AMSI 2013: MEASURE THEORY
Handout 3
Measurable Sets and Borel Measures

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INTRODUCTION

It is built into the definition of a measure μ that it be countably subadditive:

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) \leqslant \sum_{j=1}^{\infty} \mu(A_j)\,.$$

What we would hope for is that μ be *countably additive* on pairwise disjoint sets:

$$A_j \cap A_k = \emptyset \text{ for } j \neq k \quad \Longrightarrow \quad \mu\left(\bigcup_{j=1}^{\infty} A_j\right) \stackrel{?}{=} \sum_{j=1}^{\infty} \mu(A_j)\,.$$

In particular, we would like μ to be *additive*:

$$A \cap B = \emptyset \quad \Longrightarrow \quad \mu\,(A \cup B) \stackrel{?}{=} \mu(A) + \mu(B)\,.$$

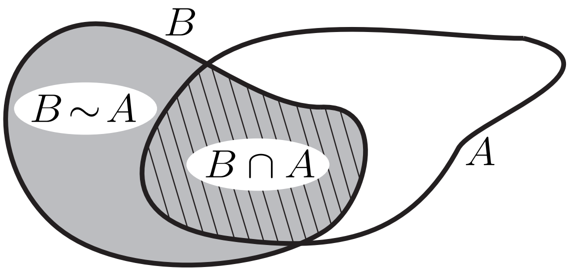
It is easy to show that the Dirac measure and the anything-is-wonderful measure are countably additive on all sets. And, given a (possibly uncountable) collection $\{\mu_j\}$ of countably additive measures and $\{c_j \geqslant 0\}$ then $\sum c_j\mu_j$ is countably additive; so, in particular, counting measure is countably additive on any set X. But, unfortunately, this is generally not the case. For example, the anything-will-do measure is additive only if $A = \emptyset$ or $B = \emptyset$, and similarly negative conclusions hold for the other simple measures we defined. As we discuss below, the situation for Lebesgue measure is trickier: but, if we assume the axiom of choice (or the continuum hypothesis), then we have to accept that Lebesgue measure is not in general additive.

However, even if a measure μ is not additive, it will still be additive when restricted to a suitable, hopefully large, collection of well-behaved sets. To that end, we define for any measure μ the collection of *μ-measurable* sets. We show that the μ-measurable sets form a natural algebraic class (theorem 10) and that μ is countably additive on this class (theorem 6). This class may still be small (for the anything-will-do measure, for example), but we establish a natural condition, *Carathéodory's criterion*, which implies that a measure on a metric space is countably additive on the very large class of *Borel sets* (theorem 11). Many naturally defined measures satisfy Carathéodory's criterion; in particular, we show that Lebesgue measure satisfies the criterion (corollary 12), and thus that Lebesgue measure is what we term a *Borel measure*.

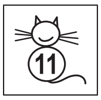
MEASURABLE SETS

Definition: Suppose μ is a measure on X. Then $A \subseteq X$ is *μ-measurable* (or just *measurable*, if the context is clear) if A splits every $B \subseteq X$ in an additive way:

$$\mu(B) = \mu(B \cap A) + \mu(B \sim A) \qquad \text{for all } B \subseteq X.$$



We begin with some simple observations:

- By subadditivity, $\mu(B) \leqslant \mu(B \cap A) + \mu(B \sim A)$, so only the $\geqslant$ is ever at issue.
- Trivially, $\emptyset$ and X are measurable.
- If A is μ_j-measurable for each μ_j in a (possibly uncountable) collection $\{\mu_j\}$, and if each $c_j \geqslant 0$, then A is $(\sum c_j \mu_j)$-measurable.
- 

If $\mu(A) = 0$ then A is measurable. That is, null sets are measurable.

Clearly, measurability captures some notion of additivity. Indeed, if A and B are disjoint, and if either set is measurable, then $\mu(A \cup B) = \mu(A) + \mu(B)$. It is perhaps surprising that measurability is sufficient to also establish countable additivity:

THEOREM 6: Suppose μ is a measure on X. If $\{A_j\}$ is a sequence of pairwise disjoint μ-measurable sets then

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)\,.$$

PROOF:

Part 1: We first use induction to prove finite additivity:

$$\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)\,.$$

The base case is the trivial equality $\mu(A_1) = \mu(A_1)$. Then, for the inductive step, we use A_{k+1} to split $A_1 \cup \cdots \cup A_{k+1}$. Since A_{k+1} is disjoint from the other sets,

$$\begin{aligned} \mu(A_1 \cup \cdots \cup A_{k+1}) &= \mu(A_{k+1}) + \mu(A_1 \cup \cdots \cup A_k) && \text{(measurability of } A_{k+1}\text{).} \\ &= \mu(A_1) + \cdots + \mu(A_k) + \mu(A_{k+1}) && \text{(inductive hypothesis).} \end{aligned}$$

Part 2: We now prove countable additivity. That LHS $\leqslant$ RHS is exactly countable subadditivity, and so we only need to prove that RHS $\leqslant$ LHS. By part 1 and monotonicity,

$$\sum_{j=1}^{n} \mu(A_j) = \mu\left(\bigcup_{j=1}^{n} A_j\right) \leqslant \mu\left(\bigcup_{j=1}^{\infty} A_j\right)\,.$$

Taking the limit as $n \to \infty$, this gives

$$\sum_{j=1}^{\infty} \mu(A_j) \leqslant \mu\left(\bigcup_{j=1}^{\infty} A_j\right)\,,$$

which is exactly what we wanted to prove.

