What is the best way to lace your shoes?

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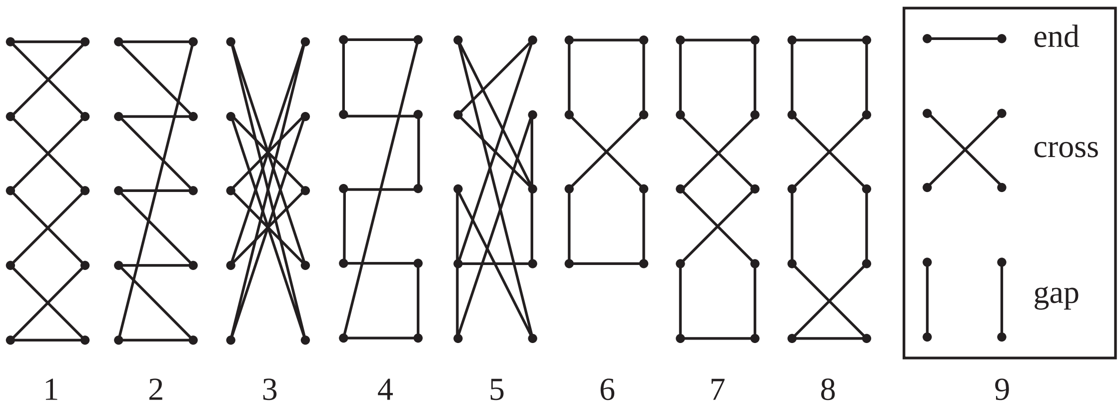
Learning to lace and tie shoes is a difficult hurdle that everybody has to overcome in the course of growing up. However, how many people have ever asked themselves whether there are better ways to perform these tasks than the ones we commonly use, and have been using since time immemorial? Have a look at one of your shoes. If it is laced, then the lacing you see is almost certainly either a *crisscross lacing* or a *straight lacing*; see Figures 1 and 2. In the following, we identify the shortest, as well as the strongest, among all possible lacings of idealised shoes. It may come as a surprise that the shortest lacings are not among the lacings commonly used. On the other hand, you may be relieved to find that the crisscross and straight lacings are indeed the strongest lacings.



Figures 1–9: (1) the crisscross 5-lacing; (2) a straight 5-lacing; (3) the 5-lacing of maximal length; (4) a lacing used by the Canadian armed forces; (5) a generic 5-lacing; (6) a bowtie 4-lacing; (7, 8) two bowtie 5-lacings; (9) the three building blocks for bowtie lacings.

The *2n* eyelets of an idealised shoe are the points of intersection of two vertical lines and *n* equally spaced horizontal lines in the plane. The two columns of eyelets are one unit apart, and two adjacent rows of eyelets are a positive distance *h* apart. An *n-lacing* of our shoe is a closed path in the plane consisting of *2n* line segments whose end points are the *2n* eyelets. Furthermore, we require that, given any eyelet *e*, at least one of the two segments ending in it be not contained in the same column as *e*. This condition ensures that every eyelet genuinely contributes towards pulling the two sides of the shoe together. Virtually all lacings that are actually used satisfy this property.

We call a lacing *dense* if, given any eyelet *e*, none of the two segments ending in it is contained in the same column as *e*; that is, a dense lacing zigzags back and forth between the two columns of eyelets. Figures 1–8 show a number of 5-lacings and one 4-lacing. Among these lacings, only the first three are dense. Many lacings that are used for display purposes are not dense. Also, as was noted in [5], the use of the kind of non-dense lacing shown in Figure 4 used to be standard practice in the Canadian armed forces.

In the following, we call a segment of a lacing a *vertical* if its end points are both contained in the same column, and, otherwise, we call the segment a *diagonal*. Furthermore, we define the *length* of an *n*-lacing to be the sum of the lengths of the segments it consists of. Finally, we assume that *n* is at least 2.

Using standard combinatorial techniques, we find that the number of *n*-lacings is

$$\frac{(n!)^2}{2}\sum_{k=0}^{m}\frac{1}{n-k}\binom{n-k}{k}^2,$$

where $m=n/2$ for even *n*, and $m=(n-1)/2$ for odd *n*. Furthermore, the number of dense *n*-lacings is

$$\frac{n!(n-1)!}{2}.$$

So, for example, there are 382,838,400 different 7-lacings. But, which are the "best" among these 7-lacings?

The Shortest Lacings

In [1], Halton proved that the crisscross *n*-lacing is the shortest among all those dense *n*-lacings where a horizontal segment connects the top pair of eyelets–the most interesting case, because this is where shoes are usually tied. Later, in [2], Misiurewicz gave a very short proof of this result. In fact, he showed that this result stays true even if the eyelets are not fully aligned.

