elements of I, then $F_J^+(F(p)_J) = F^+(p)_J$, and $F_J^-(F(p)_J) = F^-(p)_J$. If s is less than all elements of I and n is odd (even), then $F_J^+(F(p)_J) = F^+(p)_J$ $(= F^-(p)_J)$ and $F_J^-(F(p)_J) = F^-(p)_J$ $(= F^+(p)_J)$.*

So, given an n-unisolvent set, we first try to embed it in a larger n-unisolvent set and then concentrate on the functions that interpolate a point $p \in (I \setminus J) \times \mathbb{R}$. This will provide us with a separating set. It is not known whether all n-unisolvent sets can be embedded into larger n-unisolvent sets in this way. Examples of n-

unisolvent sets on half-open intervals that cannot be extended are known for any $n \geq 2$ (see [Pol, 2.8.1]).

What if we concentrate on points $p \in J \times \mathbb{R}$, that is, what happens if we move the point from the 'outside' $(I \setminus J) \times \mathbb{R}$ to the 'inside' $J \times \mathbb{R}$?

Theorem 2. *Let F_1 and F_2 be n-unisolvent sets, $n \geq 2$, on the interval I, and let $p = (s,t) \in I \times \mathbb{R}$. Furthermore, let $F_1(p) = F_2(p)$. Then $F_1(p) \cup F_1^+(p) \cup F_2^-(p)$ and $F_1(p) \cup F_2^+(p) \cup F_1^-(p)$ are n-unisolvent sets.*

Proof. Let $x_0, x_1, x_2, \dots, x_{n+1}$ be $n+2$ distinct points in I such that $x_0 < x_1 < \dots < x_{n+1}$ and such that $x_0 < s < x_{n+1}$. We have to show that for any n-unisolvent set F on I with $F(p) = F_1(p) = F_2(p)$ we can tell by just looking at $F(p)$ what the sign of $t - F(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n|s)$ is going to be. If s is one of the x_i's, this is certainly the case. We may therefore assume that $x_j < s < x_{j+1}$ for some j. Let $f(x) = F(s, x_2, x_3, \dots, x_n, t, y_2, y_3, \dots, y_n|x)$ if $j \leq 1$, and let $f(x) = F(x_2, \dots, x_j, s, x_{j+1}, \dots, x_n, y_2, \dots, y_j, t, y_{j+1}, \dots, y_n|x)$ if $j > 1$; hence $f \in F(p)$. Furthermore, let sig be the sign of $y_1 - f(x_1)$. If $sig = 0$, then $F(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n|x) = f(x)$ belongs to $F(p)$. We may therefore assume that $sig \neq 0$. By Lemma 1, the function $F(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n|x) - f(x)$ has exactly $n-1$ zeros $(x_2, x_3, \dots, x_n)$ and changes signs at every single one of these zeros. Therefore, we know that if $j = 0$, then $F(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n|x)$ is contained in $F^+(p)$, $F^-(p)$ if and only if $sig = 1$, or $sig = -1$, respectively. If $j \geq 1$, then $F(x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n|x)$ is contained in $F^+(p)$, $F^-(p)$ if and only if $sig(-1)^{j-1} = 1$, or $sig(-1)^{j-1} = -1$, respectively. □

Corollary 2. *Let F_1 and F_2 be n-unisolvent sets, $n \geq 2$, on the interval I, and let $p_1 = (s, t_1)$ and $p_2 = (s, t_2)$ be two distinct points in $I \times \mathbb{R}$ such that p_1 lies above p_2, that is, $t_1 > t_2$. Furthermore, let $F_i(p_1, p_2)$ be the set of all $f \in F_i$ such that $t_2 \leq f(s) \leq t_1$. If $F_1(p_1,p_2) = F_2(p_1,p_2)$, then $F_1^+(p_1) \cup F_1(p_1,p_2) \cup F_2^-(p_2)$ is an n-unisolvent set.*

