MATHS CHALLENGE SOLUTIONS – PART 2

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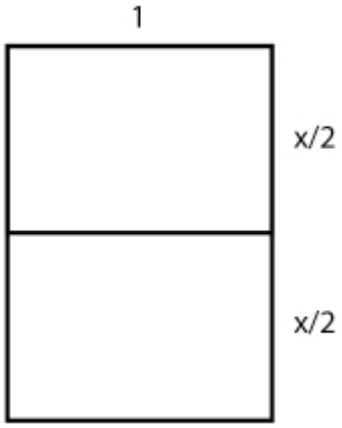
21. Wanted: a rectangle that when cut in half leaves two rectangles of the same shape as the original. What are its proportions? Have you ever seen a rectangle like this?

Anwer: The large rectangle has width 1 and length x, and is to have the same proportions as the two half-triangles. From the diagram we see that $x/1 = 1/(x/2)$. Solving gives $x^2 = 2$, and so $x = \sqrt{2}$.



Therefore the ratio of our mystery piece of paper is $\sqrt{2}$ to 1. A4 paper and other standard paper sizes have these proportions. The pattern starts with A0 paper, chosen to have an area of 1 square meter. We'll leave it for you to work out the dimensions of A0 paper, and the area and dimensions of A4 paper.

22. This square array of numbers is very special. Why?

96	11	89	68
88	69	91	16
61	86	18	99
19	98	66	81

Answer: It's a magic square, and is also a magic square upside down. In both cases all rows, columns and diagonals sum to 264.

23. 0.1818181818… , 0.2727272727… , 0.3636363636…

What's the next number? What do all these numbers have in common? Of course there is an obvious answer, but can you see what lies underneath?

Answer: All are fractions with denominator 11. To see this, consider the first recurring decimal 0.1818181818… . Calling the decimal *N*, then

$100N$ = 18.18181818… = 18 + N.

Subtracting, we see $99N$=18, giving $N = 18/99 = 2/11$.

This is the simplest approach to analysing any repeating decimal. See also Problem 40.

24. What is this number?

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}}$$

Answer: Set *M* = the monster. Squaring both sides gives $M^2 = 2 +$ monster $= 2 + M$. This equation has solutions 2 and –1. If the monster really makes sense as a number (which it does) then it is clearly positive, and so it must be 2.

25. In the movie *Die Hard: With a Vengeance*, Bruce Willis and Samuel L. Jackson are presented with a 5 gallon jug and a 3 gallon jug and a water fountain. They are then given 30 seconds to fill the larger jug with exactly 4 gallons of water, before they'll be blown to smithereens. Your task is to find a solution (your 30 seconds starts now!).

If instead Bruce and Sam start with a 15 gallon jug and a 6 gallon jug, and are again after exactly 4 gallons of water in one of the jugs, can they still do it? Under what conditions can you solve this sort of puzzle?

Answer: Fill the 5-Jug, move 3 gallons to the 3-Jug and empty the 3-Jug. Move the remaining 2 gallons from the 5-Jug to the 3-Jug. Fill the 5-Jug again, and use 1 gallon to fill the 3-Jug, leaving 4 gallons in the 5-Jug.

The second problem is impossible: since both jug volumes are multiples of 3, any water amount we can produce must also be a multiple of 3. This also gives the clue as to when such problems can be solved. For details see http://www.qedcat.com/mathsnacks/perfect_puzzles.html, and our column *Gallons of Strife* on http://www.qedcat.com/column/index.html.

26. What symbol comes next?

Answer: Every symbol is a digit and its mirror image glued together. Next comes 8, glued to itself.

27. There are two jugs of the same size, one containing water and one containing wine. A cup of water is taken from the first jug and added to the wine. Then, a cup of the mixture is taken from the second jug and added to the water. Is there now more water in the wine or wine in the water?