Using the symmetries of the configuration of eyelets, it is possible to design a powerful list of local shortening rules and use these to identify the so-called *bowtie n-lacings* (defined below) as the shortest *n*-lacings. Furthermore, generalising Holton's and Misiurewicz's results, we can show that the crisscross *n*-lacing is the shortest dense *n*-lacing (even if the eyelets are not fully aligned, just as in Misiurewicz's paper).

The bowtie *n*-lacings are composed of three different types of building blocks called an *end*, a *cross*, and a *gap*; see Figure 9. An end is a horizontal segment connecting either the top pair or bottom pair of eyelets. This means that a lacing contains at most two ends. For example, the crisscross *n*-lacing can be partitioned into two ends and $n–1$ crosses. If n is even, the maximal number of gaps in an *n*-lacing is $n/2$. If n is odd, the maximal number of gaps is $(n–1)/2$. Now, a bowtie *n*-lacing is an *n*-lacing that can be partitioned into crosses, the maximal number of gaps, and two ends. It is an easy exercise to check that if n is even, there is exactly one bowtie *n*-lacing, and if n is odd, there are exactly $(n+1)/2$ different bowtie *n*-lacings. As an example, Figure 6 shows the unique bowtie 4-lacing, while Figures 7 and 8 show two bowtie 5-lacings.

The third bowtie 5-lacing (3=(5+1)/2) is the horizontal mirror image of the bowtie 5-lacing in Figure 8.

We only remark that it is also possible to identify the longest dense *n*-lacings for general *n*. For example, Figure 3 shows the longest dense 5-lacing.

The Strongest Lacing

When you pull on the ends of a shoelace, it acts like a pulley. We are interested in finding out which lacings are the best pulleys. Let's focus on what we mean by this.

Ideally, the tension along the shoelace is a positive constant T. This tension gives rise to a total tension T_h of the pulley in the horizontal direction; that is, the direction in which the two sides of the shoe are being pulled together. This total tension T_h is the sum of all horizontal components of T along the different segments of the lacing. Then, a strongest *n*-lacing is an *n*-lacing that maximises T_h.

Clearly, if we are dealing with a vertical segment, the horizontal component of T corresponding to this segment is 0, and if we are dealing with a diagonal segment of length l, then this component is T/l (recalling that the distance between the two columns of eyelets is 1).

Given an *n*-lacing that contains verticals, it is always possible to find an *n*-lacing that is stronger by replacing two verticals contained in different columns by suitably chosen diagonals. This implies that any strongest *n*-lacing must be dense. In that case, if $1_1, 1_2,\ldots,1_{2n}$ denote the lengths of the different segments in a dense *n*-lacing, what we wish to maximise is the sum

$$\sum_{i=1}^{2n} \frac{1}{l_i}.$$

Compare this to the problem of determining the shortest dense *n*-lacing, where the object is to minimise the sum

$$\sum_{i=1}^{2n} l_i .$$

Clearly, we expect the solutions to both problems to consist of short segments, because small l_i values give rise to a strong lacing in the first case, and to a short lacing in the second. Indeed, we have already convinced ourselves that this is exactly what happens in the second case.

The shortest dense *n*-lacing is independent of the distance *h* between two adjacent rows of eyelets. In contrast, the strongest *n*-lacing does depend on *h*. Of course, the unique dense 2-lacing is the strongest 2-lacing, so let $n>2$. Then, we can prove that there is a positive value h_n, such that the strongest *n*-lacings are: (1) the crisscross *n*-lacing, for $h<h_n$; (2) the crisscross *n*-lacing and the straight *n*-lacings, for $h=h_n$; and (3) the straight *n*-lacings, for $h>h_n$.

Table 1 lists approximate values of h_n for the first few *n*. What is interesting is that for many real shoes with *n* pairs of eyelets, the ratio of the distance between adjacent rows of eyelets and the distance between the columns is very close to h_n. This means that no matter whether you prefer to lace straight or crisscross, you get close to maximising the total horizontal tension when you pull on the two ends of one of your shoelaces.

n	3	4	5	6	7	8	9	10
h_n	.9029	.7412	.6450	.3794	.3309	.4931	.4625	.4372

Table 1: The vertical separation h_n for which the crisscross *n*-lacing is as strong as the straight *n*-lacing.

How to Tie the Knot

We've looked at the strongest way of *lacing* shoes. What about the strongest way of *tying* the laces? A moment's thought will yield the following answer; see also [3]. Most people place one *half-granny* knot on top of another to tie a shoelace (let's not worry about the loops as they are not essential here). This results in either a *granny knot* or a *reef knot*, depending on whether the two half-knots have the same or opposite orientations. The granny knot is notoriously unstable and should be avoided, whereas the reef knot is very stable. As we have seen, thousands of years of trial and error have resulted in our using the strongest ways to lace our shoes. Unfortunately, the same cannot be said about the way we tie our shoelaces as many, if not the majority, of people use granny knots to perform this task.

References

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