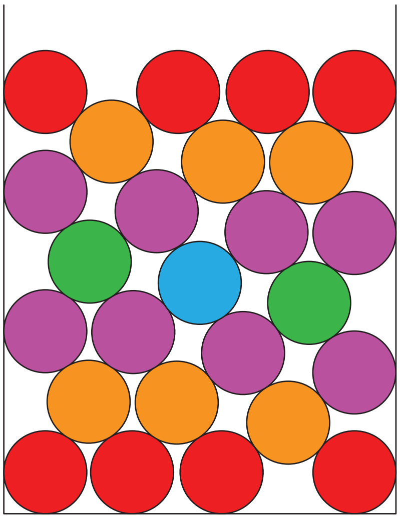
Stacking wine bottles revisted

Burkard Polster, burkard.polster@monash.edu



Here are 25 bottles in a wine rack. We've placed the four bottles at the bottom so that the two on the far left and right touch the vertical sides. Then we stacked the remaining bottles row by row as shown.

Since the bottles at the bottom are not equally spaced the other rows end up not being level, . . . except for the top seventh row. Surprisingly, this will always be the case, no matter where we place the middle two bottles in the bottom row.

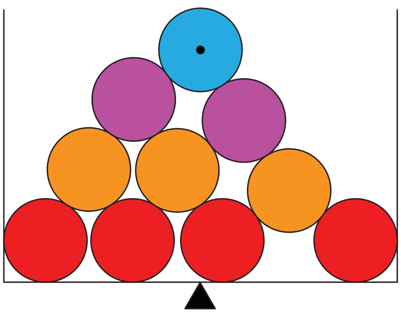
More generally, if bottles are stacked like this, starting with n bottles in the bottom row that are not too widely spaced, then the $2n$-1st row of bottles will be level.

This pretty theorem was discovered by Charles Payan in 1989. The n=3 case featured as problems in Velleman and Wagon's fabulous collection of math club problems *Which way did the bicycle go?* (1996) and in *Honsberger's Mathematical Diamonds* (2003). The *Cut-the-Knot* website has a number of pages with very nice interactive applets dedicated to this theorem; look for the page entitled *More bottles in a wine rack and follow the links on this page*. Also part of these pages are proofs of the general theorem and some generalizations by Nathan Bowler. The only other published proof I am aware of appears in a post by David Robbins to the Math Forum under the topic *Stacking bottles in a crate*.

None of the published proofs make for light reading and one of the aims in this note is to present proofs of the main results mentioned on *Cut-the-knot* that should be very close to those "in the book", proofs that I hope are easy to read and worthy of such pretty results. We'll also be considering a number of natural generalizations of the bottle stacking problem.

Here is the plan. In order to not get bogged down in notation, at least to start with we'll always restrict ourselves to discussing the case of four bottles in the bottom row and seven rows, as in our example. This case features all arguments that also take care of the general case.

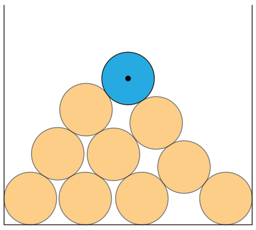
We first show that all our stacks have a half-turn symmetry. This symmetry is apparent in our sample stack – the picture of our stack consisting of seven rows stays unchanged when you turn it 180 degrees. Since a half-turn like this turns the level bottom row into the seventh row it follows that the seventh row has to be level, too, which is what we want to show.



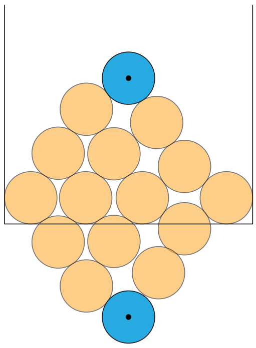
Our discussion starts out with an idea that Adam Brown used to prove a closely related result: If we stack bottles in a pyramid, as shown on the left, the center of the blue bottle at the top is exactly halfways between the two sides of the rack. (A. Brown, A circle stacking theorem, Math. Magazine 76 (2003), 301-302.)

Here we go...

