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Centering small generalized polygons—projective pottery at work[☆]

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Abstract

The generalized triangle, quadrangle, and hexagons of order 2 are small point-line geometries that play a role in the theory of generalized polygons and buildings that is comparable to that of the Fano plane in the theory of projective planes. Virtually everybody working in discrete mathematics is familiar with the generalized triangle of order 2 aka the Fano plane, the smallest projective plane, $\text{PG}(2,2)$, or the unique symmetric block design with parameters $2-(7,3,1)$, and its elementary representation as the geometrical configuration of an equilateral triangle together with its three medians and inscribed circle. The main purpose of this paper is to derive an elementary representation of the generalized hexagons of order 2 which extends naturally to a special representation of the projective space $\text{PG}(5,2)$. Along the way, we also derive a similar representation of the generalized quadrangle of order 2 which turns out to be equivalent to the well-known representation of the quadrangle in terms of synthems and duads and which extends naturally to a representation of $\text{PG}(3,2)$.

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1. Introduction

To make a pot on a pottery wheel the potter slaps a homogeneous lump of clay in the middle of a turning pottery wheel, centers the clay and shapes it into a pot. Finally, he

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decorates the pot while the wheel is still turning. A number of construction techniques in finite geometry are very reminiscent of this process. Instead of a lump of clay we use a finite point-line geometry and instead of the pottery wheel we use a cyclic group of automorphisms to ‘center’ the geometry. Now we ‘decorate’ this centered geometry by emphasizing some orbits of interesting substructures of the geometry under the cyclic group. Often, interesting properties of the geometry are easily discerned once a geometry is centered and decorated in this way. Finite pottery also proves to be very useful when it comes to constructing appealing pictures and models of finite geometries; see [12] for many examples of such pictures and models. The finite pottery in this paper falls in the last category.

We center the projective spaces $\text{PG}(3,2)$ and $\text{PG}(5,2)$ using automorphisms of orders 5 and 7, respectively. Right in the middle of these centered projective spaces we find a centered generalized quadrangle of order 2 and one of the two generalized hexagons of order 2. Translating these centered generalized polygons into sets of diagrams, we arrive at very compact, elementary descriptions of these geometries. We also identify a number of important properties, substructures and interconnections of these geometries.

2. Generalized polygons

In the following a (point-line) *geometry* will consist of a nonempty set of *points* and a nonempty set of subsets of the point set called *lines*. Every point is contained in at least two lines and every line contains at least two points.

Two geometries are *isomorphic* if and only if there is a bijection between the point sets of the two geometries that induces a bijection between their line sets.

Every graph can be interpreted as a geometry. Here the vertices of the graph are the points, and associated with every edge is a line consisting of the two vertices contained in this edge. In particular, an *ordinary n -gon* is a geometry that is isomorphic to the geometry of vertices and edges of a regular n -gon in the plane.

The *incidence graph* of a geometry is the graph whose vertices are the points and lines of the geometry. Two vertices are connected by an edge if and only if they correspond to a point and a line such that the point is contained in the line.

Let \mathcal{G} be a geometry with point set P and line set L . A geometry \mathcal{G}' is *contained in \mathcal{G}* , if its point set P' is a subset of P and all its lines are intersections of lines in L with P' .

The *dual* of \mathcal{G} is constructed by interchanging the roles of points and lines in \mathcal{G} . More precisely, its points are the lines of \mathcal{G} , and to every point of \mathcal{G} corresponds a line of the dual consisting of all lines in \mathcal{G} containing this point. A geometry is called *self-dual* if it is isomorphic to its dual.

A *generalized n -gon* $\mathcal{G}, n \geq 2$, is a geometry which satisfies the following two axioms:

- (Q1) \mathcal{G} does not contain any ordinary k -gons for $2 \leq k < n$.
- (Q2) Given two points, two lines, or a point and a line, there is at least one ordinary n -gon in \mathcal{G} that contains both objects.

Generalized n -gons were introduced by Tits [15]. Generalized 2-gons, 3-gons, 4-gons, etc., are also called generalized digons, triangles, quadrangles, etc., respectively. Similarly, generalized n -gons are also referred to as *generalized polygons*.

A generalized polygon is called *thick* if every point is contained in at least 3 lines and every line contains at least 3 points. An ordinary n -gon is not thick. From the above definition it is clear that the dual of a (thick) generalized n -gon is also a (thick) generalized n -gon.