We now want to show the measurable sets form a natural collection. We define

$$\mathcal{M}_\mu = \{A \subseteq X : A \text{ is } \mu\text{-measurable}\}\,.$$

Then, as a first step,

LEMMA 7: Suppose μ is a measure on X. Then $\emptyset, X \in \mathcal{M}_\mu$, and:

(i) $A \in \mathcal{M}_\mu \implies \sim A \in \mathcal{M}_\mu$;

(ii) $A_1, A_2 \in \mathcal{M}_\mu \implies A_1 \cup A_2 \in \mathcal{M}_\mu$;

(iii) $A_1, A_2 \in \mathcal{M}_\mu \implies A_1 \cap A_2 \in \mathcal{M}_\mu$.

Thus $\mathcal{M}_\mu$ is non-empty and is closed under complements, finite unions and finite intersections. Such a collection of sets is called an *algebra* (or, in some texts, a *field*).

PROOF: We have already noted that $\emptyset$ and X are measurable. And, since $\sim A$ splits any set the same as A, it is clear that $\mathcal{M}_\mu$ is closed under complements. Also, by De Morgan's law, (i) and (ii) imply (iii). So, it just remains to show that $\mathcal{M}_\mu$ is closed under unions.

Suppose A_1 and A_2 are measurable. Then, for $B \subseteq X$,

$$\begin{aligned}
\mu(B) &= \mu(B \cap A_1) + \mu(B \sim A_1) && (A_1 \text{ splits } B \text{ additively})\\
&= \mu(B \cap A_1) + \mu((B \sim A_1) \cap A_2) + \mu((B \sim A_1) \sim A_2) && (A_2 \text{ splits } B \sim A_1 \text{ additively})\\
&= \mu(B \cap A_1) + \mu(B \cap (A_2 \sim A_1)) + \mu(B \sim (A_1 \cup A_2)) && (\text{tidying up})\\
&\geqslant \mu(B \cap (A_1 \cup A_2)) + \mu(B \sim (A_1 \cup A_2)) && (\text{subadditivity}).
\end{aligned}$$

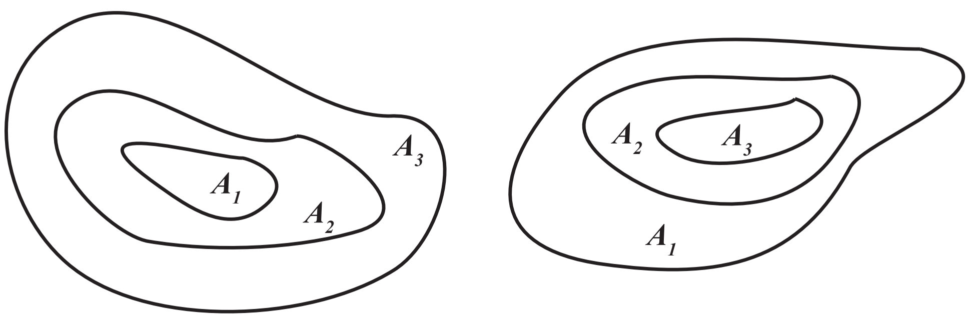
This is exactly what we wanted to prove.

We want to show that $\mathcal{M}_\mu$ is a σ-*algebra*, that it is actually closed under *countable* unions and intersections. This takes some work, and we first prove some preliminary results, which are independently important.

THEOREM 8 (Continuity of Measures): Suppose μ is a measure on X and that $\{A_j\}_{j=1}^\infty$ is a sequence of μ-measurable subsets of X.

(a) If $A_1 \subseteq A_2 \subseteq \ldots$ then $\mu\left(\bigcup_{j=1}^\infty A_j\right) = \lim_{j\to\infty} \mu(A_j)$.

(b) If $A_1 \supseteq A_2 \supseteq \ldots$ and if $\mu(A_1) < \infty$ then $\mu\left(\bigcap_{j=1}^\infty A_j\right) = \lim_{j\to\infty} \mu(A_j)$.



REMARKS

1. If we don't assume $\mu(A_1) < \infty$ then it is easy to give counterexamples to (b).[1] For example, let μ_0 be counting measure on $\mathbb{N}$ and let $A_j = \{j, j+1, \ldots\}$. Then each $\mu(A_j) = \infty$ and $\mu(\cap A_j) = \mu(\emptyset) = 0$.

2. Theorem 8 can also be applied to a general (non-monotonic) sequence $\{A_j\}_{j=1}^{\infty}$ of measurable sets:

(a′) $\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} A_j\right)$.

(b′) $\mu(A_1) < \infty \quad \Longrightarrow \quad \mu\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{n\to\infty} \mu\left(\bigcap_{j=1}^{n} A_j\right)$.

PROOF OF THEOREM 8: To prove (a), we use lemma 7 to split A_j as a disjoint union of measurable sets and apply theorem 6:

$$\begin{aligned} & A_j = A_1 \cup (A_2 \sim A_1) \cup (A_3 \sim A_2) \cup (A_j \sim A_{j-1}) \\ \Longrightarrow \quad & \mu(A_j) = \mu(A_1) + \mu(A_2 \sim A_1) + \mu(A_3 \sim A_2) + \mu(A_j \sim A_{j-1}) . \end{aligned}$$

Thus, by theorem 6 again,

$$\lim_{j\to\infty} \mu(A_j) = \mu(A_1) + \sum_{j=1}^{\infty} \mu(A_{j+1} \sim A_j) = \mu\left(A_1 \cup \bigcup_{j=1}^{\infty} (A_{j+1} \sim A_j)\right) = \mu\left(\bigcup_{j=1}^{\infty} A_j\right) .$$