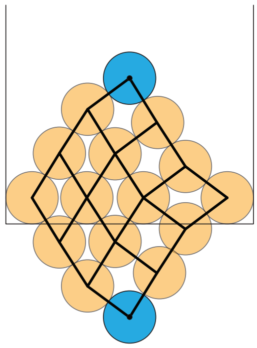
Stacks have a half-turn symmetry



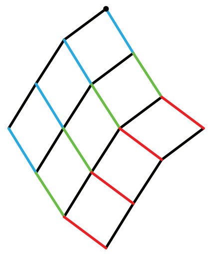
Consider the pyramid of bottles sitting on top of the bottom row.



Build a second pyramid pointing down. It is the horizontal mirror image of the one pointing up.

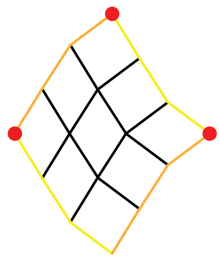


Connect the centers of touching circles as shown. All connections have the same length and form nine rhombi.

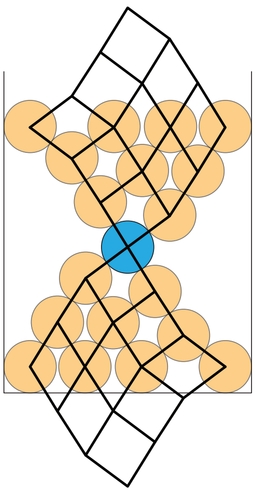


The three blue edges are parallel. And so are the three green and red ones.

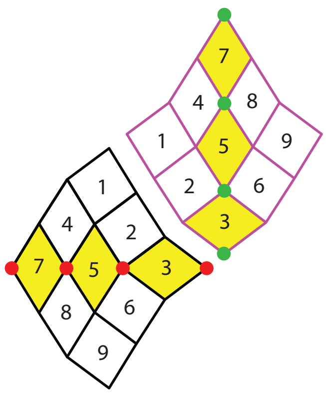
Hence the two yellow sides of the mesh are translates of each other.



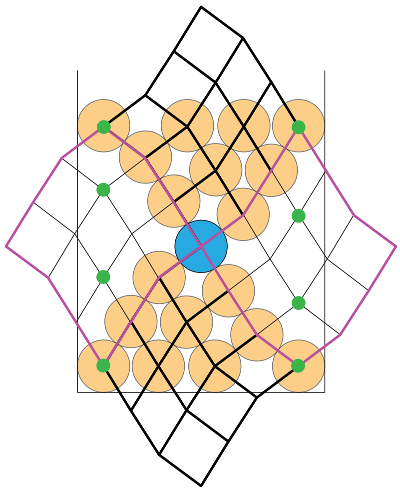
Since the orange sides are horizonal mirror images of the yellow sides it follows that the three red points form an isosceles triangle. **Hence the blue circle is halfways between the two sides of the rack.**



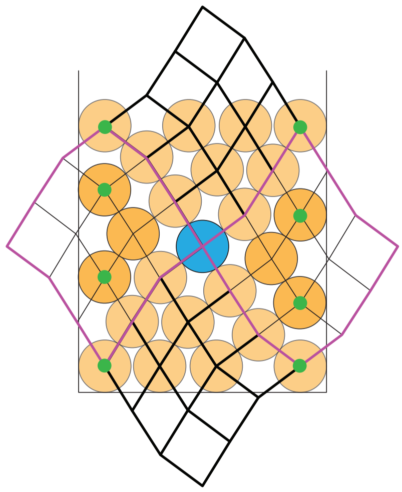
Make a copy of the pyramid, turn it 180 degrees and make its tip coincide with that of the first pyramid. Because of the halfways property of the tip it fits into the rack.



Reassemble the nine rhombi of the mesh into a second mesh as shown. Here corresponding rhombi in the two meshes end up being translates of each other. Therefore, since the red points are horizonally aligned, the green points are vertically aligned.

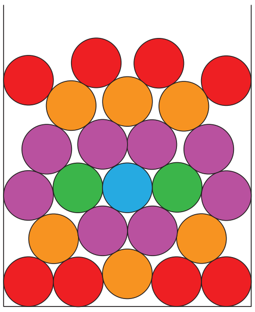


The new mesh and a half-turned copy of this new mesh seamlessly fill the gaps between the two other meshes.