By Axiom Q1, a generalized n -gon, $n > 2$, does not contain any digons. This just means that two of its points are connected by at most one line and that two of its lines intersect in at most one point. This observation plus Axiom Q2 imply that in a generalized triangle two points are contained in exactly one line and two lines intersect in exactly one point. This shows that the thick generalized triangles are just the *projective planes*.

A generalized n -gon is *finite* if its point set is a finite set. A finite generalized n -gon \mathcal{G} is of *order* (s, t) , $s, t \geq 1$, if every line contains $s + 1$ points and every point is contained in $t + 1$ lines. If $s = t$, we also say that \mathcal{G} is of order s . The ordinary n -gon is the unique generalized n -gon of order 1. If \mathcal{G} is not an ordinary n -gon, and if it has an order, then $n = 3, 4, 6, 8$, or 12, and, if $n = 12$, then \mathcal{G} is not thick; see [5].

The projective planes of order s are exactly the thick generalized triangles of order s . In particular, the Fano plane is the unique generalized triangle of order 2. The incidence graph of a generalized n -gon of order t is a generalized $2n$ -gon of order $(1, t)$.

There is a unique generalized triangle of order 2 and a unique generalized quadrangle of order 2. Both are self-dual geometries; see [8]. There are two generalized hexagons of order 2. These are the so-called split Cayley hexagon $\mathcal{H}(2)$ and its dual; see [3]. In the following, we will refer to the Fano plane, the generalized quadrangle of order 2, and the generalized hexagon $\mathcal{H}(2)$ as the TRIANGLE, the QUADRANGLE, and the HEXAGON, respectively. For these and other information about generalized polygons see [14] or [16].

3. The Fano plane and its incidence graph

In this section we recall an elementary representation of the Fano plane on a regular 7-gon and derive a labelling of the incidence graph of this projective plane that we will need in Section 6 of this paper.

In [12] we used finite pottery to model small geometries that admit automorphisms of order n on regular n -gons. Fig. 1 shows two possible pictures of the Fano plane, or equivalently the TRIANGLE, that we arrive at by centering the plane using a Singer cycle (a Singer cycle of a projective plane is a cyclic group of automorphisms of the plane that acts sharply transitively on the point set of the plane). Its points are the 7 points of the 7-gon and its lines are the images of one of the two triangles under successive rotations through $360/7^\circ$ degrees around the center of the 7-gon. Of course, one Singer cycle of this plane corresponds to the cyclic group of order 7 of rotations around the center of the 7-gon. One more Fano plane is hiding in this picture. Its points are the lines of the left Fano plane and its lines are the lines of the right Fano

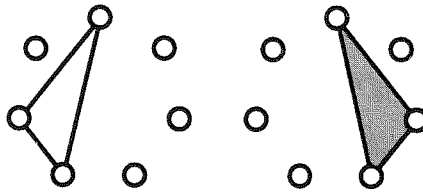


Fig. 1. Two centered Fano planes.

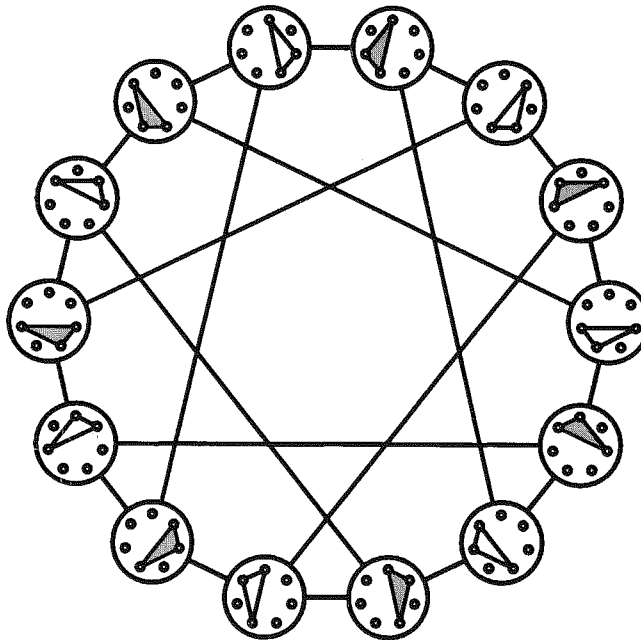


Fig. 2. The incidence graph of the Fano plane.

plane. Here a point is incident with a line if and only if the corresponding triangles have exactly one point in common. Fig. 2 shows the incidence graph of the Fano plane whose vertices have been labelled with the two different kinds of triangles. It is easily verified that this incidence graph is a generalized hexagon of order $(1,2)$. In fact, it is very easy to draw ordinary hexagons in this geometry.