Of course, this result does not tell us anything really new since, in this case, $F_1^+(p_1) \cup F_1(p_1,p_2) \cup F_2^-(p_2) = F_1^+(p_2) \cup F_1(p_2) \cup F_2^-(p_2)$. Still, it shows that separating sets can be much "thicker" than the separating sets considered in Theorems 1 and 2; the set $F_1(p_1,p_2)$ "separates" the n-unisolvent set F_1 like $F_1(p_2)$. This suggests

Corollary 3. *Let F_1 and F_2 be n-unisolvent sets, $n \geq 2$, on the interval I and let H and K be $(n-1)$-unisolvent common subsets of both F_1 and F_2 such that all elements of H are strictly convex with respect to K. Furthermore, let $[H,K]_i$ denote the set of all $f \in F_i$ such that f is convex with respect to K and concave with respect to H. If $[H,K]_1 = [H,K]_2$, then $F_1^+(H) \cup [H,K]_1 \cup F_2^-(K)$ is an n-unisolvent set.*

Again this result does not say anything new since, as a consequence of Proposition 1, we find that $F_1^+(H) \cup [H,K]_1 \cup F_2^-(K) = F_1^+(K) \cup K \cup F_2^-(K)$.

3. Separating sets in unrestricted n-unisolvent sets

In this section we show that if we substitute the word 'n-unisolvent' in Theorems 1 and 2 by 'unrestricted n-unisolvent', then both theorems are still true.

Let I be an interval. A SIC (set of initial conditions) S of order $m \in \mathbb{N}$ on I is an ordered triple (X, Λ, Y) where $X = \{x_1, x_2, \ldots, x_k\}$ is a set of distinct points in I, $\Lambda = \{\lambda_{x_1}, \lambda_{x_2}, \ldots, \lambda_{x_k}\}$ is a set of positive integers such that $\lambda_{x_1} + \lambda_{x_2} + \ldots + \lambda_{x_k} = m$ and $Y = \{Y_{x_1}, Y_{x_2}, \ldots, Y_{x_k}\}$ where $Y_{x_i} = \{y_{x_i}^{(0)}, y_{x_i}^{(1)}, \ldots, y_{x_i}^{(\lambda_{x_i}-1)}\}$, $i = 1, 2, \ldots, k$ is a set of real numbers. We abbreviate all this by writing $S = \{x_1, \ldots, x_k | \lambda_{x_1}, \ldots, \lambda_{x_k} | y_{x_1}^{(0)}, \ldots, y_{x_k}^{(\lambda_{x_k}-1)}\}$. A function $f : I \to \mathbb{R}$ *satisfies* S if

$$f^{(j)}(x_i) = y_{x_i}^{(j)}, \qquad i = 1, 2, \ldots, k, \qquad j = 0, 1, \ldots, \lambda_{x_i} - 1.$$

Here $f^{(0)} = f$ and $f^{(j)}$, $j \geq 1$ denotes the jth derivative of f.

Following Hartman [Ha], we call a set F of $n-1$ times continuously differentiable functions on I an *unrestricted n-unisolvent set* if every SIC of order n on I is satisfied by exactly one $f \in F$. Note that an unrestricted n-unisolvent set is automatically an n-unisolvent set since a function satisfying a SIC of the form $\{x_1, \ldots, x_n | 1, 1, \ldots, 1 | y_{x_1}^{(0)}, y_{x_2}^{(0)}, \ldots, y_{x_n}^{(0)}\}$ interpolates the n points $(x_1, y_{x_1}^{(0)})$, $(x_2, y_{x_2}^{(0)}), \ldots, (x_n, y_{x_n}^{(0)})$.

The geometries on surfaces associated with unrestricted n-unisolvent sets form an important class of nested orthogonal arrays (see [Pol]). For F to be an unrestricted n-unisolvent set just means that it solves the Hermite interpolation problem of order n [LJR].

The Chebyshev systems that give rise to unrestricted n-unisolvent sets of functions are called *extended Chebyshev systems* [KS]. The set $\{1, x, x^2, \ldots, x^{n-1}\}$ is the classical example for such a Chebyshev system.