Adding circles centered at the vertices of the new meshes clearly reconstructs the stack of wine bottles we started with and shows that it has the half-turn property.

Stacks have a half-turn symmetry unless . . .

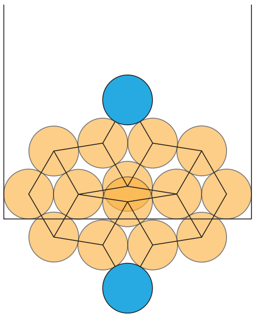


Sometimes in wine racks that allow almost five bottles to be placed in the bottom row things go wrong and the bottles in the seventh row don't line up, as in this example. Is there something wrong with our proof?

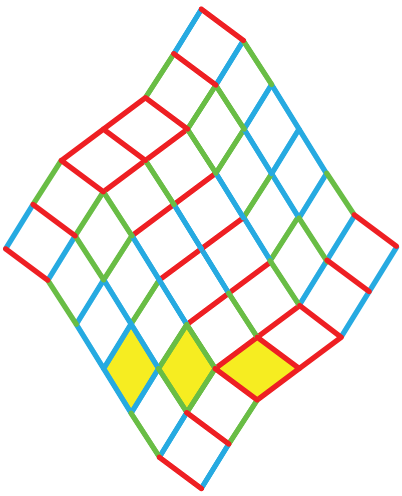
We now show that there won't be any thin or flat rhombi if our three yellow rhombi at the bottom are not flat.

From our construction it is clear that every edge in the big mesh we end up with is parallel to one of the edges the yellow rhombi as illustrated on the right.

This implies that the top angle of **any** of the rhombi will be at least as big as the smallest top angle of the yellow rhombi and won't be larger than the largest among the top angles of the yellow rhombi.

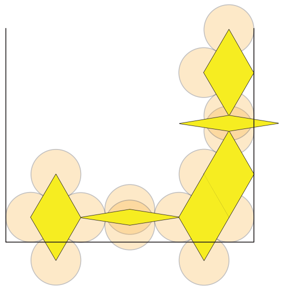


Note that in the corresponding mesh the rhombus in the middle is too flat to accomodate the four circles centered at its vertices without overlap. This is not a problem yet because the circles attached at the bottom are not part of the stack.

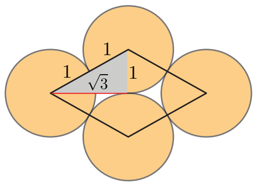


Since we started our construction with non-overlapping circles in the bottom row we can be sure that none of the yellow rhombi is thin.

Consequently, as long as none of the yellow rhombi is flat the seventh row will be level.

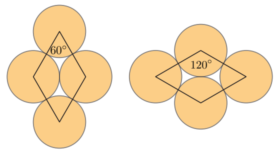


However, in our construction the rhombi corresponding to the bottom row do occur again, as highlighted above. And in its second incarnation the flat rhombus clearly causes our proof to break down.



In terms of the spacing of the circles at the bottom this can be expressed as follows: If the circles are of radius 1 there won't be any flat rhombi if the spacing of centers of adjacent circles in the bottom row is not greater than $2\sqrt{3}$.

What should also be obvious at this stage is that the absence of thin or flat rhombi in a stack is equivalent to the stack "looking like" our example, that is, the stack having the following three properties:



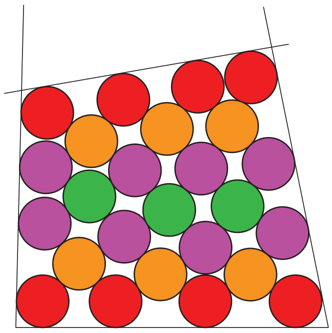
Let's call a rhombus thin if its top angle is less than 60 degrees and flat if its top angle is greater than 120 degrees.

For us to be sure that our proof works and we therefore have a level seventh row, we have to ensure that there are no thin or flat rhombi.