We remark only that in Fig. 1 the triangle on the right is an oval in the Fano plane generated by the triangle on the left, and vice versa. Maybe you are interested in the following more general setting: ‘Two projective planes are defined on the same point set such that lines of one are ovals in the other, and vice versa, and such that a third projective plane can be defined using the lines of both given planes as above’. If so, then have a look at [7,10,11].

4. Centering of projective spaces

In this section we summarize some results about projective spaces that we will use to center the QUADRANGLE and the HEXAGON. Most of the facts mentioned are common knowledge; see also [6,9].

4.1. Centering $\text{PG}(n, 2)$ via the shift map

Consider the nonzero points of the affine space $\text{AG}(n + 1, 2)$ as the points of the projective space $\text{PG}(n, 2)$. Then, given two points in $\text{PG}(n, 2)$, the third point on the unique connecting line of the two points is the sum of the two points. Every permutation of the standard basis elements of $\text{AG}(n + 1, 2)$ corresponds in a unique way to an automorphism of the associated projective space. In particular, the cyclic shift to the right

$$\sigma_n : \text{AG}(n, 2) \rightarrow \text{AG}(n, 2) : (a_0, a_1, \dots, a_{n-1}) \mapsto (a_{n-1}, a_0, a_1, \dots, a_{n-2})$$

is induced by a cyclic permutation of the standard basis vectors. Note that the only fixed point of $\text{PG}(n, 2)$ under the cyclic shift σ_{n+1} is the point

$$p_* = (1, 1, \dots, 1).$$

4.2. $\text{PG}(n, 2)$ as a geometry of subsets of an $(n + 1)$ -gon

Let O_n be a regular n -gon in the plane, one of whose symmetry axes is a vertical, and whose uppermost vertex is situated on this vertical axis. Label this uppermost vertex by 0, and label the remaining vertices from 1 to $n - 1$ by proceeding in the clockwise direction. Given a point p in $\text{PG}(n, 2)$, we identify it with the subset of O_{n+1} consisting of all those vertices i such that the entry x_i of p is 1. In this way, it is possible to represent $\text{PG}(n, 2)$ as a *subset geometry* on $n + 1$ vertices. Note that in this representation the points of the geometry are the nonempty subsets of vertices of O_{n+1} and that the shift σ_{n+1} translates into a rotation of O_{n+1} through $360/(n + 1)$ degrees around its center.

4.3. The quotient space

Let n be a fixed even number. Consider the quotient space of $\text{PG}(n, 2)$ with respect to the point p_* . This space is isomorphic to $\text{PG}(n - 1, 2)$. The points and lines of this space are the lines and planes in $\text{PG}(n, 2)$ containing p_* . Given a point p in $\text{PG}(n, 2)$, let $C(p)$ be its *complement*, that is, $p + p_*$, the unique third point on the line connecting p and p_* .

We identify the points of $\text{PG}(n - 1, 2)$ in the natural way with the points of $\text{PG}(n, 2)$ having at most $n/2$ entries that are 1's. Here the point $\{p, C(p), p_*\}$ in $\text{PG}(n - 1, 2)$ (this is a line in $\text{PG}(n, 2)$) gets identified with p if p has at most $n/2$ entries that are

1's, and with $C(p)$ otherwise. With this identification in place, the following map is an endomorphism of geometries.

$$\pi_{n+1} : \text{PG}(n, 2) \setminus \{p_*\} \rightarrow \text{PG}(n-1, 2):$$

$$p \mapsto \begin{cases} p & \text{if at most } n/2 \text{ entries of } p \text{ are 1's} \\ C(p) & \text{otherwise.} \end{cases}$$

The shift map σ_{n+1} is also an automorphism of $\text{PG}(n-1, 2)$ and commutes with π_{n+1} .