A set G of $n-1$ times continuously differentiable functions on the interval I is said to have *the property of unique n initial values* if all SICs of the form $\{x_1 | n | y_{x_1}^{(0)}, y_{x_1}^{(1)}, \ldots, y_{x_1}^{(n-1)}\}$ on I are satisfied by exactly one $f \in G$.

The following important result is due to Hartman [Ha]

Lemma 3. *A set of $n-1$ times continuously differentiable functions on an open interval is an unrestricted n-unisolvent set if and only if it is an n-unisolvent set and it has the property of unique n initial values.*

We need one more

Lemma 4. *Let F be an unrestricted n-unisolvent set, $n \geq 2$ on I, let H be an unrestricted $(n-1)$-unisolvent subset of F, and let K be the set of all functions in F that satisfy a given SIC $\{x_1|n|y_{x_1}^{(0)}, y_{x_1}^{(1)}, \dots, y_{x_1}^{(n-2)}\}$ of order $n-1$ on I. Then K has the following properties:*

1. *Every point $(s,t) \in I \times \mathbb{R}$, $s \neq x_1$ is interpolated by exactly one element in K.*
2. *There is exactly one element $h \in H$ that is contained in K.*
3. *A function $f \in K \setminus \{h\}$ is contained in $F^+(H)$ $(F^-(H))$ if and only if $f(x) - h(x)$ is positive (negative) for any (and therefore all) $x > x_1$.*
4. *If $f \in F^+(H) \cap K$ and $x < x_1$, then $f(x) - h(x)$ is positive (negative) if n is odd (even).*
5. *If $f \in F^-(H) \cap K$ and $x < x_1$, then $f(x) - h(x)$ is negative (positive) if n is odd (even).*

Proof. (1) and (2) follow immediately from the definition of complete unisolvence. (3), (4), and (5) are corollaries of [Ma3, Corollary 1 and the remark following this corollary]. □

Theorem 1*. *Let F_1 and F_2 be unrestricted n-unisolvent sets, $n \geq 2$, on the interval I and let H be a common unrestricted $(n-1)$-unisolvent subset of F_1 and F_2. Then both $H \cup F_1^+(H) \cup F_2^-(H)$ and $H \cup F_1^-(H) \cup F_2^+(H)$ are unrestricted n-unisolvent sets.*

Proof. Theorem 1 and the Lemma 3 show that we only have to demonstrate the following: Given any SIC $S = \{x_1|n|y_{x_1}^{(0)}, y_{x_1}^{(1)}, \dots, y_{x_1}^{(n-1)}\}$ of order n, and any unrestricted n-unisolvent set F that contains H, looking at H alone suffices to determine whether the uniquely determined function $f \in F$ satisfying S will be contained in $F^+(H)$, H, or $F^-(H)$.

Let $h \in H$ be the uniquely determined function that satisfies the SIC $\bar{S} = \{x_1|n-1|y_{x_1}^{(0)}, y_{x_1}^{(1)}, \dots, y_{x_1}^{(n-2)}\}$ of order $n-1$. Both $y_{x_1}^{(n-1)} - h^{(n-1)}(x_1)$ and $f(x) - h(x)$, $x > x_1$ have the same sign. Hence, by the previous lemma, f will belong to $F^+(H)$, H, or $F^-(H)$ if and only if $y_{x_1}^{(n-1)} - h^{(n-1)}(x_1)$ is positive, zero, or negative. □

We do not know whether or not an $(n-1)$-unisolvent subset of an unrestricted n-unisolvent set is always an unrestricted $(n-1)$-unisolvent set. The set $\operatorname{span}\{1, x^3\}$ is a $(n-2)$-unisolvent subset of the unrestricted 4-unisolvent set $\operatorname{span}\{1, x, x^2, x^3\}$ that is not unrestricted 2-unisolvent since the SIC $\{0|2|0,1\}$ is not satisfied by any function in the set.

The sets F_i, $i = 1, 2$ in Example 1 are actually unrestricted 3-unisolvent and their common subset is unrestricted 2-unisolvent. Hence Example 1 is actually an example of what happens in Theorem 1*.

