AMSI 2013: MEASURE THEORY
Handout 9 - "Temporary"
Hausdorff Measure

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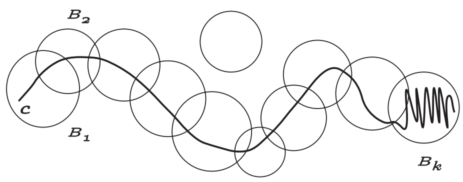
February 6, 2013

These notes are clumsy and brief. The point is to get the definition of Hausdorff measure down, list the basic properties (a number as easy boxed pussycats), and state the Vitali Covering Theorem. In particular, we only outline the proof that Hausdorff n measure and Lebesgue n measure agree on $\mathbb{R}^n$, and we don't discuss the Area Formula. **The material is examinable essentially in the manner given here, and nothing deeper or weirder.**

In a sentence, the idea behind geometric measure theory is to generalize the notion of "n-dimensional submanifold", allowing one to consider limits and subsequently to obtain existence (compactness) theorems. The fundamental notion is that of the *n-dimensional volume* of a (possibly nasty) subset of $\mathbb{R}^p$. (Recall that Lebesgue measure gives a notion of p-dimensional volume in $\mathbb{R}^p$, but Lebesgue gives no notion of lower dimensional volume in $\mathbb{R}^p$). This is the role played by *n-dimensional Hausdorff measure*, $\mathscr{H}^n$. To motivate the definition, consider a curve c in $\mathbb{R}^2$.



Covering c by balls (i.e. disks), we can hope that

$$\text{Length}\,(c) \overset{?}{\approx} \sum_{j=1}^{k} \text{diam}\,(B_j)\,.$$

There are two obvious problems with this approximation to Length (c):

(i) The sum may be too large because of wasted overlap or poorly placed balls. To compensate for this we need to take an *inf* over possible coverings. Of course this issue also arises in the definition of Lebesgue measure.

(ii) The sum may be two small because one big ball can cover a lot of lengthy wriggling of c. To compensate for this we need to progressively consider coverings of c consisting of smaller and smaller sets. This issue does *not* arise in the definition of Lebesgue measure.

We note also:

(iii) For a technical reason (discussed below) it is helpful to consider coverings by arbitrary sets C_j rather than just balls B_j. (See Remark (b) after Theorem 48 below).

(iv) In approximating/defining n-dimensional volume, the quantity $\operatorname{diam} B_j$ is replaced by $\omega_n \left(\frac{\operatorname{diam} C_j}{2}\right)^n$, where $\omega_n = \mathscr{L}^n\left(B_1(0)\right) = \operatorname{Vol}(\text{unit } n\text{-ball})$. (To see this quantity is reasonable, consider C_j a ball cutting off a piece of an n-plane that passes through the centre of C_j).

(v) For the usual reasons, we want to allow coverings by contain countably many sets.

Juggling all this motivation, we come up with:

Definition ($\mathscr{H}^n_\delta$-approximating measure, $\mathscr{H}^n$-measure)

For $n \geq 0$, $0 < \delta \leq \infty$ and $A \subseteq \mathbb{R}^p$. we define

$$\boxed{\begin{aligned} &\mathscr{H}^n_\delta(A) = \inf\left\{\sum_{j=1}^{\infty} \omega_n \left(\frac{\operatorname{diam} C_j}{2}\right)^n : A \subseteq \bigcup_{j=1}^{\infty} C_j,\ \operatorname{diam} C_j \leq \delta\right\} \\ &\mathscr{H}^n(A) = \lim_{\delta \to 0^+} \mathscr{H}^n_\delta(A) \end{aligned}}$$

Remarks

(a) Since $\mathscr{H}^n_\delta(A)$ increases as δ decreases, $\mathscr{H}^n(A)$ is well-defined.

(b) We take $\omega_0 = 1$. This is justified by Theorem 1(vi) below.

(c) n need not be an integer in the above definitions, though it usually will be for us. When n is not an integer we take ω_n to be any positive constant.

The following accumulation of facts shows that Hausdorff measure in general is well-behaved and in particular agrees with other notions of n-dimensional volume in familiar special cases.

Theorem 48 (Fundamental properties of Hausdorff measure)

(i) 57 $\mathscr{H}^n_\delta$ and $\mathscr{H}^n$ are measures.

(ii) 58 $\mathscr{H}^n$ is a Borel regular measure. If $E \subseteq \mathbb{R}^p$ is $\mathscr{H}^n$-measurable with $\mathscr{H}^n(E) < \infty$ then the restriction $\mathscr{H}^n \llcorner E$ is Radon.