We now prove (b). First of all, for any j,

$$\begin{aligned} A_j \supseteq \bigcap_{j=1}^{\infty} A_j \quad & \Longrightarrow \quad \mu(A_j) \geqslant \mu\left(\bigcap_{j=1}^{\infty} A_j\right) \\ & \Longrightarrow \quad \lim_{j\to\infty} \mu(A_j) \geqslant \mu\left(\bigcap_{j=1}^{\infty} A_j\right) . \end{aligned}$$

To prove the reverse inequality, note that

$$\emptyset = A_1 \sim A_1 \subseteq A_1 \sim A_2 \subseteq A_1 \sim A_3 \subseteq \ldots .$$

[1] It is not as easy to construct counterexamples showing the necessity of assuming measurability in theorem 8. However, this follows from Vitali's example of a Lebesgue non-measurable set. See 18.

So, by (a) and De Morgan's law (taking complements within A_1),

$$\lim_{j\to\infty} \mu(A_1 \sim A_j) = \mu\left(\bigcup_{j=1}^{\infty}(A_1 \sim A_j)\right) = \mu\left(A_1 \sim \bigcap_{j=1}^{\infty} A_j\right).$$

Using the finiteness of $\mu(A_j) \leqslant \mu(A_1) < \infty$, we can estimate both sides of this equation:

$$\begin{cases} \mu(A_1 \sim A_j) = \mu(A_1) - \mu(A_j) & \text{(additivity on measurable sets),} \\ \mu\left(A_1 \sim \bigcap_{j=1}^{\infty} A_j\right) \geqslant \mu(A_1) - \mu\left(\bigcap_{j=1}^{\infty} A_j\right) & \text{(subadditivity).} \end{cases}$$

Thus,

$$\lim_{j\to\infty} (\mu(A_1) - \mu(A_j)) \geqslant \mu(A_1) - \mu\left(\bigcap_{j=1}^{\infty} A_j\right)$$

$$\Longrightarrow \quad \lim_{j\to\infty} \mu(A_j) \leqslant \mu\left(\bigcap_{j=1}^{\infty} A_j\right),$$

as desired.

Definition: Suppose μ is a measure on X and $B \subseteq X$. Then we define the *restriction* of μ to B, $\mu \llcorner B : \wp(X) \to [0, \infty]$ by

$$\mu \llcorner B(A) = \mu(B \cap A).$$

The usefulness of this definition is made clear by the following lemma.

LEMMA 9: Suppose μ is measure on X and $B \subseteq X$. Then:

(a) $\mu \llcorner B$ is a measure on X;

(b) If $A \subseteq X$ is μ-measurable then A is also $\mu \llcorner B$-measurable.

(c) If B is μ-measurable and $A \subseteq B$ is $\mu \llcorner B$-measurable, then A is μ-measurable.

We emphasise that in (a) and (b) of this lemma there is no hypothesis that B is μ-measurable.[2] Also, though $\mu \llcorner B$ is obviously trivial outside of B, this new measure is still defined on all of X.[3] In particular

$$\mu \llcorner B(X) = \mu(B \cap X) = \mu(B).$$

[2]Note as well that, whether or not B is μ-measurable, B will always be $\mu \llcorner B$-measurable.

[3]In certain contexts it is more natural to regard $\mu \llcorner B$ as a measure on B. It is easy to see that the measurability of $C \subseteq B$ is the same with either interpretation.

Finally, we can prove:

THEOREM 10: Let μ be a measure on X. Then $\mathcal{M}_\mu$ is a σ-*algebra* (or σ-*field*). That is, $\mathcal{M}_\mu$ is non-empty, and is closed under complements, countable unions and countable intersections.

REMARK: This is where many measure theory texts begin, by defining a measure to be a countably additive set function on a σ-algebra of subsets of X. Of course, the chosen meaning of the word "measure" makes no practical difference to the work to be done.

PROOF OF THEOREM 10: By lemma 7 and De Morgan's law, we just have to show $\mathcal{M}_\mu$ is closed under countable unions: if $\{A_j\}_{j=1}^\infty$ is a sequence of measurable subsets of X, we want to show $\displaystyle\bigcup_{j=1}^\infty A_j$ is also measurable. So, for any $B \subseteq X$ we need to show

$$\mu(B) \geqslant \mu\left(B \cap \left(\bigcup_{j=1}^\infty A_j\right)\right) + \mu\left(B \sim \left(\bigcup_{j=1}^\infty A_j\right)\right)$$

If $\mu(B) = \infty$ this is trivial, so we assume $\mu(B) < \infty$. In this case we calculate

$$\begin{aligned}
&\mu\left(B \cap \left(\bigcup_{j=1}^\infty A_j\right)\right) + \mu\left(B \sim \left(\bigcup_{j=1}^\infty A_j\right)\right) \\
=&\mu\llcorner B\left(\bigcup_{j=1}^\infty A_j\right) + \mu\llcorner B\left(\sim\bigcup_{j=1}^\infty A_j\right) && \text{(definition of } \mu\llcorner B\text{)} \\
=&\mu\llcorner B\left(\bigcup_{j=1}^\infty A_j\right) + \mu\llcorner B\left(\bigcap_{j=1}^\infty \sim A_j\right) && \text{(De Morgan)} \\
=&\lim_{n\to\infty}\left[\mu\llcorner B\left(\bigcup_{j=1}^n A_j\right) + \mu\llcorner B\left(\bigcap_{j=1}^n \sim A_j\right)\right] && \text{(lemma 9, theorem 8 and } \mu\llcorner B(X) < \infty\text{).} \\
=&\lim_{n\to\infty}\left[\mu\llcorner B\left(\bigcup_{j=1}^n A_j\right) + \mu\llcorner B\left(\sim\bigcup_{j=1}^n A_j\right)\right] && \text{(De Morgan)} \\
=&\lim_{n\to\infty}\mu\llcorner B(X) && \text{(additivity with respect to } \mu\llcorner B\text{)} \\
=&\mu(B)\,.
\end{aligned}$$