Answer: There is the same amount of liquid in each jug. So whatever amount of water is in the wine jug, that amount of wine is missing and must be in the water jug. For a very nice discussion of the ways to think about this problem, see http://www.geocities.com/CapitolHill/Lobby/7049/wine.htm

28. In how many different ways can you make change for a dollar using 5, 10, 20 and 50 cent coins?

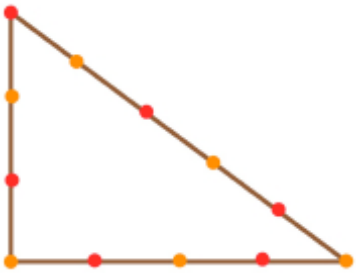
Answer: There are 49 ways of giving change. The easiest approach to listing the possibilities is to focus first on how many 50 cent pieces and how many 20 cent pieces are used. This gives the cases 50-50, 50-20-20, 50-20, 50, 20-20-20-20-20, 20-20-20-20, 20-20-20, 20-20, 20, and "none". For each of these cases, the number of possibilities is the number of 10 cent pieces which can then be included, plus 1. So, the number of possibilities in each case is: 1, 2, 4, 6, 1, 3, 5, 7, 9, 11. These sum to 49.

29. In the movie *Fermat's Room*, four potential victims have to solve the following problem, or die a horrible death: How can you measure exactly 9 minutes using two hourglasses, one a 4 minute hourglass and the other a 7 minute hourglass? The hourglasses can be turned over at any stage, but always have to be running: you cannot lay them on their sides.

Answer: Start both hourglasses. When the 4-glass stops, turn it over. When the 7-glass stops, there is one minute left in the 4-glass. Turn the 7-glass over. When the 4-glass runs out, 8 minutes have passed and 1 minute has run in the 7-glass. Turn the 7-glass over.

30. Your friend has a necklace with 12 equally spaced beads. Impress her by using her necklace to make a perfect right angle.

Answer: Form a 3–4–5 triangle.



31. At least one of John's two children is a girl. What are the chances that they are both girls?

Answer: The four possible outcomes of a two–kids "experiment" are

Boy Boy, Boy Girl, Girl Boy, Girl Girl.

The last three outcomes contain "at least one girl". Therefore the chances of John having two girls is 1/3.

32. Starting with a rectangular bar of chocolate you and your chocoholic friend take turns breaking the bar along the lines separating the squares. You keep breaking pieces until only individual squares are left. The person who makes the last break gets to eat all the chocolate. Who wins?

Answer: If there are *N* squares of chocolate then it will take *N-1* breaks to finally create these *N* pieces. So, If there is an even number of squares then the person who breaks first wins, and otherwise the second person wins. For details see

http://www.qedcat.com/mathsnacks/chocoholic.html

33. Three playing cards are placed in a row. To the right of a King there is a Queen or two Queens. To the left of a Queen there is a Queen or two Queens. To the left of a Heart there is a Spade or two Spades. To the right of a Spade there is a Spade or two Spades. What are the three cards?

Answer: The three cards are King of Spades, Queen of Spades and Queen of Hearts. The information indicates there are at least two queens, and at least two spades, from which the precise cards follows.

34. What number is this?

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}}$$

Answer: Let M = monster. Then $M = 1 + 1/M$. Multiplying both sides by M gives a quadratic equation. The only positive solution to this equation is $(1 + \sqrt{5}\,)/2$, the golden ratio.

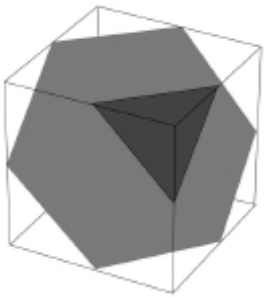
Infinitely nested fractions such as these are known as *continued fractions*. They are a little out of fashion now but were once a very active area of research. For studying irrational numbers, continued fractions are in fact much more useful than decimal expansions. For example, in 1768 Johann Lambert used them to give the first proof, and probably the simplest proof, that π is irrational.

35. A boat can travel under its own power at 20 kilometers per hour. Wanda takes the boat downstream, with a river flowing at 10 kilometers per hour, and then she brings the boat back upstream again. What is the average speed of the boat for the trip?

Answer: Wanda travels at 30 km/hour downstream and 10 km/hour upstream. So, if the stretch of river Wanda travels on is L km long, then it takes her $L/30$ hours to go downstream and $L/10$ hours to return upstream. This means that the average speed over the whole trip is $2L/(L/30 + L/10) = 15$ (km/h). Note that since the distance L cancels out, the answer does not depend upon how far Wanda travels down the river.

