A NOVEL APPROACH TO LIMITS

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There is an unending debate about how limits should be taught in first year university classes, and specifically whether $\epsilon - \delta$ proofs should be taught. Without addressing the latter question,[1] I want to summarise one approach to the teaching of limits. This is from the 2006 (printed and sold) lecture notes, by an apparently very popular lecturer, from a reputedly good Australian university.[2]

In this particular subject limits are taught from the intuitive angle, but the students are briefly confronted with the formal concept of a limit, and given one assignment question to do. The formal approach begins with the following definition:

> A function $f(x)$ has the limit L as $x \to a$ if for each choice of ϵ there exists a $\delta < \epsilon$ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$.

A purported definition of an accepted concept is either right or wrong, and this definition – of the most important concept in analysis – is wrong. More thoroughly, compared to the logical structure of the standard definition, this one sentence contains three or four errors,[3] the most stark being the inclusion of $x = a$ in consideration of the limit.[4] The definition is sufficiently egregious, and it is probably enough to indicate the nature of what follows, but I'll quickly detail the subsequent discussion, first in the example which accompanies the definition, and then in the associated assignment question.

After giving the definition, the writer uses it to prove that $\lim\limits_{x\to 1}(2x+3) = 5$. In terms of the definition, they rephrase the problem so that: given ϵ, then δ must satisfy

$$(*) \qquad 5 - \epsilon < f < 5 + \epsilon \qquad \text{whenever} \qquad 1 - \delta < x < 1 + \delta\,.$$

The writing of f instead of $f(x)$ can be dismissed as a typo, but one cannot ignore the lines of calculation which then follow. These lines are the standard calculations for a linear limit but, leaving out the intermediate implications, they make the claim that

$$5 - \epsilon < f < 5 + \epsilon \qquad \Longrightarrow \qquad 1 - \delta < x < 1 + \delta\,.$$

[1] The answer is "yes".

[2] The notes were again on sale in 2007, 2008, 2009 and 2010, with were no significant changes for the purposes of this article. In 2011, the formal discussion of limits was eliminated entirely.

[3] The count depends upon whether one considers the superfluous condition $\delta < \epsilon$ to be an error.

[4] In the 2007, 2008, 2009 and 2010 versions, the other errors in the definition have been corrected, but this error remains.

This claim for correctly chosen δ is of course valid, but of course the claim does *not* "show that we can take $\delta \leqslant \epsilon/2 < \epsilon$". Also, apart from the critical confusion between an implication and an equivalence, and the needless noting that $\delta < \epsilon$, what is stark is the needlessly vague inequality, rather than simply defining $\delta = \epsilon/2$.

The subsequent discussion gets worse, when they ask "What happens when we shrink ϵ down to zero?". The answer is that "we also find that the x values are squeezed down to the single number $x = 1$". This is not a parenthetical remark, but rather is indicated to be core of the proof. In summary, the writer has started out correctly but then, perhaps guided by their incorrect definition, they have mistaken the desired direction of the $\epsilon - \delta$ dependence.

The students were then assigned a question, to prove that $\lim_{x\to 2} x^2 = 4$. If the unaided students had been expected to give the standard $\delta = \min\left(1, \frac{\epsilon}{5}\right)$ proof, this would have been exceedingly tough. Instead, it was wrong.

The provided solution to the assignment question indicates that the expected approach was directly along the lines of $(*)$, to find δ_1 and δ_2 such that

$$4 - \epsilon < f < 4 + \epsilon \qquad \text{whenever} \qquad 2 - \delta_1 < x < 2 + \delta_2\,.$$

That is, we should have

$$(\dagger) \qquad \begin{cases} \delta_1 = 2 - \sqrt{4-\epsilon} \\ \delta_1 = -2 + \sqrt{4+\epsilon}\,, \end{cases}$$

and then we take

$$\delta = \min(\delta_1, \delta_2)\,.$$

Though this can be considered a natural approach to the problem, it definitely should not be left without comment. There are subtle issues concerning the validity of the proof,[5] and it is important to remark that, at best, it can only apply to monotonic functions. In particular, it gives no sense of how to prove the algebraic limit laws.

In any case, even on their own terms, the writer gets the proof wrong. First of all, they take the incorrect root, falsely claiming that $\delta_1 = 2 + \sqrt{4-\epsilon}$. This leads them to incorrectly claim that $\delta_2 < \delta_1$, from which they make the needlessly vague claim that we can choose $0 < \delta < -2 + \sqrt{4+\epsilon}$. They then implicitly use the continuity of $\sqrt{x}$ to make the irrelevant conclusion that $\delta \to 0$ as $\epsilon \to 0$. QED.

The continued existence of limit notes such as these suggests a number of conclusions, and raises a number of questions. I shall leave the pondering of these as exercises for the reader.

[5]There is the minor issue that we need $\epsilon \leqslant 4$. Much more fundamental, the existence of the roots requires something like the Intermediate Value Theorem, i.e. the continuity of x^2, which is what we're trying to prove. One can get around this, but at minimum one needs to use something akin to the Archimedean Property of the real numbers.