

Strike-me-out: a proof

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Problem 1. *Take the sequence of positive integers starting from 1. Then strike out every third element, and make a cumulative sum of the remaining sequence:*

$$\begin{array}{cccccccccccc} 1 & 2 & \cancel{3} & 4 & 5 & \cancel{6} & 7 & 8 & \cancel{9} & 10 & \dots \\ \sum : & 1 & 3 & & 7 & 12 & & 19 & 27 & & 37 & \dots \end{array}$$

Now strike out every second element of the new sequence, and make a cumulative sum of the remaining sequence:

$$\begin{array}{cccccccc} & 1 & \cancel{3} & 7 & \cancel{12} & 19 & \cancel{27} & 37 & \dots \\ \sum : & 1 & & 8 & & 27 & & 64 & \dots \end{array}$$

The result is the cubes.

Proof. In the second step we struck out every second element of the cumulative sum. Looking back at the original sequence, the sequence $\{1, 7, 19, 37\}$ is actually

$$\begin{aligned} 1 &= 1 \\ 7 &= 1 + (2 + 4) \\ 19 &= 1 + (2 + 4) + (5 + 7) \\ 37 &= 1 + (2 + 4) + (5 + 7) + (8 + 10) \\ &\dots \end{aligned}$$

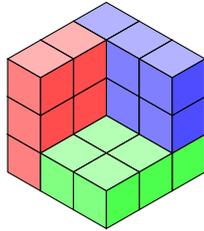
Ignoring the initial 1, each new term that is added is a sum of numbers either side of a multiple of 3, so 2 times that multiple of 3. In other words,

$$\begin{aligned} 1 &= 1 \\ 7 &= 1 + 3 \times 2 \times (1) \\ 19 &= 1 + 3 \times 2 \times (1 + 2) \\ 37 &= 1 + 3 \times 2 \times (1 + 2 + 3) \\ &\dots \end{aligned}$$

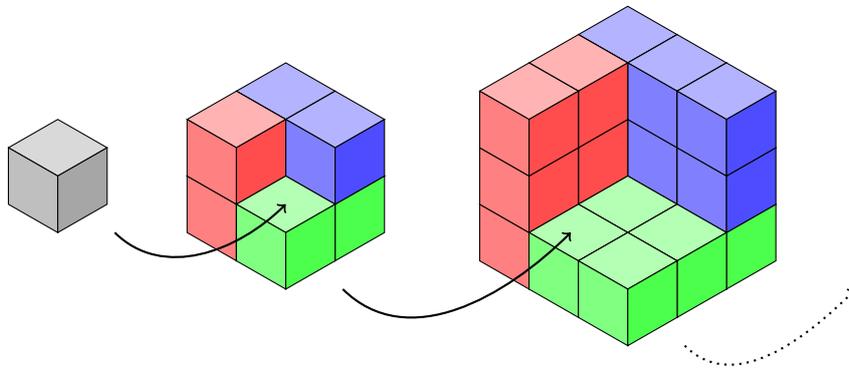
In the brackets we have the triangular numbers, which when multiplied by 2 give a rectangle:

$$\begin{aligned} 1 &= 1 \\ 7 &= 1 + 3 \times (2 \times 1) \\ 19 &= 1 + 3 \times (3 \times 2) \\ 37 &= 1 + 3 \times (4 \times 3) \\ &\dots \end{aligned}$$

Now each of these numbers is 1 plus 3 rectangles, which can be thought of as small cubes arranged in the following way (the 1 is in the hidden corner):



Thus, summing these numbers is constructing a cube as follows:



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