36. It is easy to slice a cube with a flat plane to expose an equilateral triangular face. What about a square? A regular pentagon? A regular hexagon?

Answer: All except the regular pentagon are possible. Squares are easy. The diagram shows where equilateral triangles and regular hexagons are hiding. For some nice class materials on this, see http://www.schoolmath3d.org/e/teacher/unit01/.



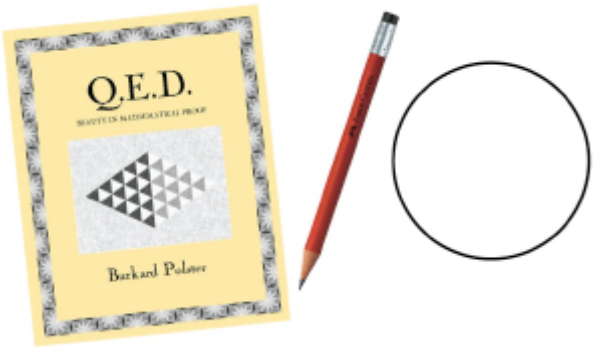
37. Your pocket contains 36 coins, and you empty your pocket onto a table. What are the chances that an even number of heads comes up?

Answer: No matter how many coins you toss there is always a 50-50 chance of an even number of heads coming up. One way to see this is to imagine tossing up all but one of the coins. Whether the number of heads at that stage is even or odd, there is a 50-50 chance of the final coin changing the number of heads, and thus a 50-50 chance that the final is even.

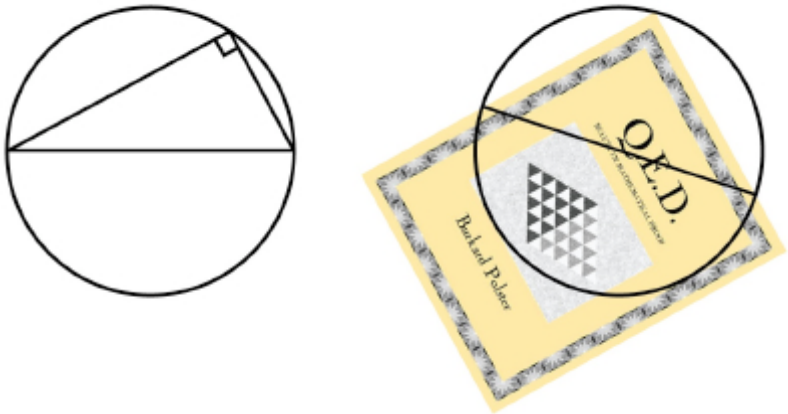
38. You receive three boxes of lollies. One contains only mints, the second only chocolates, and the third contains a mixture of the two. The boxes are meant to have labels identifying the contents, but the labels have been mixed up and each box is now incorrectly labeled. What is the minimum number of lollies you need to sample to decide which box is which?

Answer: It is important to note that the boxes definitely do *not* carry their proper labels. Then, choosing one lolly from the box labeled Mixed will suffice. Suppose the lolly chosen is a mint (we can give an exactly analogous argument if a chocolate is chosen). Then we know that this lolly is not from the Mixed box, so it must be from the Mint box (labeled Mixed). Then, since the chocolates cannot be in the box labeled Chocolates, they must be in the box label Mints. That leaves the mixed lollies in the third box, labeled Chocolates.

39. You find a circle drawn on the floor and your life depends on finding the exact centre of this circle. The only tools you have are a book and a pencil. How can you do this?



Answer: This is a neat application of what is known as Thales's theorem. Set the corner of the book at a point on the circumference of the circle, and draw a right angle using the edges of the book. The two points at which the right angle intersects the circle mark the endpoints of a diameter, which you can draw in using the book. Draw a second diameter in the same way. The point of intersection of the two diameters is the center of the circle.



40. What is the following mystery number?

3.142857142857142857…

Answer: 3.142857142857142857… = 22/7: anyone who answered π fails immediately! This problem can be approached the same as problem 23. If we call the decimal 3 + *N*, then 100000*N* = 142857 + *N*. Solving for *N* gives 3 + *N* = 3 + 142857/999999, which reduces to 3 1/7.