AMSI 2013: MEASURE THEORY
Handout 5
Integration

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INTRODUCTION

Given a measure μ on X and a function $f : X \to \mathbb{R}^*$, we now want to define the *integral*

$$\int f \, \mathrm{d}\mu \, .$$

The key properties we aim for are: the area-of-a-rectangle behaviour on characteristic functions,

$$\int \chi_A \, \mathrm{d}\mu = \mu(A) \, ;$$

and *linearity*,

$$\begin{cases} \displaystyle\int f + g \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu \, , \\ \displaystyle\int cf \, \mathrm{d}\mu = c \int f \, \mathrm{d}\mu \end{cases}$$

The integrals of characteristic functions will be built into the definition at the first step, but additivity will be much less obvious. With our approach,[1] the integral of a measurable function is automatically defined, subject to concerns about ∞. However, the proof of additivity comes quite late, as a consequence of the key convergence theorems. Not surprisingly, there are other roads to Rome: we'll point out some of these along the way.

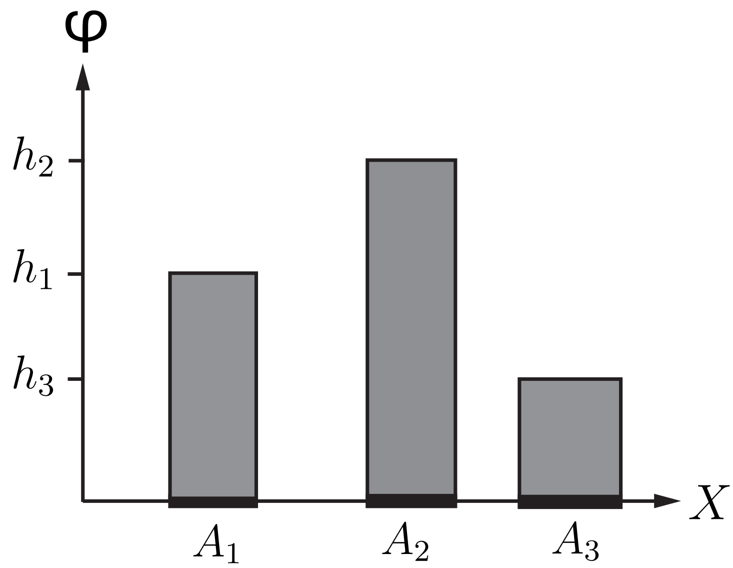
[1] We'll largely follow the approach in §11 of *Real Analysis* by H. Royden (Prentice Hall, 3rd ed., 1988), and §2 of *Real Analysis* by G. Folland (Wiley, 1984).

SIMPLE FUNCTIONS

Definition: Suppose μ is a measure on X. A function $\phi\colon X\to\mathbb{R}^*$ is *simple* (more precisely, μ-*simple*) if ϕ is measurable, $\phi \geqslant 0$ a.e., and if ϕ has countable range $Y = \{h_1, h_2, \dots\}$ off of a null set $N\subseteq X$: that is, $\phi^{-1}(Y) = X\sim N$.

This definition is slightly cumbersome, but it has the advantage of being measure-theoretic: a simple function ϕ remains simple when changed (or is undefined) on a null set. Eliminating the null set, we then have the canonical representation of ϕ as a sum of characteristic functions:

$$\begin{cases} \phi = \sum_{j=1}^{\infty} h_j \cdot \chi_{A_j} \qquad h_j > 0\,, \\ A_j = \phi^{-1}(\{h_j\}) = \{x \in X : \phi(x) = h_j\}\,. \end{cases}$$



We have specified that each h_j is positive in this representation, which is legitimate since the possible existence of a term $h_j = 0$ adds nothing to the sum. Note also that the measurability of ϕ implies the measurability of each A_j. We also emphasise, what characterises such a representation as being canonical is that each A_j is non-empty, and

$$j \neq k \quad \Longrightarrow \quad \begin{cases} h_j \neq h_k\,, \\ A_j \cap A_k = \emptyset\,. \end{cases}$$

Given such a representation of ϕ, it is clear that we want to define

$$(✠) \qquad \boxed{\int \phi \,\mathrm{d}\mu \;=\; \sum_{j=1}^{\infty} h_j \mu(A_j)}$$

We make some simple remarks.

- It is clear that $\int \phi$ is independent of the null set N. In particular, if ϕ is simple and $\psi = \phi$ a.e. then ψ is simple and $\int \phi = \int \psi$.
- It is more standard to define simple functions to have finite rather than countable range, and to not necessarily be nonnegative. Also, one can restrict simple functions to be real-valued. None of these variations makes a substantial difference to the arguments that follow.
- In the definition of the integral, we use the convention $0 \cdot \infty = 0$ if need be. That is, the integral on a null set will be 0, even for an infinite valued function.
- We write $\int \phi$ if the context is clear.
- $\int \phi \, d\mu \geqslant 0$, with equality iff $\phi = 0$ almost everywhere.

The definition of $\int \phi \, d\mu$ for a simple function ϕ is clear and unambiguous, but it is important to realise that we *only* know the value of $\int \phi$ if ϕ is written in canonical form. For example, suppose $\phi = \sum h_j \chi_{A_j}$ and $\psi = \sum m_k \chi_{B_k}$ are simple functions in canonical form. Then $\psi + \phi$ is simple, since there are at most countably many possible outputs $h_j + m_k$. However, we do not automatically have the canonical representation of $\phi + \psi$. Consequently, the proof of the following lemma is a bit fiddly.

25 **LEMMA 16**: Suppose ϕ and ψ are simple functions on X. Then $\phi + \psi$ is simple and

$$\int \phi + \psi \, d\mu = \int \phi \, d\mu + \int \psi \, d\mu \qquad \text{(additivity)}.$$

If ϕ and ψ are simple with $\phi \leqslant \psi < \infty$ a.e., then $\psi - \phi$ is also simple. Writing $\psi = \phi + (\psi - \phi)$, it is then immediate from the previous lemma that

$$\phi \leqslant \psi \text{ a.e.} \quad \Longrightarrow \quad \int \phi \, d\mu \leqslant \int \psi \, d\mu \qquad \text{(monotonicity)}.$$

For this argument we assumed $\psi < \infty$ a.e., but if this is not the case then the monotonicity conclusion is trivial.