1) The stack naturally splits into long rows and short rows, with long rows containing one more bottle than short rows, the first row being a long row, and long and short rows alternating.
2) The two outer bottles in long rows touch the sides of the rack. No bottle in a short row touches the sides.
3) Every bottle that touches a side touches the first or last bottles in the row above and below. All other bottles touch two consecutive bottles in the row above and two consecutive bottles in the row below.

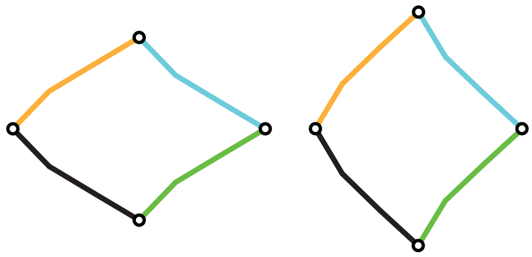
Let's call any stack, even the fancy ones coming up, **well-behaved** if they satisfy these three properties.

Racks with tilted sides

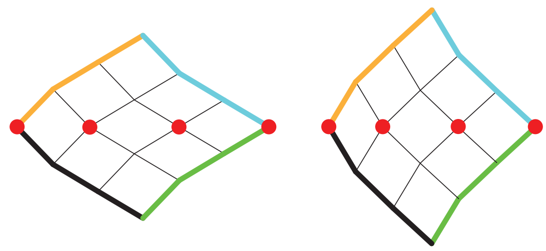


In the following we'd like to show that even if a rack has tilted sides, any well-behaved stack inside it will have a top row of bottles that is aligned.

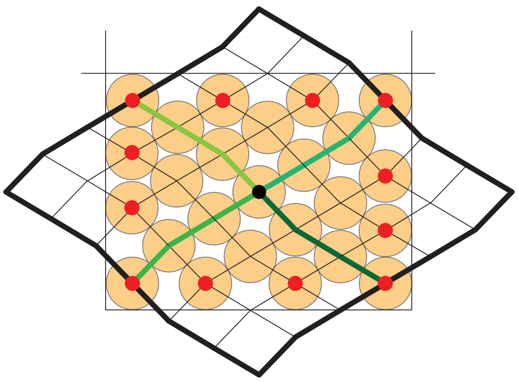
The idea for our proof is to use the same bottom row as in the tilted rack to build a stack inside a vertical rack. Then we transform this vertical stack together with its underlying meshes into the tilted stack. Then the way the meshes transform will show at a glance that the stack in the tilted rack does have the desired property.



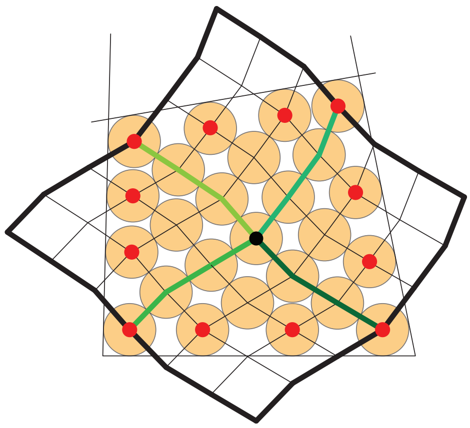
Think of the four sides of one of our meshes as a flexible frame consisting of four rigid pieces, hinged together at the corners, as indicated. It is clear that all the shapes that this frame can be flexed into have a mirror symmetry and that opposite sides of the frame will always be translates of each other.



Then it is easy to see that any of these shapes spans one of our mirror-symmetric meshes and in the following we will think of the mesh flexing together with the frame. Note also that the red vertices will always be aligned.

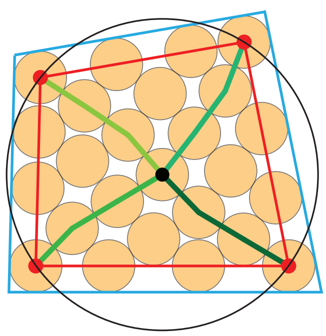


Here is our vertical stack. We are dealing with four of the special meshes glued together along the green sides. As we rotate the top two green sides around their common black end point the four meshes tranform, and so does the associ-ated stack of wine bottles and the encasing rack.



