e is not a quadratic irrational

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We prove here that e is not the root of a non-zero quadratic equation with integer coefficients.[1] We begin with the well known series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^m}{m!} + \cdots . \tag{1}$$

Setting $x = 1$ then gives

$$e = e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} + \cdots . \tag{2}$$

1 e is irrational

We'll first use (2) to give the familiar proof that e is irrational. Assume, by way of contradiction, that $e = \frac{a}{b}$ with a and b positive integers. Using b to determine a cut-off, (2) gives

$$\begin{cases} \dfrac{a}{b} = e = \left[1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{b!}\right] + \dfrac{\text{SMALL}}{b!}\,, \\[2ex] \text{SMALL} = \dfrac{b!}{(b+1)!} + \dfrac{b!}{(b+2)!} + \dfrac{b!}{(b+3)!} + \cdots . \end{cases} \tag{3}$$

Clearly *SMALL* is positive, and cancelling out the $b!$ with the denominators, we have

$$\text{SMALL} = \frac{1}{(b+1)} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots \ < \ \frac{1}{(b+1)} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \cdots .$$

The latter sum is an infinite geometric series, which sums to $\frac{1}{b+1}/\left(1 - \frac{1}{b+1}\right) = \frac{1}{b}$. So,

$$0 < \text{SMALL} < \frac{1}{b} \leqslant 1\,. \tag{4}$$

Now, multiplying (3) by $b!$, we have

$$\text{INTEGER} \ = \ \text{INTEGER} \ + \ \text{SMALL}$$

But by (4), SMALL is strictly between 0 and 1, which is a contradiction.

[1] We're fleshing out here the details of Conway's and Guy's sketch-proof in *The Book of Numbers*, p 253 (Copernicus, 1998).

2 e is not a quadratic irrational

We'll now show that e cannot solve the equation

$$a - be + ce^2 = 0 \tag{5}$$

with a, b, c integers, not all 0. Rearranging, (5) implies

$$\frac{a}{e} + ce = b\,. \tag{6}$$

To show (6) is impossible, we'll write e and $\frac{1}{e}$ as almost-fractions. For a positive integer m to be chosen later, we first use (3) and (4) to write

$$\begin{cases} e = \dfrac{\text{INTEGER}}{m!} + \dfrac{\text{SMALL}}{m!}\,, \\ 0 < \cdot\text{SMALL} < \dfrac{1}{m}\,. \end{cases} \tag{7}$$

Next, we need a similar expression for $\frac{1}{e}$, though in this case the *small* error will alternate in sign. This expression comes from first setting $x = -1$ in (1), giving

$$\frac{1}{e} = e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^m}{m!} + \cdots\,. \tag{8}$$

Below we use a standard alternating series calculation to prove that (8) gives

$$\begin{cases} \dfrac{1}{e} = \dfrac{\text{INTEGER}}{m!} + \dfrac{(-1)^{m+1}\text{small}}{m!}\,, \\ 0 < \cdot\text{small} < \dfrac{1}{m+1}\,. \end{cases} \tag{9}$$

Substituting (7) and (9) into (6), multiplying by $m!$ we find

$$\text{INTEGER} + \left[c \cdot \text{SMALL} + (-1)^{m+1} a \cdot \text{small}\right] = \text{INTEGER}\,. \tag{10}$$

Clearly we can make the magnitude of the small stuff less than 1 by choosing m large. So, as long as all the small stuff doesn't cancel to 0, (10) gives a contradiction. But the non-cancellation is easy to ensure. First, if one of $a = 0$ or $c = 0$ then the small stuff is automatically non-zero. Otherwise, we simply choose m odd if a and c have the same sign, and m even if a and c have opposite signs.

Finally, we show how (8) leads to (9). Stopping the series at the mth term, (8) gives

$$\frac{1}{e} = \left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^m}{m!}\right] + \frac{(-1)^{m+1}\text{small}}{m!}$$

where

$$\text{small} = \frac{1}{(m+1)} - \frac{1}{(m+1)(m+2)} + \frac{1}{(m+1)(m+2)(m+3)} - \cdots\,.$$

Now we just have to note that the alternating terms in *small* are strictly decreasing in size. So, grouping in pairs,

$$\text{small} > \left[\frac{1}{(m+1)} - \frac{1}{(m+1)(m+2)}\right] + [*** - ****] + \cdots \;>\; 0\,.$$

Similarly, splitting off the first term and then grouping in pairs,

$$\text{small} < \frac{1}{(m+1)} - \left[\frac{1}{(m+1)(m+2)} - \frac{1}{(m+1)(m+2)(m+3)}\right] - [**** - *****] + \cdots \;<\; \frac{1}{(m+1)}\,.$$

Done.