4.4. $\text{PG}(n-1, 2)$ as a geometry of subsets of an $(n+1)$ -gon

Keeping all this in mind, it is clear when n is even how to represent $\text{PG}(n-1, 2)$ as a subset geometry on the regular $(n+1)$ -gon O_{n+1} . The points in this representation are the nonempty subsets of vertices of O_{n+1} containing no more than $n/2$ elements.

5. Centering the QUADRANGLE

The QUADRANGLE has 15 points and 15 lines and that 15 is also the number of points in $\text{PG}(3, 2)$. Every one of the lines of the QUADRANGLE contains 3 points and every point is contained in 3 lines.

The following quadric contained in $\text{PG}(4, 2)$ is globally fixed by the shift σ_5^* :

$$\mathcal{Q} = \left\{ (x_0, x_1, x_2, x_3, x_4) \mid \sum_{0 \leq i < j \leq 4} x_i x_j = 0 \right\}.$$

A point in $\text{PG}(4, 2)$ is contained in \mathcal{Q} if and only if either 3 or 4 of its components are 1's. The quadric \mathcal{Q} contains the following 3 points and the 3 lines and their images under powers of σ_5 . To be more precise, if x_0 is one of the points or lines below, then $x_k = \sigma_5^k(x_0)$, $k = \{1, 2, 3, 4\}$.

$$\begin{aligned} a_0 &= (0, 1, 1, 1, 1), & A_0 &= (a_0, c_1, c_4), \\ b_0 &= (1, 0, 1, 1, 0), & B_0 &= (a_0, b_0, c_0), \\ c_0 &= (1, 1, 0, 0, 1), & C_0 &= (a_0, b_2, b_3). \end{aligned}$$

The geometry of points and lines of \mathcal{Q} is isomorphic to the QUADRANGLE; see [8]. The map π_5 maps the points of this geometry bijectively onto the points of $\text{PG}(3, 2)$ (as described in the previous section) and gives a representation of the QUADRANGLE right in the center of this projective space. We consider $\text{PG}(3, 2)$ and this subgeometry as subset geometries of the regular 5-gon. Then there are essentially 3 different points corresponding to a_0, b_0 , and c_0 , above, as indicated by the diagrams in Fig. 3. Every single one of the diagrams stands for 5 different points.

Furthermore, there are essentially 3 different lines corresponding to A_0, B_0 and C_0 , above, as indicated by the diagrams in Fig. 4. Again, every single diagram stands for 5 lines.

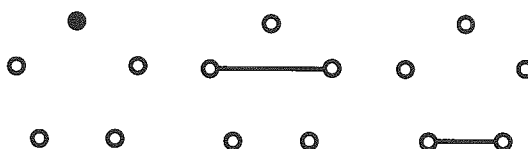
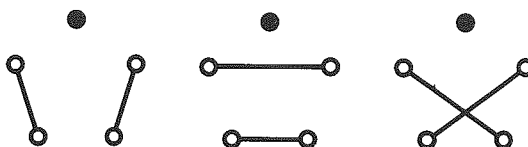
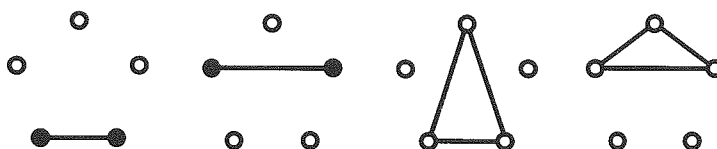
Fig. 3. The points of $PG(3,2)$ and the QUADRANGLE.

Fig. 4. The lines of the QUADRANGLE.

Fig. 5. The remaining lines of $PG(3,2)$.

The 20 lines in $PG(3,2)$ that are not contained in the quadrangle correspond to the 4 diagrams in Fig. 5.

Using the rotated images of the diagrams in Fig. 3 as labels, it is possible to construct the appealing picture of the QUADRANGLE in Fig. 6. This picture is called the *doily* and is due to Payne. Use the diagrams in Fig. 5 to complete Fig. 6 to a picture of the projective space. For a detailed discussion of the resulting model of $PG(3,2)$, see Chapter 5 of the ‘Geometrical Picture Book’ [12].