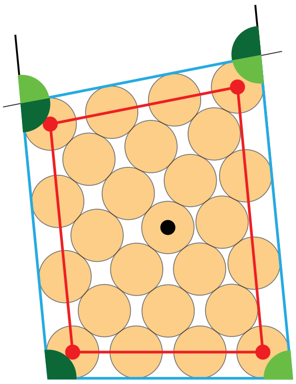
We rotate until the encasing rack is the one we are really interested in.

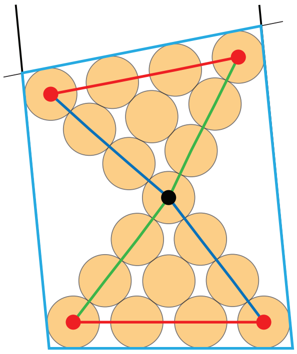
Note that is is because of the way the red points line up, we can be sure that the sides as well as the top of the stack align.



Since the four red points are at equal distance from the black point the red quadrilateral is cyclic. This means that opposite angles in this quadrilateral add up to 180 degrees. Since the sides of the blue quadrilateral are parallel to those of the red one it has the same angles as the red one. **This means that the lines across the tops of all well-haved stacks inside a give rack are parallel.**

Racks with sides that tilt at the same angle

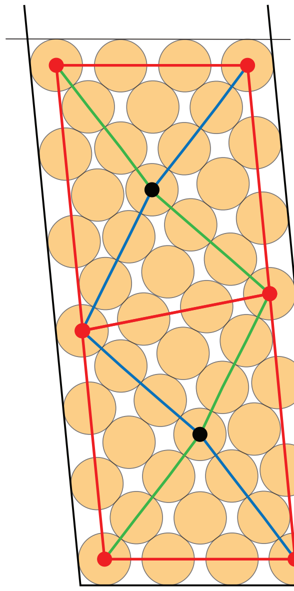




A special case of a rack with tilted sides is that of the two sides being parallel. Then the fact that opposite angles of the blue quadrilateral add up to 180 degrees implies that our stack forms an isosceles trapezium.

Also we conclude that, as in the case of racks with vertical sides the top inverted pyramid in the picture on the right is just a rotated copy of the bottom pyramid.

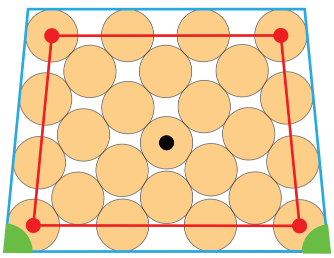
This means that if we keep stacking bottles beyond the top row things will repeat as indicated on the right and we'll eventually end up with a horizontal row of bottles. If we start with n bottles in the bottom row this horizontal row will be row $4n–3$.

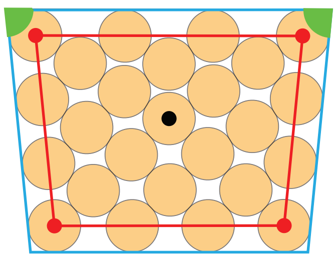


As usual all this only works if the stack we are dealing with is well-behaved.

Up to this point in our account our main contribution to the "theory" of stacking bottles has been to provide some (hopefully) easily accessible explanations for why the tops of well-behaved stacks line up.

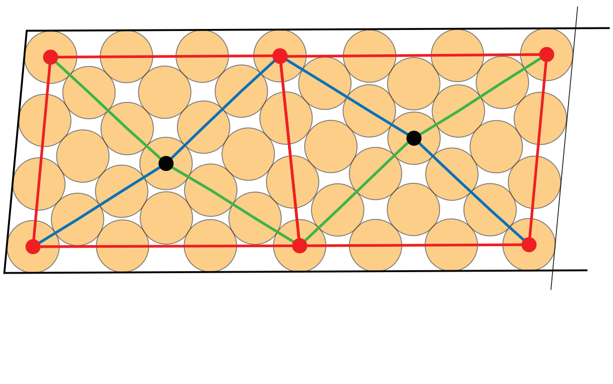
For the rest of this article let's discuss some things that have not been discussed elsewhere.





