√2 is Irrational

Burkard Polster and Marty Ross

We want to prove that $\sqrt{2}$ is irrational. That is, we want to show that it is impossible to write $\sqrt{2} = N/M$, with N/M whole numbers. First of all, if we square both sides and multiply by M^2, then what we're doing is showing that there is no positive whole number solution to the equation
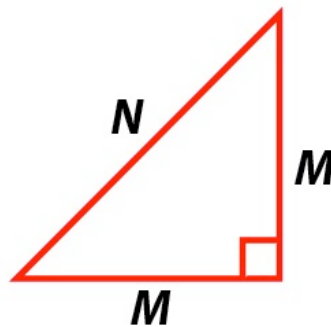
$$2M^2 = N^2.$$

We'll give two proofs of this, one geometric and one numerical. At the end, we give some references, for these and other proofs.

Geometric Proof

This proof uses what is called *the impossibility of infinite descent*. Suppose that we start with some positive whole number N, and then we have another positive whole number P, but with $N > P$. And then suppose we have Q, with $N > P > Q$, and so on. The impossibility of infinite descent says that this sequence cannot go on forever: since all the whole numbers are positive, we can only step down a finite number of times before we hit 1.

Now, what we do is show that if we have a whole number solution to $2M^2 = N^2$ then this will start off an *infinite* descending sequence of positive whole numbers. Since this is impossible, the fractional equation must also be impossible, and that means $\sqrt{2}$ is irrational.

To begin, we think of the equation in terms of a triangle:

By Pythagoras' Theorem, solving $2M^2 = N^2$ is the same as finding an isosceles right-angled triangle with hypotenuse N and shorter sides M. What we do is show that any time we have such a triangle, with M and N whole numbers, then we can always construct a smaller isosceles right-angled triangle, still with whole number sides. So, we could repeat the construction to give a still smaller triangle, and a smaller triangle again, and so on. That gives us the infinite descent, which we know is impossible. And, since the chain of triangles is impossible, the chain could never have started: we can conclude that there could never have been the first triangle to begin with.

So, supposing we have an isosceles right-angled triangle with whole number sides, we have to show how to construct the smaller triangle. To do this, consider the following diagram. The circular arc shows how to position the new edge so that the small triangle is right-angled. And, since a second angle is 45°, the third angle must be as well, and so the small triangle is isosceles.
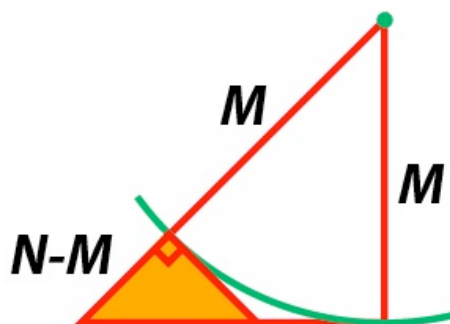
We just have to show that this small triangle has whole number sides. The circular arc shows that the smaller side has length $N - M$. To figure out the length of the hypotenuse, we just have to note that the three segments highlighted in the diagram below all have this same length $N - M$.
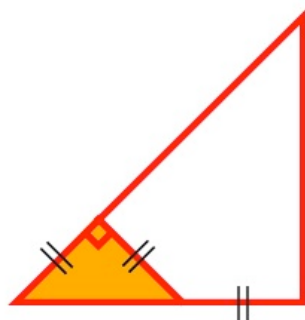
So, the small triangle must have hypotenuse length $M - (N{-}M)$, which equals $2M{-}N$, definitely a whole number.

There is also a purely numerical version of the above proof. The geometric proof amounts to showing that if $\sqrt{2} = N/M$ then also $\sqrt{2} = (2M - N)/(N - M)$. To show this new fractional representation directly, we can calculate

$$\sqrt{2} = \frac{\sqrt{2}(\sqrt{2} - 1)}{\sqrt{2} - 1} = \frac{2 - \sqrt{2}}{\sqrt{2} - 1} = \frac{2 - \frac{N}{M}}{\frac{N}{M} - 1} = \frac{2M - N}{N - M}$$

Numerical Proof

This proof uses the idea of writing whole numbers in base three instead of our usual base ten. As an illustration, the number fifteen is normally written as 15, amounting to 1 x 10 + 5 x 1. In base three, this same number would be written as 120, amounting to 1 x 9 + 2 x 3 + 0 x 1.

The important thing to note is: *written in base three, the first non-zero digit of a square number is always a 1*. For example, the square of fifteen is two hundred and twenty-five, which we would normally write as $15^2 = 225$. However, in base three, this is written as $120^2 = 22100$, with 22100 coming from 2 x 81 + 2 x 27 + 1 x 9 + 0 x 3 + 0 x 1. In this example, the 1 in the "nines place" is the first non-zero digit.

This fact about squares written in base three follows easily from the base three products 1 x 1 = 1 and 2 x 2 = 11, combined with the normal rules for multiplication.

With that background, the proof is now really easy. We want to show that the equation $N^2 = 2M^2$ is impossible for whole numbers N and M. But suppose now that M and N are written in base three. Then M^2 and N^2 both end in a 1. That means $2M^2$ ends in a 2, and so can't possibly equal N^2.

References

The geometric proof has been written up by Tom Apostol in *American Mathematical Monthly* **107** (2000), pp841-842. The proof seems to have been known at least a century earlier.

John Conway and Richard Guy present an origami version of Apostol's proof in *The Book of Numbers* (Springer, 1995, p183).

The numerical version of Apostol's proof appears in *A Course of Modern Analysis* by E. T. Whittaker and G. N. Watson (Cambridge, 4th ed. 1927, p5)

The base three numerical proof appears in *American Mathematical Monthly* **63** (1956), p247. It is due to Robert Gauntt, who was an undergraduate at the time.

Many other proofs can be found at http://www.cut-the-knot.org/proofs/sq_root.shtml.