BOREL MEASURES

We now know that a measure μ is countably additive on a σ-algebra $\mathcal{M}_\mu$ of subsets of X. However, for any specific measure - Lebesgue measure, for instance - we still have to consider how large or how small $\mathcal{M}_\mu$ can be. For example, let us define

$$\mathcal{N}_\mu = \{A \subseteq X : \text{either } \mu(A) = 0 \text{ or } \mu(\sim A) = 0\} .$$

$\mathcal{N}_\mu$ is a σ-algebra.

Thus, we have the following chain of σ-algebras of subsets of X:

$$\{\emptyset, X\} \subseteq \mathcal{N}_\mu \subseteq \mathcal{M}_\mu \subseteq \wp(X) .$$

However, this chain can trivialise: for the anything-will-do measure, $\{\emptyset, X\} = \mathcal{N}_\mu = \mathcal{M}_\mu$. Luckily, there is a simple condition, satisfied by Lebesgue measure along with many other natural measures, which guarantees that $\mathcal{M}_\mu$ is a very large collection of sets.

Definition: Suppose that (X, d) is a metric space. For $A, B \subseteq X$, we define the *distance* from A to B:[4]

$$\text{dist}(A, B) = \inf\{d(a, b) : a \in A, b \in B\} .$$

Of course, if A and B intersect then $\text{dist}(A, B) = 0$, but this may be so even if $A \cap B = \emptyset$; for example $\text{dist}(\mathbb{Q}, \sim \mathbb{Q}) = 0$. Also, if A or B is empty then $\text{dist}(A, B) = \infty$: this is obviously rather arbitrary, but in practice is consistent with the vacuous cases that can arise.

We now have

THEOREM 11 (Carathéodory's Criterion): Suppose that (X, d) is a metric space and that μ is a measure on X such that

$$(*) \qquad \text{dist}(A, B) > 0 \quad \Longrightarrow \quad \mu(A \cup B) = \mu(A) + \mu(B) .$$

Then all closed subsets of X are μ-measurable.[5]

Carathéodory's Criterion will take some work to prove, but the converse is easy: if closed sets are μ-measurable, and if $\text{dist}(A, B) > 0$, then the closure $\overline{A}$ splits $A \cup B$ additively, immediately giving $(*)$.

[4] Though we are using the language of "distance", there is no claim that $\text{dist}(A, B)$ is a metric on $\wp(X)$. In fact, the triangle inequality fundamentally fails for $\text{dist}(A, B)$.

[5] The conclusion of theorem 11 is purely topological, suggesting that perhaps the theorem can be generalized. This is indeed the case, in fact even beyond topological spaces. See, for example, §12.8 of *Real Analysis* by H. Royden (3rd ed., Prentice Hall, 1988).

The immediate relevance of Carathéodory's criterion is:

COROLLARY 12: Lebesgue measure $\mathscr{L}^m$ satisfies Carathéodory's criterion $(*)$. Consequently all closed subsets of $\mathbb{R}^m$ are $\mathscr{L}^m$-measurable.

We prove theorem 11 and corollary 12 below, but we first discuss the consequences of these results. The open sets of X are exactly the complements of the closed sets. Thus, by theorem 10, if all closed sets are measurable, then so are all open sets. Further, countable unions of closed sets would also be measurable; these are classically known as $\mathcal{F}_\sigma$ sets. Similarly, countable intersections of open sets, known as $\mathcal{G}_\delta$ sets, would also be measurable. And so on, to $\mathcal{F}_{\sigma\delta}$ sets and $\mathcal{G}_{\delta\sigma}$ sets, and so forth. Thus, starting with the closed sets being measurable, we can generate a huge collection of measurable sets.

What is not clear is when we stop. We have the idea of building up from the closed (or open) sets, of the closed sets "generating" a huge family, but it is not obvious how to precisely define the complete collection of sets obtained. This can indeed be done, using the somewhat deep concept of *transfinite induction*.[6] We'll avoid this, instead enclosing the collection of generated sets from above.

To do this, suppose $(X, \mathcal{T})$ is a topological space, and let

$$\mathcal{C} = \{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra of subsets of } X, \text{ and } \mathcal{A} \text{ contains all closed sets in } X\,.\}$$

$\mathcal{C}$ is non-empty, since $\wp(X) \in \mathcal{C}$. We then define

$$\mathcal{B} = \mathcal{B}(X) = \bigcap \mathcal{C} = \{A \subseteq X : A \in \mathcal{A} \text{ for all } \mathcal{A} \in \mathcal{C}\}\,.$$

$\mathcal{B}$ is called the collection of *Borel subsets* of X. Clearly, $\mathcal{B}$ includes all closed subsets of X.
14 And, it is easy to prove that $\mathcal{B}$ is a σ-algebra.

Then, as argued above, $\mathcal{B}$ will also contain all open sets, and the large class of sets generated by the repeated operations of countable unions and countable intersections.