Theorem 2*. *Let F_1 and F_2 be unrestricted n-unisolvent sets, $n \geq 2$, on the interval I, and let $p = (s,t) \in I \times \mathbb{R}$. Furthermore, let $F_1(p) = F_2(p)$. Then $F_1(p) \cup F_1^+(p) \cup F_2^-(p)$ and $F_1(p) \cup F_2^+(p) \cup F_1^-(p)$ are unrestricted n-unisolvent sets.*

Proof. As in the proof of Theorem 1* we have to prove: Given any SIC $S = \{x_1|n|y_{x_1}^{(0)}, y_{x_1}^{(1)}, \dots, y_{x_1}^{(n-1)}\}$ of order n, and any unrestricted n-unisolvent set F that contains $F(p)$, looking at $F(p)$ alone suffices to determine whether the uniquely determined function $f \in F$ satisfying S will be contained in $F^+(p)$, $F(p)$, or $F^-(p)$.

If $s = x_1$ and $t > y_{x_1}^{(0)}$, $t = y_{x_1}^{(0)}$, or $t < y_{x_1}^{(0)}$, then, clearly, $f \in F^+(p)$, $f \in F(p)$, or $f \in F^-(p)$, respectively. Let $x_1 < s$ and let h be the uniquely determined function in $F(p)$ that satisfies the SIC $\bar{S} = \{x_1|n|y_{x_1}^{(0)}, y_{x_1}^{(1)}, \dots, y_{x_1}^{(n-2)}\}$ of order $n-1$. Then f will be contained in $F^+(p)$, $F(p)$, or $F^-(p)$ depending on whether $y_{x_1}^{(n-1)} - h^{(n-1)}(x_1)$ is positive, zero or negative. In the case $s < x_1$ a similar argument applies. □

There are many more shades of unisolvence and corresponding notions of convexity in between n-unisolvence and unrestricted n-unisolvence; see, for example, the papers by Mathsen and Umamaheswaram. For many of these it should be possible to generalize the results in this paper. Also, by restricting ourselves to open intervals, we have been able to avoid a lot of messy notation and proofs. By giving a counterexample, Mathsen [Ma1] showed that Lemma 3, which made the proof of Theorem 2* so easy, cannot be generalized to the case of unrestricted n-unisolvent sets on closed intervals.

4. Periodic and half-periodic n-unisolvent sets

In this section a *periodic function* f is a continuous function $[-\pi, \pi] \to \mathbb{R}$ such that $f(-\pi) = f(\pi)$. A *half-periodic (or antiperiodic) function* f is a function $[-\pi, \pi] \to \mathbb{R}$ such that $f(-\pi) = -f(\pi)$. A (half-)periodic function $f : [-\pi, \pi] \to \mathbb{R}$ is continuously differentiable if its restriction to $(-\pi, \pi)$ is differentiable, the left and right derivatives at $-\pi$ and π, respectively, exist, and the derivative of f is a continuous (half-)periodic function.

A set F of (half-)periodic functions is called a *(half-)periodic n-unisolvent set* if the restriction of F to the half-open interval $[-\pi, \pi)$ is an n-unisolvent set.

A set F of $n-1$ times differentiable (half-)periodic functions is called a *(half-) periodic unrestricted n-unisolvent set* if the restriction of F to the half-open interval $[-\pi, \pi)$ is an unrestricted n-unisolvent set.

It turns out that periodic n-unisolvent sets exist only if n is odd, half-periodic n-unisolvent sets exist only if n is even (see [Pol, Proposition 2.6], see also [Cu, Corollary on p. 1016] for the periodic case). This implies, of course, that periodic (or half-periodic) n-unisolvent set cannot have $(n-1)$-unisolvent subsets. So, separating subsets of the first kind do not exist in this case. The classical example of an unrestricted periodic $(2k+1)$-unisolvent set is $\text{span}\{1, \sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin kx, \cos kx\}$ (see [Jo]), and an example of an unrestricted half-periodic $(2k)$-unisolvent set is $\text{span}\{\sin\frac{x}{2}, \cos\frac{x}{2}, \sin 3\frac{x}{2}, \cos 3\frac{x}{2}, \sin 5\frac{x}{2}, \cos 5\frac{x}{2}, \dots, \sin(2k-1)\frac{x}{2}, \cos(2k-1)\frac{x}{2}\}$ (see [Pol]).