INTEGRABLE AND SUMMABLE FUNCTIONS

Definition: Suppose μ is a measure on X and that $f : X \to \mathbb{R}^*$ is measurable.

(a) If $f \geqslant 0$ we define

$$\boxed{\int f \,\mathrm{d}\mu \;=\; \sup\left\{\int \phi \,\mathrm{d}\mu : \phi \text{ simple and } \phi \leqslant f \text{ a.e.}\right\}}$$

where $\int \phi$ is defined by (✠).

(b) For general f we write

$$f = f^+ - f^- ,$$

where $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$. If either $\int f^+ \,\mathrm{d}\mu$ or $\int f^- \,\mathrm{d}\mu$ is finite we say f is μ-*integrable* and we define

$$\boxed{\int f \,\mathrm{d}\mu \;=\; \int f^+ \,\mathrm{d}\mu \;-\; \int f^- \,\mathrm{d}\mu}$$

If $\int f \,\mathrm{d}\mu$ exists and is finite, we say f is μ-*summable*.[2]

We make some easy remarks:

- We use the terms *integrable* and *summable*, and we write $\int f$, when the context is clear. Also, when we need to identify the variable of integration, we'll write
$$\int f(x) \,\mathrm{d}\mu(x) .$$
- If f is simple then, by the monotonicity conclusion following Lemma 16, this definition of $\int f$ is consistent with the definition provided by (✠).
- Any nonnegative measurable function is automatically integrable.
- We have defined $\int f$ in terms of what could be called a *lower integral*. Analogous to the case of Riemann sums, one could analogously define an *upper integral*, as an *inf* obtained over simple functions $\phi \geqslant f$. One approach, then, is to define f to be integrable if the lower and upper integrals agree.[3] It is then a fundamental theorem that, subject to ∞ concerns, measurable functions are integrable; see Proposition 24.

[2] Many texts do not directly consider what we refer to as integrable functions, and they use the term "integrable" to refer to what we are calling summable.

[3] See, for example, *Measure Theory and Fine properties of Functions*, by L. C. Evans and R. Gariepy (CRC, 1991).

The obvious property that we desire of the integral is linearity. Scaling is easy:

$$c \in \mathbb{R}^* \quad \implies \quad \int cf \, \mathrm{d}\mu = c \int f \, \mathrm{d}\mu \qquad \text{as long as both sides make sense.}$$

However, additivity is not so simple. As a first step, we have

(26) **LEMMA 17**: Suppose μ is a measure on X.

(a) If $f, g \geqslant 0$ are measurable functions on X then

$$\int f + g \, \mathrm{d}\mu \geqslant \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu \, .$$

(b) If $\{f_j \geqslant 0\}_{j=1}^{\infty}$ is a sequence of nonnegative measurable functions on X then

$$\int \left(\sum_{j=1}^{\infty} f_j \right) \mathrm{d}\mu \geqslant \sum_{j=1}^{\infty} \int f_j \, \mathrm{d}\mu$$

The reverse inequalities, and then additivity in general, will be established in the next section. For now, we note two useful consequences of Lemma 17, which hold for all (not necessarily nonnegative) integrable functions:

(27) If f and g are integrable, and if $f \leqslant g$ a.e., then

$$\int f \, \mathrm{d}\mu \leqslant \int g \, \mathrm{d}\mu \, .$$

If f and g are summable then equality occurs iff $f = g$ a.e..

(28) If f is integrable then

$$\left| \int f \, \mathrm{d}\mu \right| \leqslant \int |f| \, \mathrm{d}\mu \, .$$

Notice that a further consequence of Lemma 17(a) is, if $A \subseteq X$ is measurable and f is integrable (summable) then so is $f\chi_A$. Consequently, we can define

$$\boxed{\int_A f \, \mathrm{d}\mu = \int f \cdot \chi_A \, \mathrm{d}\mu}$$

With this notation, we now emphasise a particular consequence of Lemma 17(b) and monotonicity. Suppose that $f \geqslant 0$ is measurable and that $\{A_j\}_{j=1}^{\infty}$ is a sequence of pairwise disjoint measurable sets. The pairwise disjointness implies $\chi_{\cup A_j} = \sum \chi_{A_j}$. Then

$$\int f \mathrm{d}\mu \geqslant \int f \cdot \chi_{\cup A_j} \,\mathrm{d}\mu = \int \sum_{j=1}^{\infty} f \cdot \chi_{A_j} \,\mathrm{d}\mu \geqslant \sum_{j=1}^{\infty} \int f \cdot \chi_{A_j} \,\mathrm{d}\mu = \sum_{j=1}^{\infty} \int_{A_j} f \,\mathrm{d}\mu \,.$$

CONVERGENCE THEOREMS

Suppose μ is a measure on X and $\{f_j\}_{j=1}^{\infty}$ is a sequence of integrable functions on X. If f_j converges (pointwise) to some function f we can then ask

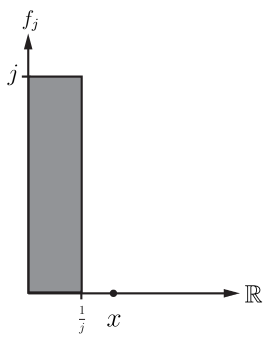
$$\int f \stackrel{?}{=} \lim_{j\to\infty} \int f_j \,\mathrm{d}\mu \,.$$

We know that f will be measurable (Proposition 15), and so if each f_j is nonnegative then f will be integrable. In this setting, the RHS at least makes sense, but we still may not have equality.[4] For example, let $\mu = \mathscr{L}$ on $\mathbb{R}$, and let

$$f_j = j \cdot \chi_{(0,\frac{1}{j})} \,.$$

Then $f_j \to f = 0$ pointwise, but

$$\int f_j \,\mathrm{d}\mathscr{L} \to 1 \neq 0 = \int f \,\mathrm{d}\mathscr{L} \,.$$



In the *Fundamental "Lebesgue is better than Riemann" Theorem* (Handout 1), we promised that equality would hold for Lebesgue measure in case the f_j were uniformly bounded on the domain $[a, b]$. The danger is ∞, in either the range or the domain, but the danger can be controlled without being so restrictive. We shall prove this result, but as a special case of the more general Dominated Convergence Theorem (Theorem 22) for general measures.