5.1. Rules for remembering the QUADRANGLE

Of course, the above representation of the QUADRANGLE can also be worded as follows:

The **QUADRANGLE** defined on 5 points
Given a set S of 5 elements, the points of the QUADRANGLE
are the nonempty subsets of S containing 1 or 2 elements. The
lines are partitions of S into 3 points of the QUADRANGLE.

If we extend S by one extra element e to a set T containing 6 elements, replace every point of the QUADRANGLE corresponding to a one-element set $\{v\}$ by $\{v, e\}$, and every line $\{\{v\}, \{w, x\}, \{y, z\}\}$ by $\{\{v, e\}, \{w, x\}, \{y, z\}\}$, we arrive at the following

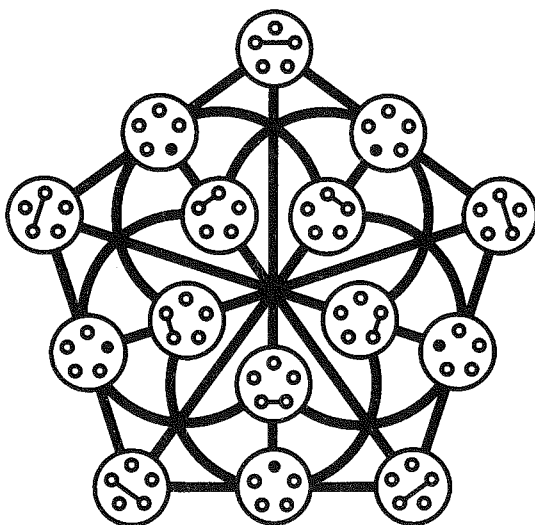


Fig. 6. A picture of the QUADRANGLE.

well-known description of our geometry in terms of duads (sets of two elements) and synthemes (partitions into duads); see [8, p. 122].

The QUADRANGLE defined on 6 points

Given a set T of 6 elements, the points of the QUADRANGLE are the subsets of T containing 2 elements. The lines are partitions of S into 3 points of the QUADRANGLE.

This also gives a way to go back and forth between the above construction of the doily and the construction of the doily in [12, Chapter 4].

5.2. An ovoid, a spread, some pentagons, and the Desargues configuration

The set of 5 points corresponding to the first diagram in Fig. 3 is an ovoid of the QUADRANGLE. This means that every line of the QUADRANGLE has exactly one point in common with this set. In $\text{PG}(3, 2)$ this set is an elliptic quadric. Also the lines of the QUADRANGLE are just the tangent lines of the quadric, that is, the lines in $\text{PG}(3, 2)$ that intersect this quadric in exactly one point.

The set of 5 lines corresponding to the diagram in the middle of Fig. 4 is a spread in the QUADRANGLE. This means that every point of the QUADRANGLE is contained in exactly one line in this set.

The sets of 5 lines each corresponding to the first and last of the diagrams in Fig. 4 are ordinary pentagons.

The Desargues configuration can be defined on a set S of 5 elements. It is the geometry whose points are the subsets of size 2 of S and its lines are all sets of points of the form $\{\{x, y\}, \{y, z\}, \{x, z\}\}$. This means that there is a nice picture of

this configuration hiding right in the middle of the above representation of $\text{PG}(3,2)$. Its points correspond to the last two diagrams in Fig. 3 and its lines to the last two diagrams in Fig. 5.

6. Centering the HEXAGON

In this section we derive an elementary representation of the HEXAGON on a regular 7-gon which parallels the representation of the QUADRANGLE on a regular 5-gon derived in the previous section. We also identify a number of prominent substructures of the HEXAGON.

The HEXAGON has 63 points and 63 lines and that 63 is also the number of points of $\text{PG}(5,2)$. Every one of the lines of the HEXAGON contains 3 points and every point is contained in 3 lines.

The following quadric contained in $\text{PG}(6,2)$ is globally fixed by the shift σ_7^* :

$$\mathcal{H} = \left\{ (x_0, x_1, x_2, x_3, x_4, x_5, x_6) \left| \sum_{0 \leq i < j \leq 6} x_i x_j = 0 \right. \right\}.$$

The quadric \mathcal{H} contains the 9 points and the 9 lines in Table 1 and their images under powers of σ_7 . To be more precise, if x_0 is one of the points or lines in Table 1, then $x_k = \sigma_7^k(x_0)$, $k = \{1, 2, 3, 4, 5, 6\}$. Note that a point is contained in \mathcal{H} if and only if exactly 1, 4 or 5 of its components are 1's. The geometry defined by these 63 points and 63 lines of \mathcal{H} is isomorphic to the HEXAGON.