Suppose now that X is a metric space, and that μ is a measure satisfying Carathéodory's criterion. Theorem 12 then implies that $\mathcal{M}_\mu$ is a σ-algebra containing all the closed sets; that is, $\mathcal{M}_\mu \in \mathcal{C}$, and thus all Borel sets are μ-measurable. We thus have the chain of σ-algebras

$$\{\emptyset, X\} \subseteq \mathcal{B}(X) \subseteq \mathcal{M}_\mu \subseteq \wp(X)\,.$$

In general, if $\mathcal{B} \subseteq \mathcal{M}_\mu$ (whether as a consequence of Carathéodory's criterion or otherwise), we say μ is a *Borel measure*.

[6] See, for example, §4.5 of *An Introduction to Measure and Integration* by I. K. Rana (2nd ed., AMS, 2002).

With our indirect definition of Borel sets it can be hard to get a handle on these sets, and thus on Borel measures. We'll give one result here, which suggests the Borel sets stay in reach of the open and closed sets.

DEFINITION: Suppose μ is a measure on X. We say μ is *finite* if $\mu(X) < \infty$. More generally, μ is *σ-finite* if we can write

$$X = \bigcup_{j=1}^{\infty} A_j \qquad \text{where each } A_j \text{ is measurable with } \mu(A_j) < \infty.$$

Since $\mathbb{R}^m$ can be written as a countable union of bounded m-boxes, it is immediate that Lebesgue measure is σ-finite. Finite measures avoid the problems with ∞, often leading to stronger results. Sometimes, but not always, these results can then be extended to σ-finite measures by simple additivity arguments.

(15) Suppose that X is a topological space, and that μ and ν are finite Borel measures on X with

$$\mu(A) = \nu(A) \qquad \text{for all open } A \subseteq X.$$

Then

$$\mu(B) = \nu(B) \qquad \text{for all Borel } B \subseteq X.$$

(16) The previous result can fail to hold if μ and ν are merely σ-finite.

(17) Suppose μ is a Borel measure on $\mathbb{R}^m$, and suppose that for any open m-box P, we have $\mu(P) = v(P)$, the volume of P. Then $\mu(B) = \mathscr{L}^m(B)$ for all Borel $B \subseteq \mathbb{R}^m$.

It is now time to get down to work, and to prove our main results.

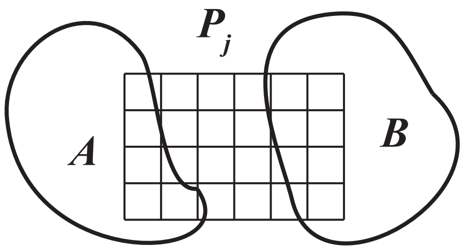
PROOF OF COROLLARY 12:

Suppose $A, B \subseteq \mathbb{R}^m$ and that $\operatorname{dist}(A, B) = \delta > 0$. Then we want to show

$$\mathscr{L}^m(A \cup B) \geqslant \mathscr{L}^m(A) + \mathscr{L}^m(B)\,.$$

Fix $\epsilon > 0$, and let $\{P_j\}_{j=1}^{\infty}$ be a covering of $A \cup B$ with closed m-boxes such that

$$\sum_{j=1}^{\infty} v(P_j) \leqslant \mathscr{L}^m(A \cup B) + \epsilon\,.$$



By subdividing each P_j into smaller m-boxes, we can keep $\sum v(P_j)$ the same while ensuring

$$\operatorname{diam}(P_j) < \delta \qquad \text{for each } j.$$

Once we've done this, no P_j can intersect both A and B, and thus we can split the covering $\{P_j\} = \{Q_k\} \cup \{R_l\}$ into separate coverings of A and B. Thus,

$$\mathscr{L}^m(A) + \mathscr{L}^m(B) \leqslant \sum v(Q_k) + \sum v(R_l) = \sum v(P_j) \leqslant \mathscr{L}^m(A \cup B) + \epsilon\,.$$

By the thrilling ϵ-lemma, we're done.

If we concentrate upon $\mathscr{L}$ for a moment, we now know

$$\mathcal{B} \subseteq \mathcal{M}_{\mathscr{L}} \subseteq \wp(X)\,.$$

One can then ask whether these inclusions are strict. In fact:

18 If $D \subseteq \mathbb{R}$ and $\mathscr{L}(D) > 0$ then D contains a non-measurable subset.[7]

19 There exist Lebesgue measurable subsets of $\mathbb{R}$ which are not Borel.[8]

[7]The original and most famous example of a set that is not Lebesgue measurable is due to Vitali, but there are many others.

[8]With our definition of Borel sets this result is not so easy. However, with the transfinite induction approach to Borel sets (see *Rana*, referenced above), the result is easily proved by a cardinality argument. By building up the Borel sets, it can be shown that $\mathcal{B}$ has the same cardinality as $\mathbb{R}$. On the other hand, any subset of the Cantor Set C is a null set, and thus Lebesgue measurable. But the standard diagonal argument shows that $\wp(C)$ has cardinality strictly greater than that of C. And, the cardinality of C can be shown to be the same as that of $\mathbb{R}$ and thus of $\mathcal{B}$. So, there are simply more Lebesgue measurable sets than Borel sets, and so some Lebesgue measurable sets (in fact, most of them) are not Borel.

Note that the minimal axioms of set theory are seemingly insufficient to prove the existence of Lebesgue non-measurable sets: one probably needs to use something akin to the uncountable axiom of choice,[9] or the continuum hypothesis.[10]

Finally, to end this Handout, we have:

PROOF OF THEOREM 11:

Let $C \subseteq X$ be closed. Then, given $B \subseteq X$, we want to show

$$(\dagger) \qquad \mu(B) \geqslant \mu(B \cap C) + \mu(B \sim C)\,.$$

This is trivial if $\mu(B) = \infty$, so we assume $\mu(B) < \infty$.