[4]The equality is very easy to prove if $\mu(X) < \infty$ and if $f_j \to f$ *uniformly* off of a null set. However, this is a very strong hypothesis to impose.

The first step, and the main work, is

THEOREM 18 (Fatou's Lemma): Suppose μ is a measure on X, and suppose that $\{f_j \geqslant 0\}_{j=1}^{\infty}$ is a sequence of nonnegative measurable functions on X, and that $f_j \to f$ almost everywhere. Then

$$\int f \mathrm{d}\mu \leqslant \liminf_{j\to\infty} \int f_j \mathrm{d}\mu\,.$$

Fatou's Lemma is not obvious. We prove it below, but we first make some comments and establish various consequences.

The example above shows that we may have strict inequality in Fatou's Lemma. Also, the *lim inf* is necessary, as the limit of the integrals need not exist.[5] As well, we don't need to hypothesise that the the sequence $\{f_j\}$ has a limit: it follows from Theorem 18 that

29 For any sequence $\{f_j\}_{j=1}^{\infty}$ of nonnegative measurable functions,

$$\int \liminf_{j\to\infty} f_j \mathrm{d}\mu \leqslant \liminf_{j\to\infty} \int f_j \mathrm{d}\mu\,.$$

THEOREM 19 (Monotone Convergence Theorem): Suppose μ is a measure on X, and suppose $0 \leqslant f_1 \leqslant f_2 \leqslant \dots$ is an almost everywhere increasing sequence of nonnegative measurable functions. Let $f = \lim f_j$. Then

$$\int f \mathrm{d}\mu = \lim_{j\to\infty} \int f_j \mathrm{d}\mu\,.$$

PROOF: f is defined a.e. (since the sequence $f_j(x)$ is increasing except for possibly a null set of x), and for each j we have $f \geqslant f_j$ a.e.. Thus $\int f \geqslant \int f_j$. Taking the limit, we have

$$\int f \mathrm{d}\mu \geqslant \lim_{j\to\infty} \int f_j \mathrm{d}\mu\,.$$

(Note that the RHS limit clearly exists in $\mathbb{R}^*$). The reverse inequality is exactly the conclusion of Fatou's Lemma.

Since we have additivity for simple functions (Lemma 16), The Monotone Convergence Theorem implies that we can integrate a series of such functions term by term:

$$\int \sum_{j=1}^{\infty} \phi_j \,\mathrm{d}\mu = \int \lim_{n\to\infty} \sum_{j=1}^{n} \phi_j \,\mathrm{d}\mu = \lim_{n\to\infty} \int \sum_{j=1}^{n} \phi_j \,\mathrm{d}\mu = \lim_{n\to\infty} \sum_{j=1}^{n} \int \phi_j \,\mathrm{d}\mu = \sum_{j=1}^{\infty} \int \phi_j \,\mathrm{d}\mu\,.$$

[5] As an example, for j odd let $f_j = j \cdot \chi_{(0,\frac{1}{j})}$ as above, and for j even let $f_j = 0$.

This leads to a practical characterization of the integral.

LEMMA 20: Suppose μ is a measure on X, and suppose $f : X \to [0, \infty]$ is nonnegative.

(a) We can write

$$f = \sum_{j=1}^{\infty} a_j \chi_{A_j} .$$

for some $a_j \geqslant 0$ and some $A_j \subseteq X$. If f is measurable then we can arrange for the A_j to be measurable.

(b) If f is measurable then, given any such representation of f, we have

$$\int f \, \mathrm{d}\mu = \sum_{j=1}^{\infty} a_j \mu(A_j) .$$

PROOF: (b) follows immediately from (a), Lemma 16 and the remark above. To prove (a), set

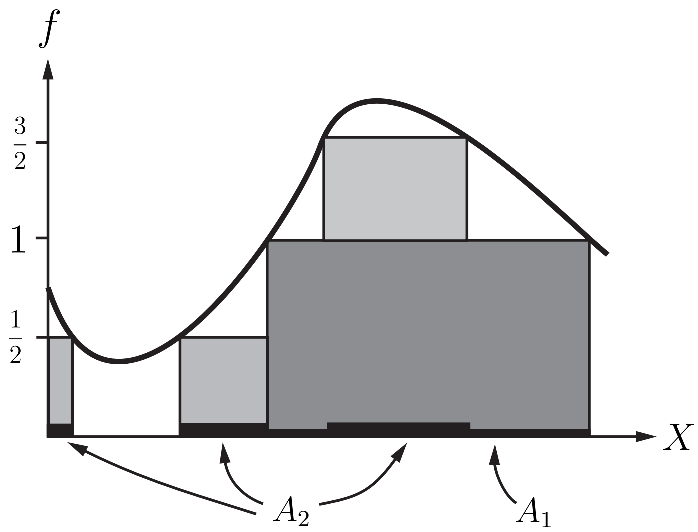
$$a_1 = 1 \text{ and } A_1 = \{x : f(x) \geqslant 1\}.$$

Then set

$$a_2 = \tfrac{1}{2} \text{ and } A_2 = \left\{x : f(x) - 1 \cdot \chi_{A_1}(x) \geqslant \tfrac{1}{2}\right\}.$$

In general, define

$$a_j = \tfrac{1}{j} \text{ and } A_j = \left\{x : f(x) - 1 \cdot \chi_{A_1}(x) - \tfrac{1}{2} \cdot \chi_{A_2}(x) - \ldots \tfrac{1}{j-1} \cdot \chi_{A_{j-1}}(x) \geqslant \tfrac{1}{j}\right\}.$$



Clearly,

$$f(x) \geqslant \sum_{j=1}^{n} a_j \chi_{A_j}(x) \qquad \text{for all } n \in \mathbb{N} \text{ and } x \in X$$

$$\Longrightarrow \quad f(x) \geqslant \sum_{j=1}^{\infty} a_j \chi_{A_j}(x) \qquad \text{for all } x \in X.$$

To prove the reverse inequality, we consider two cases.