Separating sets of the second kind exist, and Theorems 2 and 2* have obvious counterparts in the periodic and half-periodic case which are actually corollaries of these two theorems. We just state the results.

Theorem 2'. *Let F_1 and F_2 be (half-)periodic n-unisolvent sets, $n \geq 2$, and let $p = (s,t) \in [-\pi, \pi] \times \mathbb{R}$. Furthermore, let $F_1(p) = F_2(p)$. Then $F_1(p) \cup F_1^+(p) \cup F_2^-(p)$ and $F_1(p) \cup F_2^+(p) \cup F_1^-(p)$ are (half-)periodic n-unisolvent sets.*

Theorem 2*'. *Let F_1 and F_2 be unrestricted (half-)periodic n-unisolvent sets, $n \geq 2$, and let $p = (s,t) \in [-\pi, \pi] \times \mathbb{R}$. Furthermore, let $F_1(p) = F_2(p)$. Then $F_1(p) \cup F_1^+(p) \cup F_2^-(p)$ and $F_1(p) \cup F_2^+(p) \cup F_1^-(p)$ are unrestricted (half-)periodic n-unisolvent sets.*

The proofs of both theorems rely on the fact that we can argue over open subintervals of $[-\pi, \pi]$. Whenever in an argument we are dealing with a points on the boundary of $[-\pi, \pi] \times \mathbb{R}$ we can shift our point of view by a 'rotation' to a (half-) periodic n-unisolvent set that is topologically equivalent to the set we started out with and in which these boundary points are moved into the interior of the strip. We skip the details.

5. Separating sets derived from points in 3-space and separating sets of a third and fourth kind in 3-unisolvent sets

Let $B = \{1, g_2(x), g_3(x)\}$ be a *Chebyshev set* of order 3 on the interval I, that is, the real linear span of B is a 3-unisolvent set on I. Then the function $I \to \mathbb{R}^2 : x \mapsto (g_2(x), g_3(x))$ is injective and its graph K_B is a strictly convex curve in $\mathbb{R}^2$. Let $Z_B = \{(g_2(x), g_3(x), y) | x \in I, y \in \mathbb{R}\}$ be the cylinder in $\mathbb{R}^3$ with base K_B in $\mathbb{R}^2$. The graph of the function $I \to \mathbb{R} : x \mapsto a_1 + a_2 g_2(x) + a_3 g_3(x)$ (a function in F) can be interpreted as the intersection of the affine hyperplane in $\mathbb{R}^3$ that is defined by the equation $y = a_1 + a_2 x_2 + a_3 x_3$ with the cylinder Z_B. So, the set of intersections of the non-vertical hyperplanes in $\mathbb{R}^3$ is in 1-1 correspondence with the set F. We call these intersections *lines* and call two distinct points on the cylinder *parallel* if they are contained in the same vertical. Then the fact that F is 3-unisolvent can be expressed like this: For any 3 distinct, pairwise non-parallel points on the cylinder there exists a unique line that contains all of them. Let p be any point in $\mathbb{R}^3$ and let $proj(p)$ be its projection onto $\mathbb{R}^2$, and let $F(p)$ be the subset of F that corresponds to the non-vertical hyperplanes in $\mathbb{R}^3$ that contain p. A look at (1) and (2), below, shows that we just extended the definition of $F(q)$, $q \in I \times \mathbb{R}$ that we introduced in Section 2. Essentially, three different things can happen:

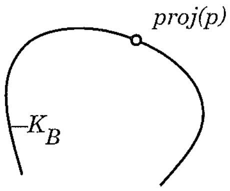

Figure 1

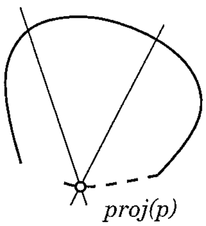

Figure 2

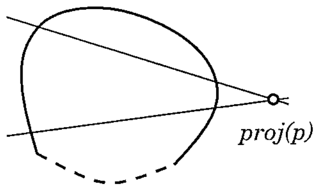

Figure 3

(1) The point *proj*(p) is contained in K_B, or equivalently, p is contained in Z_B (see Figure 1). In this case $F(p)$ is one of the separating sets of the second kind that we considered before. So, no new separating sets can be constructed in this way.