Part 1: Fixing $n \in \mathbb{N}$, let

$$C_n = \{x : \text{dist}(x, C) \leqslant \tfrac{1}{n}\}\,.$$

It follows that if $x \in C$ and $y \in \sim C_n$ then $d(x, y) > \frac{1}{n}$. Therefore

$$\text{dist}(B \cap C, B \sim C_n) > \tfrac{1}{n} > 0\,.$$

So, by Carathéodory's criterion $(*)$,

$$(\spadesuit) \qquad \mu(B \cap C) + \mu(B \sim C_n) = \mu((B \cap C) \cup (B \sim C_n)) \leqslant \mu(B)\,.$$

The plan is to let $n \to \infty$ in $(\spadesuit)$, giving $(\dagger)$. But we have to be a little tricky: note that we cannot apply Theorem 8, since we don't know $B \sim C_n$ is measurable.

Part 2: We write the gap between $B \cap C$ and $B \sim C_n$ as a union of bands R_j, where

$$R_j = B \cap \left\{x : \tfrac{1}{j+1} < \text{dist}(x, C) \leqslant \tfrac{1}{j}\right\}\,.$$

Since C is closed, this means no x in $\sim C$ can be approached by a sequence from within C. Consequently, $\text{dist}(x, C) > 0$ for any x in $\sim C$. Therefore

$$B \sim C = (B \sim C_n) \cup \bigcup_{j=n}^{\infty} R_j\,.$$

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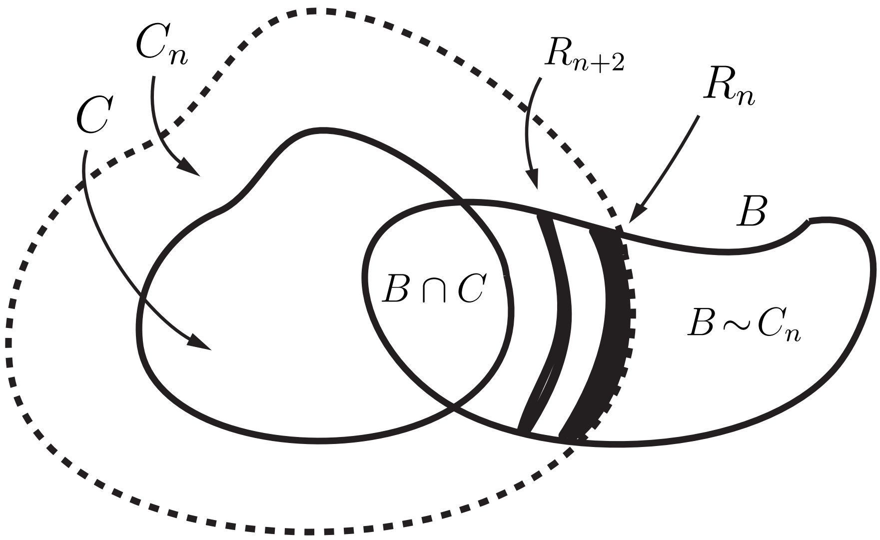
[9] See *Measure and cardinality* by J Briggs and T Schaffter, *American Mathematical Monthly,* **86** (1979), 822-835. Robert Solovay proved, under the assumption that there exists an *inaccessible cardinal*, that it is consistent with the standard axioms of set theory (i.e not including the axiom of choice or the continuum hypothesis) that all sets are Lebesgue measurable: *A model of set theory where every set of reals is Lebesgue measurable, Annals of Maths*, **62** (1970), 1-56.

[10] Stanislaw Ulam proved that the continuum hypothesis implies the existence of Lebesgue non-measurable sets: see §3.4 of *Rana*, referenced above.



So, by subadditivity,

$$(\clubsuit) \qquad \mu(B \sim C) \leqslant \mu(B \sim C_n) + \sum_{j=n}^{\infty} \mu(R_j)\,.$$

We now claim

$$(\blacklozenge) \qquad \sum_{j=1}^{\infty} \mu(R_j) < \infty\,.$$

Assuming the claim, it follows that the tail $\displaystyle\sum_{j=n}^{\infty} \mu(R_j) \to 0$. So, we can take the limit in ($\clubsuit$), giving

$$\mu(B \sim C) \leqslant \lim_{n\to\infty} \mu(B \sim C_n)\,.$$

Then, substituting into ($\spadesuit$), we obtain (†), as desired.

Part 3: It remains to prove the claim ($\blacklozenge$). For any $z \in C$ and for any x and y, the triangle inequality gives $d(x,z) \leqslant d(x,y) + d(y,z)$. Taking the *inf* over all $z \in C$, we see