The best way to check that all this is true is to have a look at Schroth's paper [13]. In this paper some finite pottery based on a cyclic automorphism of order 7 is used to construct fantastic pictures of the HEXAGON and its dual. The notation in Table 1 coincides with the notation adopted in Schroth's paper except for one coordinate transformation which takes the basis $\{g_0, g_1, g_2, g_3, g_4, g_5, g_6\}$ in Schroth's paper to the standard basis of the vector space $\text{AG}(7,2)$ (note that in Table 1 the point g_0 is the first one of the standard basis vectors).

The map π_7 maps the 63 points of the quadric bijectively onto the points of $\text{PG}(5,2)$ and gives a representation of the HEXAGON right in the center of this protective space.

Table 1
The points and lines of the HEXAGON

$a_0 = (1, 1, 0, 1, 1, 0, 1)$	$A_0 = (a_3, a_4, b_0),$
$b_0 = (0, 1, 1, 0, 0, 1, 1)$	$B_0 = (a_0, f_0, g_0),$
$c_0 = (1, 1, 1, 0, 0, 1, 1)$	$C_0 = (c_1, c_6, d_0),$
$d_0 = (0, 0, 1, 1, 1, 1, 0)$	$D_0 = (b_0, c_0, g_0),$
$e_0 = (1, 0, 1, 1, 1, 1, 0)$	$E_0 = (e_2, e_5, f_0),$
$f_0 = (0, 1, 0, 1, 1, 0, 1)$	$F_0 = (d_0, e_0, g_0),$
$g_0 = (1, 0, 0, 0, 0, 0, 0)$	$G_0 = (b_0, h_5, i_2),$
$h_0 = (0, 1, 0, 0, 1, 1, 1)$	$H_0 = (d_0, h_0, i_0),$
$i_0 = (0, 1, 1, 1, 0, 0, 1)$	$I_0 = (f_0, h_4, i_3),$

We consider $\text{PG}(5,2)$ and this subgeometry as subset geometries of the regular 7-gon. Then there are essentially 9 different points and lines, as indicated by the diagrams in Fig. 7, and every single one of the diagrams stands for 7 different objects. The labelling of the diagrams establishes the correspondence between these diagrams and the points and lines in Table 1.

6.1. Rules for remembering the HEXAGON

Note that there are four different types of points: vertices (1), edges (3), isosceles triangles (3), and Fano triangles (2) (it is because of Fig. 1 that we call the nonisosceles triangles in Fig. 7 Fano triangles). Associated with every isosceles triangle there are three different lines of the HEXAGON:

- A line that is a partition of the 7 vertices into two edges and the triangle, where the length of both edges coincides with the length of the two sides of equal length of the triangle. In Fig. 7 the lines C_0 , E_0 , and A_0 are of this type.
- A line consisting of the triangle, its apex, and its base. In Fig. 7 the lines F_0 , B_0 , and D_0 are of this type.
- A line consisting of the isosceles triangle itself, plus two Fano triangles having different orientations, such that the tip of the isosceles triangle is contained in both Fano triangles and each of the other 6 vertices is contained in exactly one of the three triangles. In Fig. 7 the lines H_0 , I_0 , and G_0 are of this type.

6.2. Incidence graph of the Fano plane

Consider the geometry whose points are the 14 Fano triangles and whose lines are the lines corresponding to the diagrams at the bottom of Fig. 7. This geometry is isomorphic to the incidence graph of the Fano plane which, remember, is a generalized hexagon of order (1,2). If we draw a picture of the incidence graph, we end up drawing the picture in Fig. 2 again. Note also that the 7 lines corresponding to one of the diagrams at the bottom of Fig. 7 form a spread of lines in the incidence graph. This means that every point of the incidence graph is contained in exactly one of these 7 lines. Consequently, the three diagrams at the bottom of Fig. 7 correspond to a partition of the line set of the incidence graph into spreads.