Did you notice that if you put any of our stacks on one of their sides (or upside down) you get another stack?

If we perform this trick on a stack inside a rack with parallel sides our above consideration show that we get a stack inside a rack with sides that are tilted the same angle towards each other or away from each other. Also, it is clear that the top row in any such stack is actually horizontal.

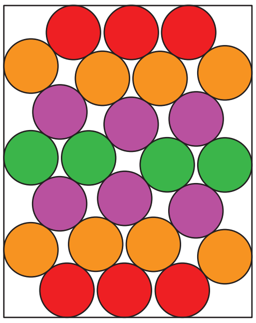


And if we perform our trick on our double stack we get something curious -- a "half-stack" inside a rack with parallel sides whose top row is horizontal. Note that if there are n bottles at the bottom the top row of this half-stack is row n.

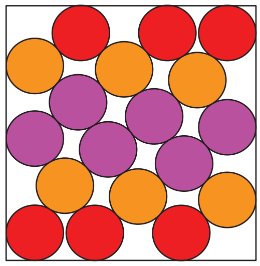
Also note that if you keep building this stack up to row $2n–1$ it will be aligned but not horizontally.

Meddling with the bottom row

In the well-behaved stacks that we've considered so far the bottles naturally split into long and short rows, with long rows containing one more bottle than short rows and long and short rows alternating. Remember that we always started with two bottles in the bottom row touching the sides of the rack. Among other things, this ensures that the bottom row is a long row.

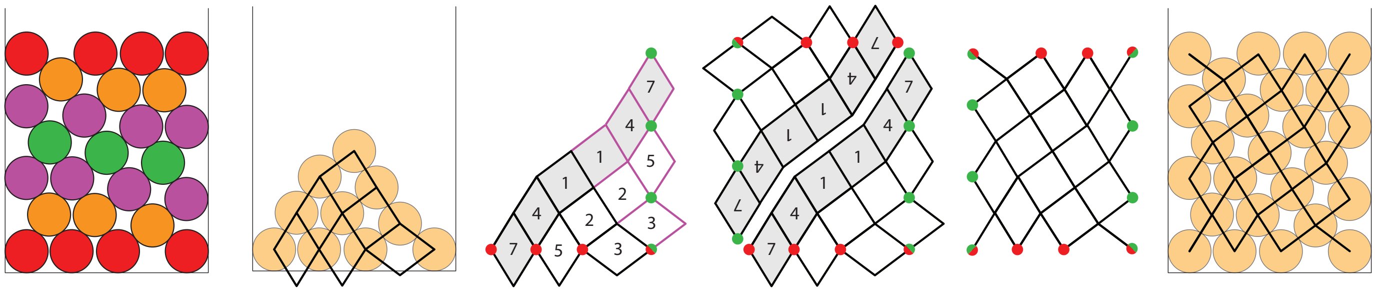


We can also build stacks in which the first row of n bottles is short with no bottle in this bottom row touching a side of the rack, as shown in the picture on the left. We'd like to show that if such a stack is well-behaved (adjust the definition of well-behaved in the obvious way), then row $2n+1$ is a level short row.



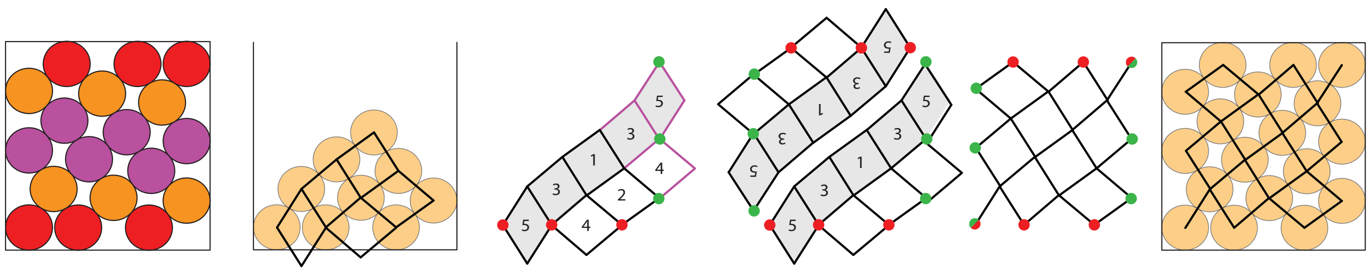
Something similar is also true for well-behaved stacks like the one shown on the right. Here all rows contain the same number of bottles and only one of the bottles in the bottom row touches a side of the rack. If we start with n bottles then row $2n$ will be level.