(iii) 59 Suppose $m > n$. Then

$$\begin{cases} \mathscr{H}^n(A) < \infty \implies \mathscr{H}^m(A) = 0\,, \\ \mathscr{H}^m(A) > 0 \implies \mathscr{H}^n(A) = \infty\,. \end{cases}$$

(iv) 60 $\mathscr{H}^n$ is invariant under isometries.

(v) 61 Generalizing (iv), if $f : \mathbb{R}^p \to \mathbb{R}^q$ is Lipschitz and if $A \subseteq \mathbb{R}^p$ then

$$\boxed{\mathscr{H}^n(f(A)) \le (\operatorname{Lip} f)^n\, \mathscr{H}^n(A)}$$

(Recall that f is Lipschitz if there is a constant $K < \infty$ such that $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in \mathbb{R}^p$. $\operatorname{Lip} f$ is the best such constant K).

(vi) 62 $\mathscr{H}^0$ is counting measure:

(vii) 63 $\mathscr{H}^p_\delta = \mathscr{L}^p$ on $\mathbb{R}^p$.

(viii) 64 If $M^n \subseteq \mathbb{R}^p$ is an embedded n-dimensional C^1-submanifold then

$$\mathscr{H}^n(M) = \operatorname{Vol}(M) \qquad \text{(e.g. by "}\sqrt{g}\text{"-definition).}$$

Remarks

(a) The import of (iii) is that for $A \subseteq \mathbb{R}^p$ there is at most one exponent n such that $0 < \mathscr{H}^n(A) < \infty$. No such exponent need exist, but we can always define the *Hausdorff Dimension* of A by

$$\dim A \equiv \sup\{n : \mathscr{H}^n(A) = \infty\} = \inf\{m : \mathscr{H}^m(A) = 0\}.$$

Also, by (vii), $A \subseteq \mathbb{R}^p \implies \dim A \leq p$. More generally, (viii) implies that (separable) immersed n-submanifolds of $\mathbb{R}^p$ are Hausdorff n-dimensional.

(b) A proof of (v) will clearly involve using f to transform coverings of A to $f(A)$. Notice than even if we begin with a covering of A by balls, the transformed covering need not consist of balls. It is for this reason that we allow coverings by arbitrary sets in the definition of $\mathscr{H}^n_\delta$.

(c) The proof of (viii) is not difficult, given (v), (vii) and the change of variables formula for Lebesgue integration. (viii) is in fact a special case of an important result, the *Area Formula*. Another day.

(d) The proof of (vii) is quite involved. The major steps are:

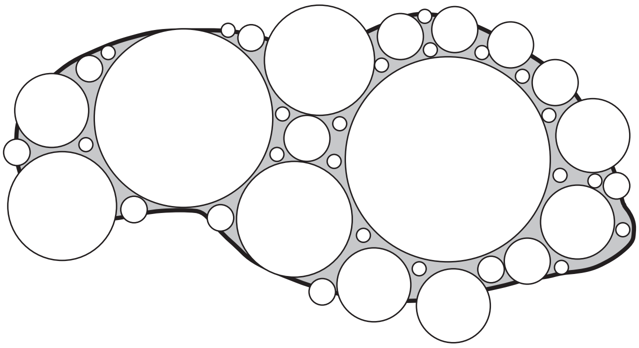
STEP 1 Prove that $\mathscr{L}^n(C) \leqslant \mathscr{H}^n_\delta(C)$ for all $C \subseteq \mathbb{R}^n$.
This is easy modulo of proof the the *Isodiametric Inequality*: $\mathscr{L}^n(C) \leqslant \omega_n \left(\frac{\operatorname{diam} C}{2}\right)^n$.

STEP 2 Prove that $\mathscr{H}^n_\delta(B) \leqslant \mathscr{L}^n(B)$ for any closed ball B.
This is trivial, using B to cover itself.

STEP 3 Prove that $\mathscr{H}^n_\delta(A) = 0$ whenever $\mathscr{L}^n(A) = 0$.
This is not difficult: given a covering of A by boxes, we chop the boxes to be small in diameter, and to not be too long and thin (the longest side is at most twice the length of the shortest side). This can be used to prove that $\mathscr{H}^n_\delta(A)$ is at most a fixed constant greater than $\mathscr{L}^n(A) = 0$.

