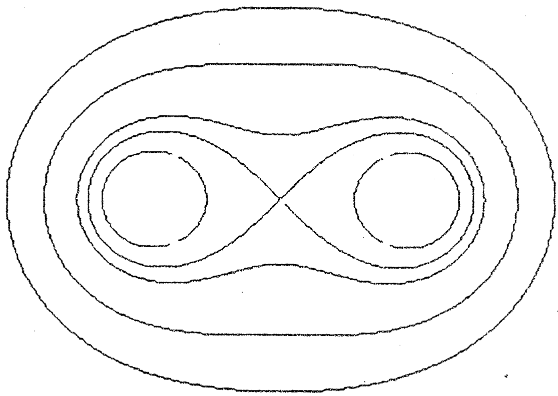
# Function

## A School Mathematics Journal

Volume 25 Part 1 February 2001



Department of Mathematics & Statistics – Monash University

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*Function* is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

*Function* deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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* $14 for *bona fide* secondary or tertiary students.

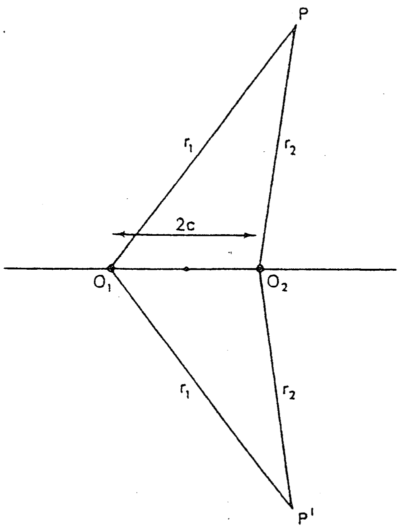
## The Front Cover

This issue's History Column deals with a number of mathematical words and their usages. One of these is the word "oval", once quite popular, now little used. However, the word has remained in mathematical use as the designation of various families of curves, even when (as the History Column makes clear) the term "oval" might be thought to be inappropriate. Our Front Cover for October, 1986 showed a family of curves known as the "Ovals of Descartes". This issue's cover shows a related family known as the "Ovals of Cassini".

Both sets of curves are most commonly expressed in terms of a co-ordinate convention that readers of *Function* will probably not have met. Take two points $O_1$ and $O_2$. These are called the *foci*. In terms of the familiar $x$-$y$ system their co-ordinates are $(\mp c, 0)$ respectively. Then $r_1$ is the distance from a point $P$ to $O_1$ and $r_2$ is the distance from $P$ to $O_2$. In more familiar terms,

$$r_1 = \sqrt{(x+c)^2 + y^2} \quad \text{and} \quad r_2 = \sqrt{(x-c)^2 + y^2}\ .$$

Such co-ordinates are called *bipolar co-ordinates*.



[Notice incidentally that these co-ordinates are ambiguous. Whatever equation gives the point $P$ will also define another point $P'$. See the diagram. This means that all curves defined in terms of bipolar co-ordinates will always be symmetric about the line $O_1 O_2$.]

The "Ovals of Descartes" are given by the equations

$$mr_1 + nr_2 = 2a,$$

where $m$, $n$ and $a$ are constants, and $a > 0$. A special case is the ellipse, for which $m = n = 1$. In that case, we get $r_1 + r_2 = 2a$, which is a well-known property of the ellipse, and the basis for the well-known "pin-and-string" for drawing it.

With the ellipse, there are two parameters involved: $a$ and $c$. The value of $a$ determines the size of the ellipse, $2a$ being the length of its longest diameter. The ratio of $c$ to $a$ tells us the shape of the ellipse. When $c$ is very small compared to $a$, then the ellipse is very nearly circular. For larger values of $c$, the flattening becomes more noticeable.

If we use, instead of the sum $r_1 + r_2$, the product $r_1 r_2$, and make that constant, we reach a set of curves $r_1 r_2 = k^2$, and these are the Cassini ovals. In the examples shown on the Front Cover, $c$ has been taken as 1, and the value of $k$ has been varied. The Cassini Ovals are special cases of another more general family of curves, called the *spiric sections*, which formed the basis of our Cover Story for *Volume* **9**, *Part* **2**.

The outermost oval drawn here has $k = \sqrt{3}$, and if $k$ were larger than this, the curve would be both larger and closer to circular. Coming in from the outermost of the curves drawn, we reach another, critical, one. This is the case $k = \sqrt{2}$. For all values of $k$ larger than this, the curve is convex: it everywhere "bulges outwards". But when $1 < k < \sqrt{2}$, the curve has a "waist" as is to be seen on the third curve, as we come in. [As the History Column remarks, such a curve would not always qualify as an "oval", but the term is applied here for consistency with the other cases which clearly do.]