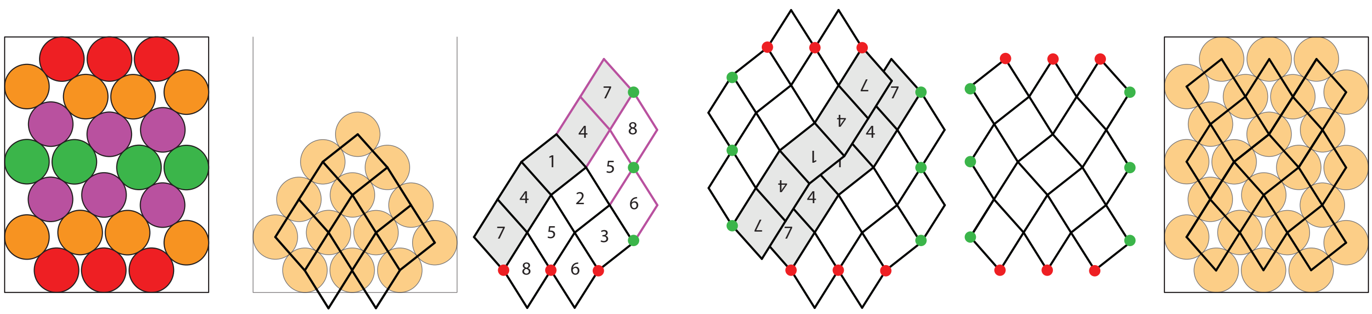
In both cases it suffices to show that these stacks have a half-turn symmetry. We begin by streamlining our proof for the original type of stack and then indicate how this streamlined proof has to be modified to turn it into proofs for the half-turn property of the two new types of stacks. Here we go, ... again.



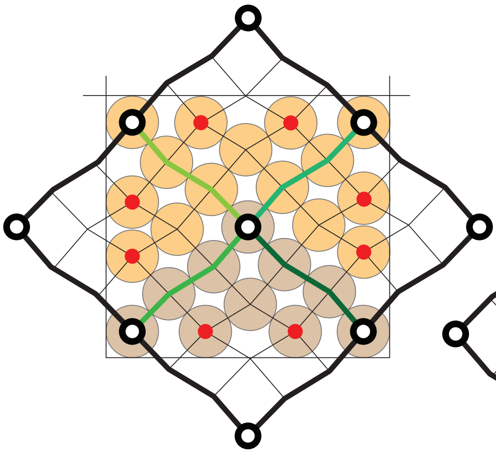
We start with the essential part of the mesh corresponding to the pyramid. We straightaway extend this mesh to one that covers half of the stack as indicated. Now it is clear that the gray band has a half-turn symmetry. We combine the mesh and a half-turned copy of this mesh. Finished!

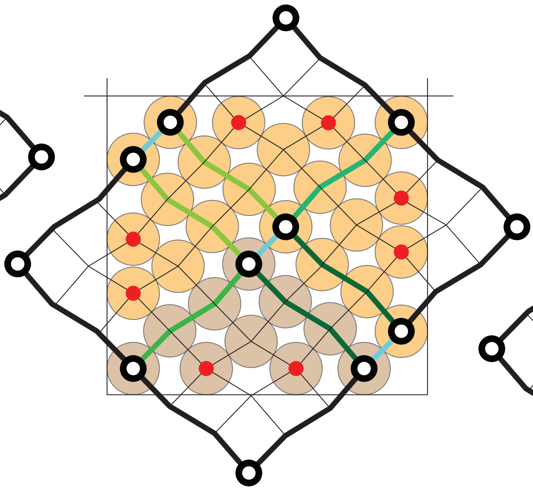
And here is how things have to be modified for the two new types of stacks. Note that in the second case we are overlapping the two gray bands.