6.3. $\text{PG}(5,2)$ and pictures

As in the case of the QUADRANGLE, Fig. 7 can be completed to a representation of $\text{PG}(5,2)$ that corresponds to the special representation of $\text{PG}(5,2)$ as a subset geometry (as described above). This is done by adding 84 more diagrams of lines to the 9 diagrams in Fig. 7. (Note that $\text{PG}(5,2)$ contains 651 lines. This means that we need a total of $651/7 = 93$ diagrams to describe all of them.)

A picture of the HEXAGON would be an analog of Fig. 6. In fact, it is clear from the way we constructed the diagrams that this picture will be Schroth's

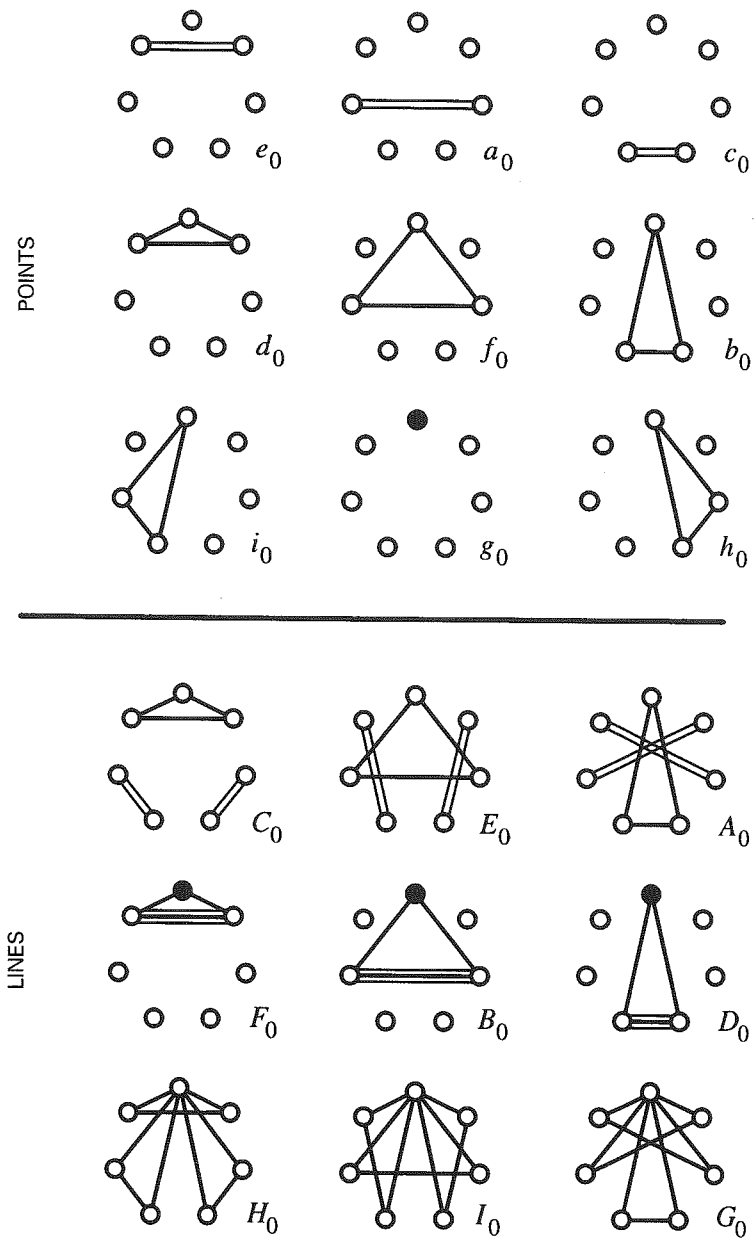


Fig. 7. The points and lines of the HEXAGON.

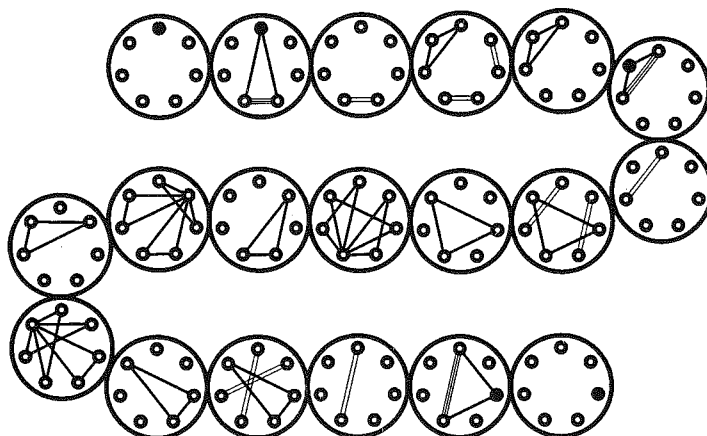


Fig. 8. A special path in the HEXAGON.