Case (i): $x \in A_j$ for all but finitely many j.

In this case,

$$\sum_{x \in A_j} a_j = \sum_{x \in A_j} \frac{1}{j} = \infty \qquad \text{(since only finitely many terms are missing)}$$

$$\implies \sum_{j=1}^{\infty} a_j \chi_{A_j}(x) = \infty \geqslant f(x) \qquad .$$

Case (ii): $x \notin A_n$ for infinitely many n.

In this case, for any such n,

$$f(x) < \sum_{j=1}^{n-1} a_j \chi_{A_j}(x) + \frac{1}{n} \leqslant \sum_{j=1}^{\infty} a_j \chi_{A_j}(x) + \frac{1}{n}\,.$$

Since this is true for infinitely many n, we have the desired inequality by the Thrilling ϵ-Lemma.

Lemma 20 provides us with the desired flexibility in evaluating integrals. In particular, it finally allows us to prove additivity:

30 **THEOREM 21**: Suppose that μ is a measure on X and that f and g are integrable functions on X. Then

$$\int f + g \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu + \int g \, \mathrm{d}\mu$$

as long as the RHS is well-defined. In particular, the sum of two summable functions is summable.

With additivity in hand we now have a version of the Monotone Convegence Theorem for infinite series: if $\{f_j\}$ is a sequence of nonnegative measurable functions then

$$\int \sum_{j=1}^{\infty} f_j \, \mathrm{d}\mu = \sum_{j=1}^{\infty} \int f_j \, \mathrm{d}\mu \qquad (f_j \geqslant 0).$$

The proof is exactly as for simple functions above.

Finally, we have a very general convergence theorem:

THEOREM 22 (Dominated Convergence Theorem): Suppose that μ is a measure on X and that $\{f_j\}$ is a sequence of measurable functions on X, with $f_j \to f$ a.e.. Suppose that there is a summable function g such that $|f_j| \leqslant g$ a.e. for each j. Then

$$\lim_{j\to\infty} \int |f - f_j| \,\mathrm{d}\mu = 0\,.$$

In particular,

$$\int f_j \to \int f\,.$$

REMARK: Setting $\mu = \mathscr{L} \llcorner [a,b]$ and $g = M = \sup\limits_{j,x}\{|f_j(x)|\}$, the Dominated Convergence Theorem gives the long-promised convergence theorem for the integrals of uniformly bounded functions.[6]

PROOF: Clearly, $|f| \leqslant g$ a.e., and so $|f - f_j| \leqslant |f| + |f_j| \leqslant 2|g|$ a.e.. So, by Fatou's Lemma,

$$\int 2g = \int \lim_{j\to\infty} (2g - |f - f_j|) \leqslant \liminf_{j\to\infty} \int (2g - |f - f_j|) = \int 2g - \limsup_{j\to\infty} \int |f - f_j|\,.$$

Subtracting $\int 2g$ from both sides, we see

$$\limsup_{j\to\infty} \int |f - f_j| \leqslant 0\,,$$

which clearly implies the first claim. The second claim follows, since $\left|\int (f - f_j)\right| \leqslant \int |f - f_j|$.

We note that the Dominated Convergence Theorem (and similarly, each of the other convergence theorems) has an obvious generalization to a family $\{f_h\}$ of functions parametrized by a real variable $h \in (0,1)$: here, we hypothesise that $f_h \to f$ a.e. as $h \to 0^+$, with each $|f_h| \leqslant g$. Supposing that $\int |f_h - f| \not\to 0$, we would then have a sequence $h_n \to 0^+$ with $\int |f_{h_n} - f| \not\to 0$, contradicting Theorem 22. We also have:

[6] In the Riemann context, our $W(x)$ example proved that the limiting function may not be integrable. However, in fact this is all that can really go wrong. See "Arzela's dominated convergence theorem for the Riemann integral" by W. A. J. Luxemburg, *American Mathematical Monthly*, **78** (1971), 970-979.

31 **THEOREM 23 (Generalized Dominated Convergence Theorem)**: Suppose that μ is a measure on X and that $\{f_j\}$ is a sequence of measurable functions on X, with $f_j \to f$ a.e.. Suppose that for each j there is a summable function g_j such that $|f_j| \leqslant g_j$ a.e., and suppose that there is a summable function g such that $g_j \to g$ a.e. and $\int g_j \to \int g$. Then

$$\lim_{j\to\infty} \int |f - f_j| \, \mathrm{d}\mu = 0 \, .$$

Finally, we provide the promised

PROOF OF FATOU'S LEMMA: With $0 \leqslant f_j \to f$ a.e., we want to show $\liminf_{j\to\infty} \int f_j \geqslant \int f$. It suffices to fix t with $0 < t < 1$, to fix a simple function $\phi \leqslant f$ a.e., and to show

$$(*) \qquad \liminf_{j\to\infty} \int f_j \geqslant t \int \phi \, .$$

Further, to simplify notation, we can assume $\phi < \infty$ everywhere. (If $f = \infty$ on a set A of positive measure, we can set $\phi = N \cdot \chi_A$ for $N \in \mathbb{N}$, and it suffices to prove $(*)$ for such ϕ).