STEP 4 Prove that $\mathscr{H}^n_\delta(C) \leqslant \mathscr{L}^n(C)$ for all $C \subseteq \mathbb{R}^n$.
This is a matter of combining the results of Steps 2 and 3, and requires the *Vitali Covering Theorem*: suppose a set $C \subseteq \mathbb{R}^n$ is finely covered by a collection $\mathscr{K}$ of closed balls of uniformly bounded and non-zero diameter.[1] Then there is a countable pairwise disjoint subcollection $\{B_k\}$ of the balls such that $\mathscr{L}^n\left(C \sim \cup_{k=1}^{\infty} B_k\right) = 0$.

[1] To be finely covered means that for every $x \in C$ and every $\delta > 0$ there is a ball $B \in \mathscr{K}$ with $x \in B$ and $\operatorname{diam}(B) < \delta$.



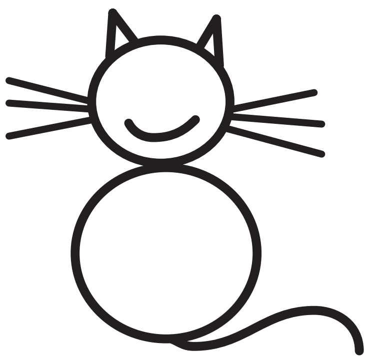
To apply the Vitali Theorem, fix $\epsilon > 0$ and choose an open set $W \supseteq C$ with $\mathscr{L}^n(W) \leqslant \mathscr{L}^n(C) + \epsilon$. (Why can we do this?) Then, let

$$\mathscr{K} = \{B : B \text{ is a closed ball, } B \subseteq W, \text{ and } \operatorname{diam} B \leqslant \delta\}\,.$$

Choosing the pairwise disjoint collection $\{B_k\}$ by the Vitali Theorem, we then can use subadditivity, and Steps 2 and 3, to calculate

$$\begin{aligned}
\mathscr{H}^n_\delta(C) &\leqslant \mathscr{H}^n_\delta\left(C \sim \bigcup_{k=1}^{\infty} B_k\right) + \sum_{k=1}^{\infty} \mathscr{H}^n_\delta(B_k) \\
&\leqslant \sum_{k=1}^{\infty} \mathscr{L}^n(B_k) \\
&= \mathscr{L}^n\left(\bigcup_{k=1}^{\infty} B_k\right) \\
&\leqslant \mathscr{L}^n(W) \\
&\leqslant \mathscr{L}^n(C) + \epsilon\,.
\end{aligned}$$

Step 4 then follows from the Thrilling Epsilon Lemma.



SOLUTIONS

58 To prove $\mathscr{H}^n$ is Borel, we apply Carathéodory's Criterion, Theorem 11. The point is,

$$\operatorname{dist}(A,B) > \delta \implies \mathscr{H}^n_\delta(A \cup B) = \mathscr{H}^n_\delta(A) + \mathscr{H}^n_\delta(B)\,,$$

since any δ-covering of $A \cup B$ splits into separate coverings of A and B. Now let $\delta \to 0$, to conclude $\mathscr{H}^n$ satisfies the Criterion.

To show that $\mathscr{H}^n$ is Borel regular, first note that in the definition of $\mathscr{H}^n_\delta$ we need only consider coverings of a given A by *closed* sets (because, for any set C, $\operatorname{diam}(\overline{C}) = \operatorname{diam} C$). This implies that for any $k \in \mathbb{N}$, we can find a Borel set $D_k \supseteq A$ for which

$$\mathscr{H}^n_{1/k}(D_k) \;\le\; \mathscr{H}^n_{1/k}(A) + 1/k\,.$$

(D_k can be a countable union of closed sets from a suitable covering of A). Setting $D = \cap D_k$, we see $D \supseteq A$ is Borel with $\mathscr{H}^n(D) = \mathscr{H}^n(A)$.

The fact that $\mathscr{H}^n \llcorner A$ will be Radon if A is $\mathscr{H}^n$-measurable with $\mathscr{H}^n(A) < \infty$ follows from the defintion of Radon measure together with Theorem 35(c)(ii).

60 $\mathscr{H}^n_\delta$, and thus $\mathscr{H}^n$ is trivially invariant under an isometry ρ, since ρ maps any covering of C to a covering of $\rho(C)$.

62 Given any $a \in \mathbb{R}^p$, it is easy to show that $\mathscr{H}^0(\{a\}) = 1$. Countable additivity then implies $\mathscr{H}^0$ is counting measure.

63 64 Eventually ...