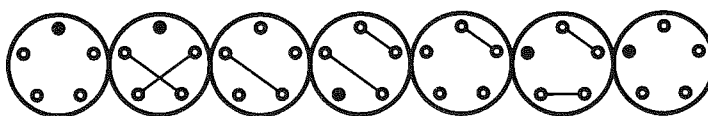


Fig. 9. A special path in the QUADRANGLE.

picture of this geometry in [13, Fig. 8] (see also the color pictures on his homepage at (<http://fbl.math.nat.tu-bs.de/top/aschroth>)).

6.4. The incidence graph of the HEXAGON

The incidence graph of the HEXAGON is the unique generalized 12-gon of order (1,2). Consider the path in the hexagon depicted in Fig. 8. It contains the different kinds of points and lines in Fig. 7 exactly once, except for its beginning and its end which are two points of the same kind. If we fit together the 7 images of this path under the cyclic group generated by σ_7 , we arrive at a path that contains every point and line of the HEXAGON exactly once and is invariant under σ_7 . This enables us to draw a picture of the incidence graph such that the vertices of the graph are the vertices of a regular 126-gon, two adjacent vertices of the 126-gon are connected by an edge, and rotations through $360/7^\circ$ around the center of the 126-gon leave the incidence graph invariant. See [12, Section 13.5] for a picture of the incidence graph that has been constructed in this way.

Fig. 9 shows a similar path in the QUADRANGLE which can be used to model the incidence graph of this geometry on a regular 30-gon such that two adjacent vertices of the 30-gon are connected by an edge, and rotations through $360/5^\circ$ around the center of the 30-gon leave the incidence graph invariant. See [12, Section 13.4] for a picture of the incidence graph that has been constructed in this way.

6.5. *A distance-2-ovoid, some 7-gons, the Coxeter graph, the Petersen graph, an extra involution, and $PG(3,2)$*

The set of points consisting of all isosceles triangles is a so-called *distance-2-ovoid* of the HEXAGON. This means that every line in the geometry contains exactly one point of this set. Note that this definition of a distance-2-ovoid coincides with the definition of an ovoid in the QUADRANGLE that we introduced in the previous section. Ovoids in generalized hexagons are defined in a different way and we only remark that the HEXAGON does not contain any ovoids; see [16] for details.

In the fourth row of diagrams in Fig. 7, each of the three diagrams A_0 , C_0 , and E_0 corresponds to a set of 7 lines that form an ordinary 7-gon. These are the 3 distinguished (mutually disjoint) 7-gons in Schroth's picture of the HEXAGON.

If we remove the distance-2-ovoid above from the HEXAGON, we are left with the union of two graphs. One is the incidence graph of the Fano plane described above. The other one is the so-called Coxeter graph. It contains the 3 disjoint 7-gons. Apart from the vertices and edges contained in these 7-gons, it also contains the one-element points of the HEXAGON and the 3 sets of 7 edges each corresponding to the 3 diagrams in the fifth row of diagrams in Fig. 7. Drawing a picture of this graph based on the 7-gons yields exactly the picture in [4, Fig. 2]. Removing the distinguished ovoid in the QUADRANGLE leaves us with the Petersen graph, a graph which is closely related to the Coxeter graph; see [1].

We also remark that our special distance-2-ovoid is an example of a so-called geometric hyperplane, and that removing a geometric hyperplane from a finite generalized polygon usually gives a connected geometry and not two connected geometries as noted above; see [2].

Note that a reflection of the regular 7-gon that our HEXAGON is modelled on translates into an automorphism of this geometry. This means that the HEXAGON admits the dihedral group of degree 7 as a group of automorphisms; see also the respective remark in [13].

There are exactly 35 points of the HEXAGON which are represented by triangles. These points are the points of a Klein quadric in $PG(5,2)$. We remark that this special set of points can be identified in a natural way with the 35 lines of the projective space $PG(3,2)$; see [9] or [6, Section 15.4] for details.

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