Write ϕ in canonical form:

$$\phi = \sum_{j=1}^{\infty} h_j \cdot \chi_{A_j} \qquad A_j = \phi^{-1}(\{h_j\}) \qquad 0 < h_j < \infty \, .$$

So

$$\int \phi \, \mathrm{d}\mu \; = \; \sum_{j=1}^{\infty} h_j \mu(A_j)$$

Changing each A_j by a null set, we can assume $f_j \to f$ and $f \geqslant \phi$ on all of $\cup_j A_j$. Then

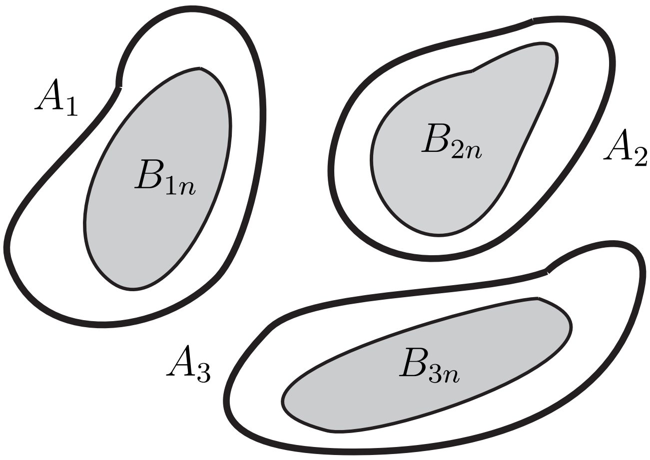
$$\begin{aligned} x \in A_j \quad &\Longrightarrow \quad f(x) \geqslant \phi(x) = h_j \\ &\Longrightarrow \quad \text{for some } n, \text{ if } k \geqslant n \text{ then } f_k(x) \geqslant t h_j \qquad (\text{since } f_k(x) \to f(x)). \end{aligned}$$

Thus, as $n \to \infty$,

$$B_{jn} \nearrow A_j$$

where

$$B_{jn} = A_j \cap \{x : f_k(x) \geqslant t h_j \text{ for all } k \geqslant n\} \, .$$



The collection $\{B_{jn}\}_j$ is pairwise disjoint, so

$$f_n \geqslant \sum_{j=1}^{\infty} f_n \chi_{B_{nj}} \,.$$

Then, since the $\{B_{jn}\}$ are measurable, Lemma 17 (see the subsequent discussion) implies

$$\int f_n \;\geqslant\; \int \left(\sum_{j=1}^{\infty} f_n \chi_{B_{jn}} \right) \;\geqslant\; \sum_{j=1}^{\infty} \int_{B_{jn}} f_n \;\geqslant\; \sum_{j=1}^{\infty} t h_j \mu(B_{jn}) \,.$$

The sum on the right increases with n (since B_{jn} increases), giving at least as good a bound for $\int f_k$ for $k \geqslant n$. Thus,

$$(*) \qquad \inf_{k \geqslant n} \int f_k \;\geqslant\; t \sum_{j=1}^{\infty} h_j \mu(B_{jn}) \,.$$

Next, by continuity of μ (Theorem 8), we have $\mu(B_{jn}) \nearrow \mu(A_j)$ as $n \to \infty$. So, taking the limit in $(*)$ as $n \to \infty$, we obtain

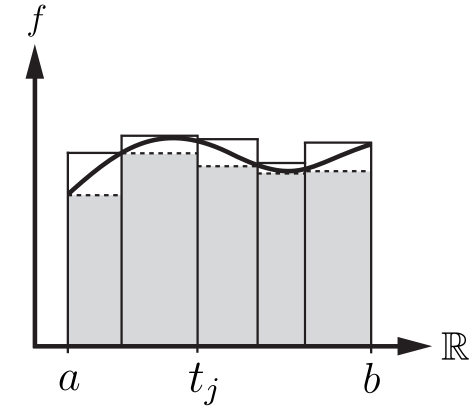
$$\liminf_{n\to\infty} \int f_n \;=\; \lim_{n\to\infty} \inf_{k \geqslant n} \int f_k \;\geqslant\; t \sum_{j=1}^{\infty} h_j \mu(A_{nj}) \;=\; t \int \phi \,,$$

as desired.

LEBESGUE, RIEMANN AND THE FUNDAMENTAL THEOREM

Suppose f is a bounded function on $[a, b]$. We want to compare the Lebesgue integrability and Riemann integrability of f. For this, we briefly recall the theory of Riemann integration.[7] We consider a *partition* $\alpha = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a, b]$. Given such a partition, we consider the *sup* and *inf* of f on each subinterval:

$$\begin{cases} u_j = \sup\{f(x) : t_{j-1} \leqslant x \leqslant t_j\}\,, \\ l_j = \inf\{f(x) : t_{j-1} \leqslant x \leqslant t_j\}\,. \end{cases}$$



This gives us the *upper and lower sums* associated with α:

$$\begin{cases} U_\alpha = \sum_{j=1}^{n} u_j(t_j - t_{j-1})\,, \\ L\alpha = \sum_{j=1}^{n} l_j(t_j - t_{j-1})\,. \end{cases}$$

If the supremum of the lower sums (over all partitions α) equals the infimum of the upper sums, we say that f is *Riemann integrable*, and we define the *Riemann integral* to be this common value:[8]

$$\sup_\alpha L_\alpha = \int_a^b f(x)\,\mathrm{d}x = \inf_\alpha U_\alpha\,.$$

The definition of our measure-theoretic integral involves no upper sums, but we do have such a characterization as a lemma:

[7] For more details, see, for example, §32 of *Elementary Analysis* by K. Ross (Springer, 1980).

[8] This is Darboux's formulation of the Riemann integral. The Riemann formulation (actually due to Cauchy) is to consider sums of the form $\sum f(t_j)(t_j - t_{j-1})$. If, as the maximum subinterval of the partition converges to 0, these sums converge to one value I, then we call I the Riemann integral of f. One can show that the Darboux formualtion is equivalent to the Cauchy-Riemann version. See §2.4 of *Theories of Integration* by D. Kurtz and C. Swartz, (World Scientific, 2004).

(32) **LEMMA 24**: Suppose μ is a measure on X, and $f : X \to \mathbb{R}^*$ is measurable. Then

(a) If $f \geqslant 0$ then

$$\int f \, \mathrm{d}\mu = \inf \left\{ \int \psi \, \mathrm{d}\mu : \psi \geqslant f \text{ a.e.}, \psi \text{ simple} \right\} .$$

(b) If f is integrable then

$$\int f \, \mathrm{d}\mu = \sup \left\{ \int \phi \, \mathrm{d}\mu : \phi \leqslant f \text{ a.e.}, \phi(X) \text{ countable} \right\}$$

$$= \inf \left\{ \int \psi \, \mathrm{d}\mu : \psi \geqslant f \text{ a.e.}, \psi(X) \text{ countable} \right\} .$$

A natural comparison of Riemann and Lebesgue integrals easily follows: if $f : [a, b] \to \mathbb{R}$ is continuous on the bounded interval $[a, b]$ then f is both Lebesgue summable and Riemann integrable, and the two integrals are equal:

$$\int_{[a,b]} f \, \mathrm{d}\mathscr{L} = \int_a^b f(x) \, \mathrm{d}x .$$

Generalizing, the definitive result comparing Riemann and Lebesgue is:[9]

(33) **THEOREM 25**: Suppose $f : [a, b] \to \mathbb{R}$ is bounded.