(2) All lines through *proj*(p) intersect K_B in at most one point (see Figure 2). In this case we get a separating set of the first kind. This choice of p can yield new such separating sets. It does not have to though: assume that our cylinder can be extended to a larger cylinder that still gives rise to a Chebyshev space and that p is contained in this cylinder, that is, the base curve can be extended to a strictly convex curve that passes through *proj*(p) (see the extension of K_B in Figure 2 by the dotted line). The larger cylinder then corresponds to an n-unisolvent set $\bar{F}$ that contains F defined on a larger interval J, the point p corresponds to a point q in $(J \setminus I) \times \mathbb{R}$ and $F(p)$ really is the same as $\bar{F}(q)_I$. On the other hand, it is easy to come up with strictly convex curves in $\mathbb{R}^2$ that cannot be extended to larger strictly convex curves; take, for example, one branch of a hyperbola. Of course, this just says that in this particular model F cannot be extended. In other models this might still be possible.

(3) Some lines through *proj*(p) intersect K_B in two points (see Figure 3). For the sake of simplicity, let us assume that K_B can be extended to a strictly convex simply closed curve $\overline{K_B}$ that contains K_B (see the extension of K_B by the dotted line in Figure 3). It is easy to see that $\overline{K_B}$ does not contain *proj*(p). This means that our original Chebyshev space can be extended to a *periodic Chebyshev space* $\bar{F}$ and the prospective separating set extends to a prospective separating set in this periodic Chebyshev space. The point p corresponds to a natural involutory homeomorphism of the cylinder over $\overline{K_B}$ to itself that exchanges points of intersection of lines through p with the cylinder. This involution will be orientation-preserving if p is situated inside the cylinder and orientation-reversing if p is situated outside the cylinder. Furthermore, our prospective separating set can be described com-

pletely by this involution as the set of all functions in $\bar{F}$ that interpolate pairs of distinct points on the cylinder that get exchanged by the involution.

We want to translate all this into the incidence geometric setting that we already mentioned before. First of all, we identify the interval $[\pi, \pi)$ with the circle $\mathbb{S}^1$ in a natural way. Now a periodic n-unisolvent set G can be seen to correspond to the incidence structure $I_G = (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C})$ whose point set is the cylinder $\mathbb{S}^1 \times \mathbb{R}$ and whose *circles*, that is, the elements of the circle set $\mathcal{C}$, are the graphs of the continuous periodic functions in G. Following [Pol], we call such an incidence structure a *periodic 3-OA* (OA stands for *orthogonal array*). It is clear that every circle in $\mathcal{C}$ is homeomorphic to $\mathbb{S}^1$. Let G be a periodic 3-OA and let γ be an involutory homeomorphism of $\mathbb{S}^1 \times \mathbb{R}$ to itself that is not the identity and that has the following properties:

(1) it maps verticals on this cylinder to verticals; and
(2) for all $p \in \mathbb{S}^1 \times \mathbb{R}$ for which $p \neq \gamma(p)$, every circle through p and $\gamma(p)$ is (globally) fixed by γ.

Following [PS2], we call γ a *pre-inversion* of G if γ is fixed-point-free and orientation-preserving. We call it *pre-reflection* if its fixed point set consists of two distinct verticals Π_0 and Π_∞. In this case let H_1 and H_2 denote the two connected components of $(\mathbb{S}^1 \times \mathbb{R}) \setminus (\Pi_0 \cup \Pi_\infty)$. In both cases $\mathcal{C}_\gamma$ denotes the set of all circles fixed by γ (this is going to be our new "separating set"). From [PS2, Lemma 2] we know:

(1) If γ is a pre-inversion and c is a circle that is not fixed by γ, then $c \cap \gamma(c) = \emptyset$.
(2) If γ is a pre-reflection and c is a circle that is not fixed by γ, then $c \cap \gamma(c) = c \cap (\Pi_0 \cup \Pi_\infty)$.