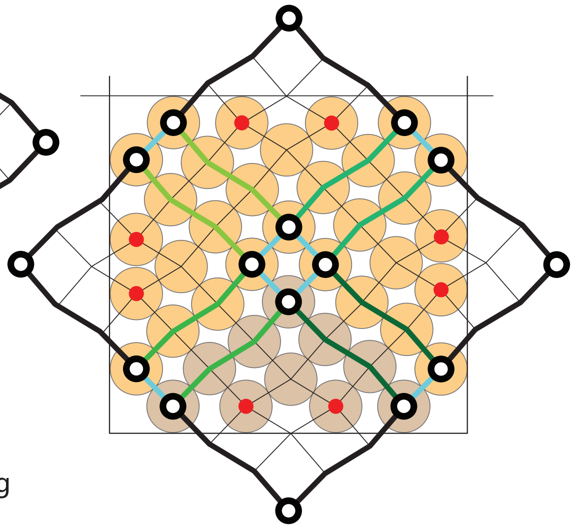




The following three diagrams show three stacks of the three different types that share the same pyramid at the bottom. Superimposed on the left diagram are the linkage and meshes that we used to show that all well-behaved stacks of the first type in tilted racks with the same bottom pyramid will have a level top row. The second and third diagrams show the corresponding linkages and meshes that can be used to show that the same is true for well-behaved stacks of the second and third types.

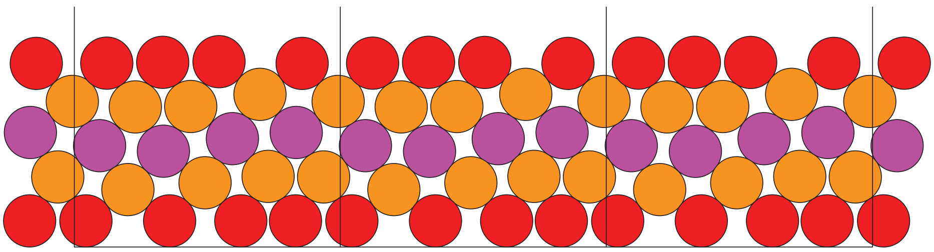




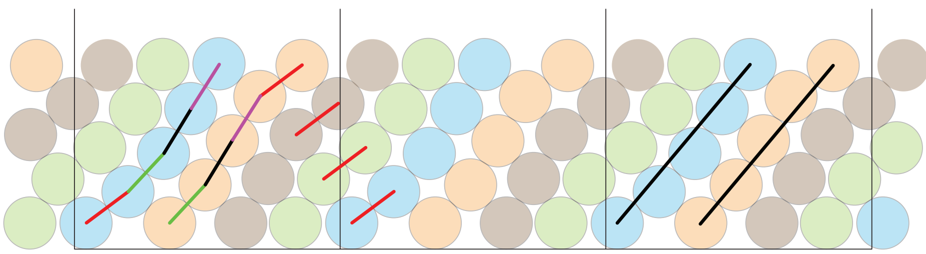


These diagrams also illustrate how closely related stacks of the three different types sharing the bottom pyramid are. Call the bottles corresponding to the top and right mesh the top half and those of the left and bottom mesh the bottom half. Then it is copies of this top and bottom part that fit together to give the stack in the middle. By combining halves of the second stack or quarters of the first stack we arrive at the third stack.

Periodic stacking



Let's have a look at stacking bottles periodically in an infinitely long rack. Then if the stack is well-behaved and the period is *n* bottles we get a level *n*+1*st* row.



To see that this is true color the diagonals slanting to the right using *n* colors as shown. Then diagonals colored the same are horizontal translates of each other.

Now focus on two adjacent diagonals (blue and orange) and, in particular, on the highlighted segments connecting the centers of adjacent bottles. Then it is clear that segments of the same color are translates of each other. From this it follows immediately that the segments connecting the midpoints of the top and bottom bottle of diagonals are horizontal translates of each other. In turn this implies that the top row is level and that the spacing of the bottles in the top row mirrors that of the bottles in the top row.