(a) Suppose f is Riemann integrable. Then f is Lebesgue measurable, and thus Lebesgue summable, and the Riemann and Lebesgue integrals are equal.

(b) f is Riemann integrable iff the set of points where f is discontinuous is a null set.

Thus, measure theory tells us exactly the limits of Riemann integration. For example, if C is the Cantor set then $\chi_C : [0, 1] \to \mathbb{R}$ is discontinuous at exactly the points of the Cantor set; by Theorem 25, χ_C must be Riemann integrable, but anything "worse" will not be. In particular, if D is a Cantorlike set of positive Lebesgue measure (see (10)), then χ_D will not be Riemann integrable. Note also that, unlike the Weird function $W(x) = \chi_{\mathbb{Q} \cap [0,1]}$, χ_D is not

[9]See, for example §2.3 *Real Analysis* by G. Folland (Wiley, 1984).

even equivalent in the a.e. sense to a Riemann integrable function: we cannot adjust χ_D on a null set to make it Riemann integrable.

An important corollary of Theorem 25 (and in practice the consideration of continuous functions usually suffices) is that we can now apply any established (or accepted) Riemann theorems. In particular, we now have the *Fundamental Theorem of Calculus*:[10] If $F : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and continuous on $[a, b]$, and if F' is Riemann integrable then

$$\int_{[a,b]} F' \, \mathrm{d}\mathscr{L} = \int_a^b F'(x) \, \mathrm{d}x = F(b) - F(a) \, .$$

However, this is being lazy. In fact, with a little work we can directly prove a Lebsgue version:

THEOREM 26 (Fundamental Theorem of Calculus - Easy Version): If $F : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and continuous on $[a, b]$, and if F' is bounded, then

$$\int_{[a,b]} F' \, \mathrm{d}\mathscr{L} = F(b) - F(a) \, .$$

Note that Theorem 26 is not the optimal result: rather than assuming F' is bounded, the result still holds if F' is merely summable.[11] However, the immediate consequences of Theorem 26 – integration by parts, substitution and so on – give us practical methods for evaluating the (Riemann = Lebesgue) integrals of concrete, everyday functions.[12]

It will also come as no great surprise that there is also a very general measure theory version of the Fundamental Theorem of Calculus, which includes the Lebesgue version as a special case. This is the theory of differentiation of measures, which we'll consider later.[13]

[10] See, for example, §34 of *Ross*, referenced above.

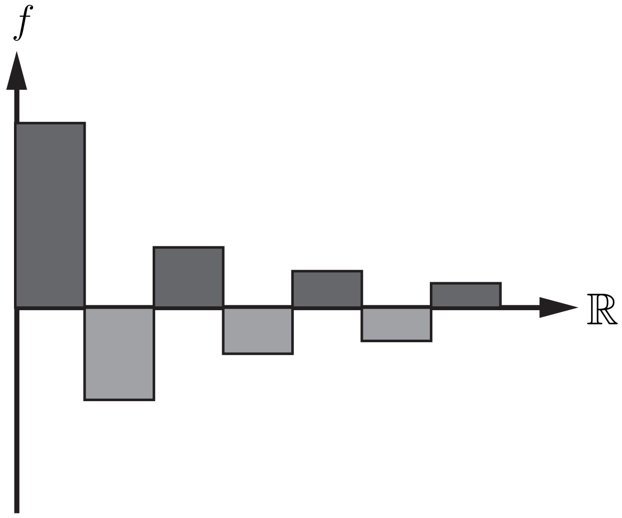
[11] See, for example, §6 of *An Introduction to Measure and Integration* by I. Rana (2nd ed., AMS, 2002).

[12] See §34 of *Ross*, referenced above. Note that none of the theorems we proved on Lebesgue integration were concerned with questions of practical computation.

[13] See, for example, Theorem 1.6.2 of *Evans and Gariepy*.

Finally, we make some quick observations about improper Riemann integrals.[14] Here one has to be more careful. For example, consider the function $f : [0,\infty) \to \mathbb{R}$ where

$$f(x) = \sum_{j=1}^{\infty} (-1)^{j+1} \frac{1}{j} \chi_{[j-1,j)}$$



f is not Lebesgue summable, since

$$\begin{cases} \displaystyle\int f^+ \, \mathrm{d}\mathscr{L} = \sum_{k \text{ odd}} \frac{1}{k} = \infty \,, \\ \displaystyle\int f^- \, \mathrm{d}\mathscr{L} = \sum_{k \text{ even}} \frac{1}{k} = \infty \,. \end{cases}$$

By contrast, f is improperly Riemann integrable, since

$$\int_0^\infty f(x)\,\mathrm{d}x = \lim_{A\to\infty} \int_0^A f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{j=1}^{n} (-1)^{j+1}\frac{1}{j} = \sum_{j=1}^{\infty} (-1)^{j+1}\frac{1}{j} \quad \text{is well-defined and finite.}$$

However, subject to summability issues, Theorem 26 still applies in the improper setting. For example if $f \geqslant 0$ is improperly Riemann integrable then f is Lebesgue integrable and the two integrals are equal. Of course, to give a precise statement and proof of this, one needs to be clear about the permitted asymptotic behaviour of f; in any standard setting, Theorem 26 together with either the Monotone Convergence Theorem or the Dominated Convergence Theorem is the basis of the proof.

[14] It is natural to consider defining directly the Riemann integrals of unbounded functions. However, this turns out to trivialize: with either the Darboux or Riemann formulation an unbounded function will have an infinite Riemann integral. See §2.2 of *Kurtz and Swartz*, referenced above. The *Henstock-Kurzweil integral* is a modern variation of the Riemann integral which allows unbounded functions to be treated directly. See §4 of *Kurtz and Swartz*.