$$\operatorname{dist}(x, C) \leqslant d(x,y) + \operatorname{dist}(y, C)\,.$$

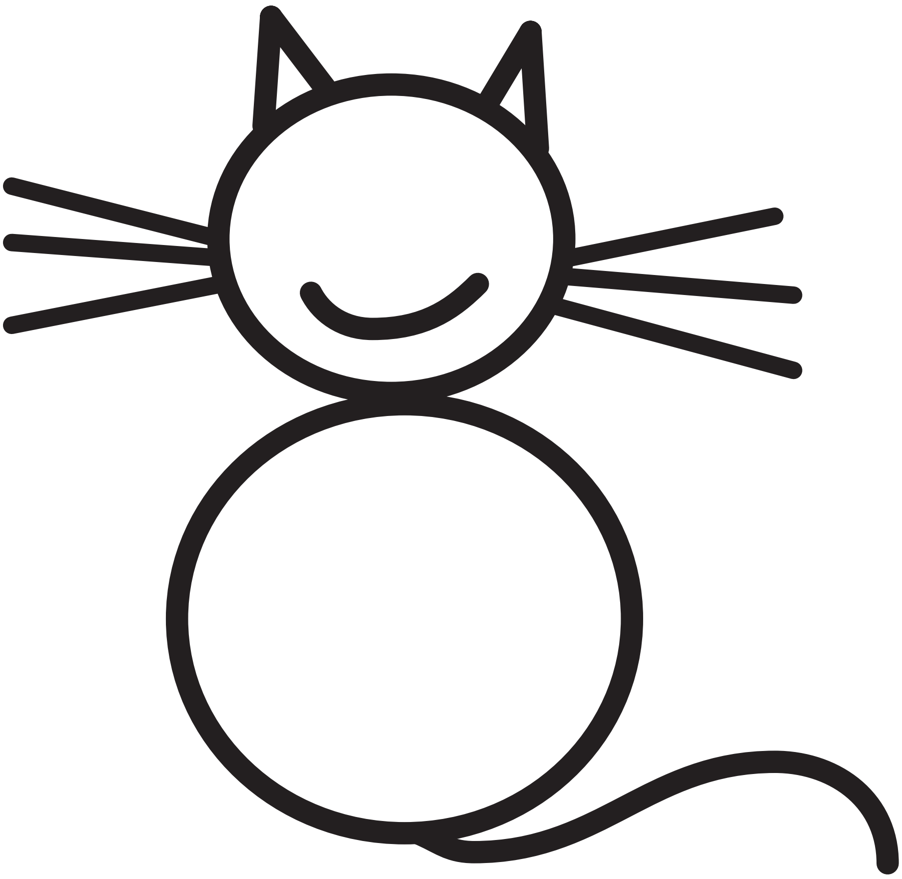
Considering $x \in R_j$ and $y \in R_k$, it follows that if $k > j+1$ then $\operatorname{dist}(R_j, R_k) > 0$. So, applying Carathéodory's Criterion ($*$) again, an obvious inductive argument gives

$$\begin{cases} \mu(R_1) + \mu(R_3) + \ldots \mu(R_{2j-1}) = \mu(R_1 \cup R_3 \cup \cdots \cup R_{2j-1}) \leqslant \mu(B)\,, \\ \mu(R_2) + \mu(R_4) + \ldots \mu(R_{2j}) = \mu(R_2 \cup R_4 \cup \cdots \cup R_{2j}) \leqslant \mu(B)\,. \end{cases}$$

So the partial sums of $\mu(R_j)$ are uniformly bounded, and taking the limit gives

$$\sum_{j=1}^{\infty} \mu(R_j) \leqslant 2\mu(B) < \infty \, .$$

This establishes (⧫), completing the proof.



SOLUTIONS

(14) Given a topological space X, we define the collection $\mathcal{B}$ of Borel subsets of X by

$$\begin{cases} \mathcal{C} = \{\mathcal{A} : \mathcal{A} \text{ is a } \sigma\text{-algebra of subsets of } X, \text{ and } \mathcal{A} \text{ contains all closed sets in } X\,, \\ \mathcal{B} = \bigcap \mathcal{C} = \{A \subseteq X : A \in \mathcal{A} \text{ for all } \mathcal{A} \in \mathcal{C}\}\,. \end{cases}$$

We want to show that $\mathcal{B}$ is a σ-algebra.

First of all $\emptyset \in \mathcal{B}$, since $\emptyset \in \mathcal{A}$ for any $\mathcal{A} \in \mathcal{C}$ (since each such $\mathcal{A}$ is a σ-algebra). Closure under complements and countable unions is just as trivial. For example, suppose $\{B_j\}$ is a sequence of sets in $\mathcal{B}$. Then, $\{B_j\}$ is a sequence of sets in $\mathcal{A}$ for any $\mathcal{A} \in \mathcal{C}$; so, since any such $\mathcal{A}$ is a σ-algebra, it follows that $\bigcap_j B_j \in \mathcal{A}$. Since this is true for all $\mathcal{A} \in \mathcal{C}$ we see $\bigcap_j B_j \in \mathcal{B}$, and thus $\mathcal{B}$ is closed under countable unions.

(15) A hard one![11] We're given finite Borel measures μ and ν on a topological space X with

$$\mu(A) = \nu(A) \text{ for all open } A \subseteq X.$$

We then ask whether $\mu = \nu$ for all Borel subsets of X. The obvious idea is to look at the collection of Borel sets where μ and ν agree:

$$\mathcal{A} = \{B \subseteq X : B \text{ is Borel and } \mu(B) = \nu(B)\}\,.$$

By assumption, $\mathcal{A}$ contains all the open sets. And, because X has finite measure, it follows that $\mathcal{A}$ is closed under complements: if $B \subseteq \mathcal{A}$ then

$$\mu(\sim B) = \mu(X) - \mu(B) = \nu(X) - \nu(B) = \nu(\sim B)$$

Also, by continuity of measures, $\mathcal{A}$ is closed under increasing sequences: if $A_1 \subseteq A_2 \subseteq \cdots$ with each $A_j \in \mathcal{A}$, then $\cup_j A_j \in \mathcal{A}$.