Let γ be a pre-inversion of G. Let $\mathcal{C}_{\gamma^+}$ ($\mathcal{C}_{\gamma^-}$) be the set of all circles $c \in \mathcal{C}$ such that on the cylinder c lies above (below) $\gamma(c)$. In the case of a pre-reflection let $\mathcal{C}_{\gamma^+}$ ($\mathcal{C}_{\gamma^-}$) be the set of all circles $c \in \mathcal{C}$ such that c lies above (below) $\gamma(c)$ on H_1. In both cases $\mathcal{C}$ is the disjoint union of $\mathcal{C}_\gamma$, $\mathcal{C}_{\gamma^+}$ and $\mathcal{C}_{\gamma^-}$. This corresponds to Corollary 1.

The following result shows that the fixed-circle sets of pre-inversions and pre-reflections correspond to separating sets of a third and fourth kind. Examples of such separating sets arise naturally as in (3) above.

Theorem 3. *Let $I_{F_1} = (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C})$ and $I_{F_2} = (\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}^*)$ be two periodic 3-OAs. Suppose both periodic 3-OAs admit the pre-inversion (or pre-reflection) γ and that $\mathcal{C}_\gamma = \mathcal{C}^*_\gamma$. Then $(\mathbb{S}^1 \times \mathbb{R}, \mathcal{C}_\gamma \cup \mathcal{C}_{\gamma^+} \cup \mathcal{C}^*_{\gamma^-})$ is a periodic 3-OA.*

The proof of this result is a straight-forward variation of the proof of Propositions 2* and 3* in [PS2] (see also [PS1]) and will be omitted here. Actually, by using the language in [Pol] we can turn Theorem 3 into a statement that is equivalent to Propositions 2* and 3* in [PS2] by adding the word 'nested' in front of every 'periodic' in Theorem 3.

It is clear that the circle set $\mathcal{C}_\gamma \cup \mathcal{C}_{\gamma^+} \cup \mathcal{C}^*_{\gamma^-}$ of the new periodic 3-OA in Theorem 3 corresponds to a periodic 3-unisolvent set.

For more details about this general incidence theoretic setting the reader is referred to the three papers we just mentioned.

References

[Cu] P. C. CURTIS, JR., *n-parameter families and best approximation*, Pacific J. Math. *9* (1959), 1013–1027.

[Ha] P. HARTMAN, *Unrestricted n-parameter families*, Rend. Circ. Mat. Palermo (2) *7* (1958), 123–142.

[HK] O. HAUPT AND H. KÜNNETH, *Geometrische Ordnungen*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 133, Springer, 1967.

[Jo] D. J. JOHNSON, *The trigonometric Hermite–Birkhoff interpolation problem*, Trans. Amer. Math. Soc. *212* (1976), 365–374.

[Ke] J. H. B. KEMPERMAN, *On the regularity of generalized convex functions*, Trans. Amer. Math. Soc *135* (1969), 69–93.

[KS] S. KARLIN AND W. J. STUDDEN, *Tchebycheff Systems: With Applications in Analysis and Statistics*, Interscience, New York, 1966.

[KZ] T. KILGORE AND R. A. ZALIK, *Splicing of Markov and weak Markov systems*, J. Approx. Theory *59* (1989), 2–11.

[LJR] G. G. LORENTZ, K. JETTER AND S. D. RIEMENSCHNEIDER, *Birkhoff interpolation*, Cambridge University Press, New York, 1984.

[LP] R. LÖWEN AND U. PFÜLLER, *Two-dimensional Laguerre planes over convex functions*, Geom. Dedicata *23* (1987), 73–85.