DIFFERENTIATING UNDER THE INTEGRAL SIGN

We end with an application of the Dominated Convergence Theorem. Suppose μ is a measure on X, $I \subseteq \mathbb{R}$ is an open interval, and $F : X \times I \to \mathbb{R}$. Assume $x \mapsto F(x,t)$ is summable for each $t \in I$ and define

$$f(t) = \int F(x,t)\,\mathrm{d}\mu(x)\,.$$

We can then ask whether

$$(*) \qquad f'(t) \stackrel{?}{=} \int D_2F(x,t)\,\mathrm{d}\mu(x)\,,$$

where

$$D_2F(x,t) = \frac{\partial F}{\partial t} = \lim_{h\to 0} \frac{F(x,t+h) - F(x,t)}{h}\,.$$

That is, can we evaluate the derivative $f'(t)$ by differentiating under the integral sign?

This is a very common step in mathematical analysis, but it is *not* always legal, even if both sides of $(*)$ are well-defined.[15]

35 For example, if we take $X = \mathbb{R}$, $\mu = \mathscr{L}\,\llcorner[0,\infty)$ and $F(x,t) = t^3 e^{-t^2 x}$, then $(*)$ fails to hold.[16]

However, we can establish quite general conditions under which $(*)$ will hold:

36 **THEOREM 27**: With the notation above, suppose

- For each $t \in I$, the function $x \mapsto F(x,t)$ is μ-summable;
- For each $t \in I$, $D_2F(x,t)$ exists for μ-a.e. $x \in X$;
- There is a summable function $M : X \to R$ with

$$\sup_{t\in I} |D_2F(x,t)| \leqslant M(x) \quad \text{for } \mu\text{-a.e. } x \in X.$$

Then f is differentiable, and $(*)$ holds for all $t \in I$.

[15] For an example of such an error made 190 years ago – and which still appears – see *Some Divergent Trigonometric Integrals* by Erik Talvila, American Mathematical Monthly **108** (2001), 432-436.

[16] This example is improper if thought of as a Riemann integral, but that's not the source of the problem. A proper example can be obtained by making the integral substitution $y = \frac{1}{x}$ on the interval $x \in (1,\infty)$.

SOLUTIONS

25 We want to prove that if ϕ and ψ are simple then $\int \phi + \psi = \int \phi + \int \psi$. Changing ϕ and ψ on null sets, we can write

$$\begin{cases} \phi = \displaystyle\sum_{j=1}^{\infty} a_j \chi_{A_j} & \{A_j\} \text{ pairwise disjoint, } \{a_j\} \text{ distinct} \\ \psi = \displaystyle\sum_{k=1}^{\infty} b_k \chi_{B_k} & \{B_k\} \text{ pairwise disjoint, } \{b_k\} \text{ distinct} \end{cases}$$

Include $A_0 = \phi^{-1}(0)$ and $B_0 = \psi^{-1}(0)$ to ensure $\cup A_j = \cup B_k = X$. Now let

$$\begin{cases} C_{jk} = A_j \cap B_k \\ d_{jk} = a_j + b_k \,. \end{cases}$$

Then

$$\phi + \psi = \sum_{j,k=0}^{\infty} d_{jk} \chi_{C_{jk}} \,.$$

This is not canonical, but we can write h_l for the (countably many) *distinct* positive values of d_{jk}, and write

$$H_l = \bigcup_{d_{jk}=h_l} C_{jk} = \{x : \phi(x) + \psi(x) = l\} \,.$$

Then we have the canonical expression

$$\begin{aligned} & \phi + \psi = \sum_{l=1}^{\infty} h_l \chi_{H_l} \\ \implies \quad & \int \phi + \psi = \sum_{l=1}^{\infty} h_l \mu(H_l) = \sum_{l=1}^{\infty} h_l \left(\sum_{d_{jk}=h_l} \mu(C_{jk}) \right) = \sum_{j,k=0}^{\infty} (a_j + b_k) \mu(A_j \cap B_k) \,, \end{aligned}$$

(using additivity on the disjoint sets C_{jk}, and rearranged the nonnegative series). So,

$$\begin{aligned} \int \phi + \psi &= \sum_{j=1}^{\infty} a_j \left(\sum_{k=0}^{\infty} \mu(A_j \cap B_k) \right) + \sum_{k=0}^{\infty} b_k \left(\sum_{j=0}^{\infty} \mu(A_j \cap B_k) \right) \\ &= \sum_{j=0}^{\infty} a_j \mu(A_j) + \sum_{k=0}^{\infty} b_k \mu(B_k) = \int \phi + \int \psi \,. \end{aligned}$$

27 Suppose f and g are integrable, and $f \leqslant g$ a.e. We want to show $\int f \leqslant \int g$, with equality iff $f = g$ a.e.

First assume $0 \leqslant f \leqslant g$ a.e. If $\phi \leqslant f$ a.e. then $\phi \leqslant g$ a.e.. So, by the definition of $\int g$, we have $\int \phi \leqslant \int g$. Taking the *sup* over such ϕ, we see $\int f \leqslant \int g$. For general f, g, write $f = f^+ - f^-$ and $g = g^+ - g^-$. Noting that $f^+ \leqslant g^+$ and $f^- \geqslant g^-$, it easily follows from the nonnegative case that $\int f \leqslant \int g$.

Next, if $\int f = \int g$ we want to show $f = g$ a.e. (the converse implication is trivial). Now, again, by splitting into positive and negative parts, we can assume $0 \leqslant f \leqslant g$. Suppose we don't have $\int f = \int g$. Then for some $\delta > 0$ and some measurable $A \subseteq X$, we have $g \geqslant f + \delta$ on A, and $\mu(A) > 0$. But let $\phi = \delta \chi_A$. Then, by Lemma 17(a),

$$\int g = \int f + (g - f) \geqslant \int f + \int g - f \geqslant \int f + \phi = \int f + \delta\mu(A) > \int f \,.$$

This contradicts our hypothesis, and thus we conclude $\int f = \int g$.