What is *not* obvious is that $\mathcal{A}$ is closed under finite unions or finite intersections. Either would do: if we could show this, then $\mathcal{A}$ would be a σ-algebra containing the open sets, and thus would contain all Borel sets. I don't think there's an easy direct way to prove $\mathcal{A}$ is closed under intersections or unions. What we do instead is a 2-step process, starting with

$$\mathcal{F} = \{B \subseteq X : B \text{ is Borel and } \mu(B \cap A) = \nu(B \cap A) \text{ for all open } A \subseteq X\}\,.$$

[11] The following proof is adapted from *Probability with Martingales* by D. Williams (Cambridge, 1991). The key result he refers to there is "Dykin's lemma" (Appendix A). We'll give the proof with as little machinery as possible, though it's not elegant.

Not that $\mathcal{F} \subseteq \mathcal{A}$ (and we hope the two collections are equal, and equal the Borel sets). By the same arguments, $\mathcal{F}$ contains the open sets, and is closed under complements and increasing sequences of sets. It is still not obvious that $\mathcal{F}$ is closed under unions and intersections, but we now define $\mathcal{G} \subseteq \mathcal{F}$ by

$$\mathcal{G} = \{B \in \mathcal{F} : B \cap D \in \mathcal{F} \text{ for all } D \in \mathcal{F}\}\,.$$

Finally, we have the desired collection: $\mathcal{G}$ contains the open sets (by definition of $\mathcal{F}$), and is closed under complements and increasing sequences. *And*, it is easy to check that $\mathcal{G}$ is closed under finite intersections. Thus $\mathcal{G}$ is a σ-algebra of Borel sets containing the open sets, and thus $\mathcal{G} = \mathcal{B}$. And, if $B \in \mathcal{G} = \mathcal{F} = \mathcal{A}$, setting $A = X$ in the definition of $\mathcal{F}$ gives us that $\mu(B) = \nu(B)$, as desired. (Phew!)

17 We're given a Borel measure μ on $\mathbb{R}^m$ $\mu(P) = v(P)$ for any open m-box P. We want to show that $\mu(B) = \mathscr{L}^m(B)$ for all Borel $B \subseteq \mathbb{R}^m$.

We first show that $\mu(U) = \mathscr{L}^m(U)$ for any open $U \subseteq \mathbb{R}^m$. Obviously, this is true if U is an open m-box. But if U and V are two bounded m-boxes then $U \cap V$ is also an m-box, and so

$$\mu(U \cup V) = \mu(U) + \mu(V) - \mu(U \cap V) = \mathscr{L}^m(U) + \mathscr{L}^m(V) - \mathscr{L}^m(U \cap V) = \mathscr{L}^m(U \cup V)\,.$$

By a simple inductive argument, it follows that μ and $\mathscr{L}^m$ agree on any finite union of m-boxes. By continuity of measures, it follows that μ and $\mathscr{L}^m$ agree on any countable union of m-boxes. But, by second countability of $\mathbb{R}^m$, any open U can be written as such a countable union of m-boxes. It follows that μ and $\mathscr{L}^m$ agree on all open sets.

The general result now easily follows from 15. This result tells us that for any open m-box P, we have $\mu \llcorner P(B) = \mathscr{L}^m \llcorner P(B)$ for any Borel set B. Writing $\mathbb{R}^m = \bigcup_j P_j$ as an increasing sequence of m-boxes, continuity of measures gives us that $\mu = \mathscr{L}^m$ on all Borel sets.

19 We show that there is a Lebesgue-measurable $A \subseteq \mathbb{R}$ which is not Borel.[12] To begin, Let C be the Cantor set, and let $D \subseteq [0,1]$ be a Cantorlike set with $\mathscr{L}(D) > 0$, as constructed in **10**. Next, we use the defining intervals to construct a natural function $f : C \to D$. If $\{I_{nj}\}$ are the intervals defining C and $\{J_{nj}\}$ are the intervals defining D then we define

$$x = \bigcap_{n=1}^{\infty} I_{n,j(n)} \quad \Longrightarrow \quad f(x) = \bigcap_{n=1}^{\infty} J_{n,j(n)} \,.$$

That is, each $x \in C$ is uniquely characterized as the intersection of a nested sequence of intervals, and then $f(x)$ is defined to be the intersection of the corresponding sequence of intervals defining D. It is easy to check that f is well-defined and a bijection. Moreover, f is easily shown to be a homeomorphism:

$$U \subseteq C \text{ is open} \quad \Longleftrightarrow \quad f(U) \subseteq D \text{ is open.}$$

(The easiest way to show this is via sequences: $x_n \to x \Longleftrightarrow f(x_n) \to f(x)$). But then, the same automatically holds for the Borel sets:

$$B \subseteq C \text{ is Borel} \quad \Longleftrightarrow \quad f(B) \subseteq D \text{ is Borel.}$$

(Note that since C and D are closed subsets of $\mathbb{R}$, subsets are Borel whether they are regarded as sets in C and D, or as sets in $\mathbb{R}$).

Now *all* subsets of the Cantor set are null, and thus measurable. Thus, if we can find a non-Borel subset $E \subseteq D$, then $f^{-1}(E)$ will be a non-Borel and measurable subset of C. But such a subset E of D is exactly given by **18**.

[12]This proof uses the Vitali construction of a non-measurable set, and thus the axiom of choice. The transfinite induction characterizations of Borel sets allow one to avoid the axiom of choice. There are also other direct proofs which avoid using choice, but I haven't yet been able to translate them into human language.