[Ma1] R. M. MATHSEN, *$\lambda(n)$-parameter families*, Canad. Math. Bull. *12* (1969), 185–191.

[Ma2] R. M. MATHSEN, *$\lambda(n)$-convex functions*, Rocky Mt. J. Math. *2* (1972), 31–43.

[Ma3] R. M. MATHSEN, *Hereditary $\lambda(n,k)$-families and generalized convexity of functions*, Rocky Mt. J. Math. *12* (1982), 753–756.

[Mo1] E. MOLDOVAN, *Sur une généralisation des fonctions convexes*, Mathematica (Cluj) *1* (1959), 49–80.

[Mo2] E. MOLDOVAN, *Introduction à l'étude comparative des ensembles de fonctions interpolatoires*, Mathematica (Cluj) *6* (1964), 145–155.

[Mo3] E. MOLDOVAN, *Sur les classes d'ensembles interpolatoires qui sont comparables par rapport a leur n-valence*, Mathematica (Cluj) *7* (1965), 319–325.

[Mo4] E. MOLDOVAN, *Propriétés des ensembles interpolatoires comparables par rapport à leur n-valence*, Mathematica (Cluj), *8* (1966), 133–136.

[Mor] M. I. MOROZOV, *On the uniform approximation of continuous functions by functions in interpolatory classes (Russian)*, Isv. Akad. Nauk. Ser. Mat. *16* (1952), 75–100.

[Ne] A. B. NEMETH, *About the extension of the domain of definition of the Chebyshev systems defined on intervals of the real axis*, Mathematica (Cluj) *11* (1969), 307–310.

[Pol] B. POLSTER, *N-unisolvent sets and flat incidence structures*, Trans. Amer. Math. Soc. *350* (1998), 1619–1641.

[Pop1] E. POPOVICIU, *Sur la convexité simultanée par rapport aux plusieurs ensembles interpolatoires*, L'analyse numérique et la théorie de l'approximation *7* (1978), 101–106.

[Pop2] E. POPOVICIU, *Sous-ensembles remarquables d'un ensemble interpolatoire*, L'analyse numérique et la théorie de l'approximation *8* (1979), 193–202.

[PS1] B. POLSTER AND G. STEINKE, *Cut and paste in 2-dimensional circle planes*, Canad. Math. Bull. *38* (1995), 469–480.

[PS2] B. POLSTER AND G. STEINKE, *The inner and outer space of 2-dimensional Laguerre planes*, J. Aust. Math. Soc., Series A *62* (1997), 104–127.

[Ri1] J. R. RICE, *The approximation of functions I*, Addison-Wesley, Reading, Mass., 1964.

[Ri2] J. R. RICE, *The approximation of functions II*, Addison-Wesley, Reading, Mass., 1969.

[Sa] H. SALZMANN, *Topological planes*, Adv. Math. *2* (1967), 1–60.

[Sc] L. L. SCHUMAKER, *Spline functions: Basic theory*, Wiley, New York, 1981.

[St] G. F. STEINKE, *Topological circle geometries*, pp. 1325–1353, in: F. Buekenhout (Ed.), Handbook of Incidence Geometry, Elsevier, 1995.

[To] L. TORNHEIM, *On n-parameter families of functions and associated convex functions*, Trans. Amer. Math. Soc. *69* (1950), 457–467.

[Um1] S. UMAMAHESWARAM, *$\lambda(n,k)$-parameter families and associated convex functions*, Rocky Mt. J. Math. *8* (1978), 491–501.

[Um2] S. UMAMAHESWARAM, *$\lambda(n,k)$-convex functions*, Rocky Mt. J. Math. *8* (1978), 759–764.

[Zi] R. ZIELKE, *Discontinuous Čebyšhev systems*, Lecture Notes in Mathematics *707*, Springer-Verlag, Berlin–New York–Heidelberg, 1979.

Burkard Polster
Department of Pure Mathematics
University of Adelaide
Adelaide, SA 5005
Australia
e-mail: bpolster@maths.adelaide.edu.au

Manuscript received: June 3, 1997 and, in final form, January 12, 1998.