29 Given nonnegative measurable functions f_j on X, we want to prove

$$\int \liminf_{j \to \infty} f_j \mathrm{d}\mu \;\leqslant\; \liminf_{j \to \infty} \int f_j \mathrm{d}\mu \,.$$

To do this, define

$$g_j = \inf_{k \geqslant j} f_k \,.$$

Then $\{g_j\}$ is a monotonic sequence of functions, and thus pointwise convergent. So, we can apply Fatou's Lemma (Theorem 18). Noting $g_j \leqslant f_j$ we conclude

$$\int \lim_{j \to \infty} g_j \mathrm{d}\mu \;\leqslant\; \liminf_{j \to \infty} \int g_j \mathrm{d}\mu \;\leqslant\; \liminf_{j \to \infty} \int f_j \mathrm{d}\mu \,.$$

By definition $\liminf f_j = \lim g_j$, and thus we're done.

30

We want to prove

$$(*) \qquad \int f+g = \int f + \int g\,,$$

as long as the RHS is well-defined. First, note that if $f, g \geqslant 0$ then the result is immediate from Lemma 20. Next, note that if the RHS makes sense then either $\int f^+$ and $\int g^+$ are both finite, or $\int f^-$ and $\int g^-$ are both finite; in the former case

$$\int (f+g)^+ \leqslant \int f^+ + g^+ = \int f^+ + \int g^+ < \infty\,,$$

and the LHS is well-defined. A similar argument holds in the latter case.

Now, for general f and g, we use additivity for nonnegative functions to decompose

$$\begin{aligned}
&\int f+g \\
&= \int (f+g)^+ - \int (f+g)^- \\
&= \int_{\left\{\begin{smallmatrix} f+g>0 \\ f>0 \\ g>0 \end{smallmatrix}\right\}} (f+g) + \int_{\left\{\begin{smallmatrix} f+g>0 \\ f>0 \\ g\leqslant 0 \end{smallmatrix}\right\}} (f+g) + \int_{\left\{\begin{smallmatrix} f+g>0 \\ f\leqslant 0 \\ g>0 \end{smallmatrix}\right\}} (f+g) + \int_{\left\{\begin{smallmatrix} f+g\leqslant 0 \\ f\leqslant 0 \\ g\leqslant 0 \end{smallmatrix}\right\}} (f+g) + \int_{\left\{\begin{smallmatrix} f+g\leqslant 0 \\ f\leqslant 0 \\ g>0 \end{smallmatrix}\right\}} (f+g) + \int_{\left\{\begin{smallmatrix} f+g\leqslant 0 \\ f>0 \\ g\leqslant 0 \end{smallmatrix}\right\}} (f+g)
\end{aligned}$$

On each of these six sets A, we can use additivity on nonnegative functions to prove $\int_A f+g = \int_A f + \int_A g$. Then, we can use additivity on the various integrals separately to prove $(*)$.

31 We assume $\{f_j\}$ are measurable functions, $f_j \to f$, $\{g_j\}$ are summable functions, $|f_j| \leqslant g_j$ a.e., $g_j \to g$ a summable function, and $\int g_j \to \int g$. We note that

$$|f| = \lim_{j\to\infty} |f_j| \leqslant \lim_{j\to\infty} |g_j| = |g| \quad \text{a.e.}$$

Therefore

$$|f - f_j| \leqslant |f| + |f_j| \leqslant |g| + |g_j|\,.$$

Thus, we can apply Fatou's Lemma to give

$$2\int g = \int \lim_{j\to\infty} g + g_j - |f - f_j| \leqslant \liminf_{j\to\infty} \int g + g_j - |f - f_j|$$
$$= 2\int g - \limsup_{j\to\infty} \int |f - f_j| \,.$$

Subtracting $2\int g$, we see $\limsup_{j\to\infty} \int |f - f_j| \leqslant 0$, which implies

$$\int |f - f_j| \to 0 \,.$$

33 These proofs are adapted from *Real Analysis* by Gerald Folland (Wiley, 1984), and *An Introduction to Measure and Integration* by Inder Rana (Narosa, 2005).

(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is Riemann integrable. Then for any $n \in \mathbb{N}$ we can find a partition such that the corresponding upper and lower sums satisfy

$$U_n \leqslant L_n + \frac{1}{n} \,.$$

Further, as n increases, we can assume that the partition chosen is a finer partition than for previous n. (Combining any two partitions only raises L and lowers U). Thinking of the lower and upper sums as (obviously measurable) functions u_n and l_n, we then have $\{u_n\}$ is an increasing sequence and l_n is a decreasing sequence. Let $u = \lim u_n$ and $l = \lim l_n$. Then l and u are measurable, and

$$l \leqslant f \leqslant u \,.$$

Everything is bounded (since f is bounded by definition of Riemann integration), and so we can apply the Dominated Convergence Theorem, to conclude

$$\int l \, \mathrm{d}\mathscr{L} = \lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = \int u \, \mathrm{d}\mathscr{L} = \int_a^b f(x)\mathrm{d}x \,.$$

It then follows from 28 that $u = l = f$ a.e., and that $\int f \mathrm{d}\mathscr{L}$ equals the Riemann integral of f.

(b) Given $f : [a, b] \to \mathbb{R}$ bounded, suppose f is Riemann integrable. Then, as in (a), we have $u = l = f$ a.e. for u and l suitable limits of upper and lower sums. But suppose x is such that $u(x) = l(x)$ and x is not a point used in defining any of the partitions (note that only countably many such points are excluded by this). Then $u_n - l_n$ will be small on a whole interval around x, as long as we stay within the same partition containing x. It follows easily that f is continuous at any such x, and thus that f is continuous a.e..

Conversely, suppose f is continuous a.e.. Let l and u be defined as in (a), with l the limit of a sequence of lower sums converging to the sup, and similarly for u. We also ensure that the size of the partition intervals converges to 0. Then it is easy to see that $l = u$ at any point where f is continuous. But then it easily follows that $\int u = \int l$ (both Riemann and Lebesgue), and thus that f is Riemann integrable.