If $k$ is further decreased, then we reach another critical value at $k=1$. This is a special case with a name in its own right; it is called "The Lemniscate of Bernoulli" and we used it on the cover of *Function* once before (see *Volume* 1, *Part* 4).

Decrease $k$ still further, and the curve separates into two parts, as is seen with the innermost member of the family, for which $k=1/\sqrt{2}$.

Cassini, after whom the curves are named, was Gian Domenico Cassini, also known as Cassini I, as there were four Cassini's, all from the same family and all achieving fame as mathematicians or astronomers. This Cassini lived from 1625 to 1712, and he proposed the Cassini Oval as a description of the path of a planet around the sun. He is also commemorated in the name of a gap in the rings of Saturn and a current space mission.

His planetary orbit of course was incorrect, and Kepler had already got the matter right in his First Law of Planetary Motion (discovered in 1605, but published in 1609). The planets move around the sun along ellipses of which the sun is one of the two foci. In 1687, Newton published his *Principia Mathematica* which used Kepler's Laws to derive the formula for his Law of Universal Gravitation. Cassini was one of Newton's opponents in this matter. He did not believe in Newton's theory.

By now, of course, we know that Kepler and Newton were right and that their opponents were wrong. But matters were not so clear back then. In fact, as the planetary orbits then known were all very nearly circular, it is not easy to distinguish the two curves, the ellipse and the appropriate Cassini Oval with any confidence. The orbit that deviates most markedly from the circular is that of Mars, for which $c/a=1/9$.

For this curve, the *sum* of the largest and the smallest value of the distance from planet to Sun is $10a/9+8a/9$. This is to be compared with the Cassini Oval where these two quantities are *multiplied* together.

If we plot this ellipse on the same axes as the Cassini spiral with $k/c=\sqrt{80}$, we got two curves we can barely distinguish, but then the same holds true for a best-fit circle. To look at the matter further, place the origin at one of the foci, as Kepler did, and then express the result in ordinary polar co-ordinates.

When this is done, the equation for the ellipse is

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

where $e$ is a shorthand for $c/a$. To a good approximation, this gives

$$r = a\{1 - e\cos\theta - e^2\sin^2\theta\}.$$

A similar approximation for the Cassini oval gives

$$r = k\{1 - \varepsilon\cos\theta - \varepsilon^2(\sin^2\theta - \frac{1}{2}\cos^2\theta)\},$$

where $\varepsilon$ is a shorthand for $c/k$.

Now consider the difference between these two expressions for $r$. This comes to (as an excellent approximation) $a - k - \frac{c^2}{2a^2}\cos^2\theta$. To get a numerical feel for this, express it as a fraction of $a$. This gives (with the values already given)

$$\left(1 - \frac{\sqrt{80}}{9}\right) - \frac{\cos^2\theta}{162}.$$

This has a numerical value varying between 0.0062 and (essentially) zero. By somewhat refining the argument and choosing a value of $k$ equal to $c\sqrt{80.5}$, we may make this error even smaller, and nowhere greater than 0.0031.

There are now two points to be made.

The first is that although Cassini was wrong and Kepler was right, the difference is actually very slight. The second is that the astronomy of the day was precise enough to distinguish the two rival theories, and so to decide the matter.

# THE TANGLED TALE OF A PILLOW PROBLEM

**Michael A B Deakin, Monash University**

Lewis Carroll, the celebrated children's author, was in real life Charles Lutwidge Dodgson, a mathematician. I told his story in *Function*, *Vol* 7, *Part* 3, and this article was reprinted with a few rather minor amendments in *Vol* **18**, *Part* **1**.

Those earlier articles gave a brief account of a paradoxical problem and solution that Carroll discussed, and got wrong. In those earlier accounts, the full resolution of the paradox was left as an exercise to the reader, but it now seems that I set too hard a problem, and so now let me set this right and give a full discussion of the source of Carroll's error.

When Carroll wrote in recreational vein, he employed his pseudonym; he wrote his more ambitious pieces of Mathematics under his real name. Thus it was that when, in 1958, some of his Mathematical Recreations were collected and published in an anthology, it was appropriate that they appeared over the name Lewis Carroll.

However, the story is actually more complicated than this. Under his real name, Dodgson published Part I (entitled *A New Theory of Parallels*) of a multi-volume work, *Curiosa Mathematica*, in 1888, and Part II (*Pillow Problems*) in 1893. [A further part, dealing with Arithmetic, was incomplete when he died in 1898.] While all this was going on, his *alter ego* Carroll published *A Tangled Tale* in 1885, *The Game of Logic* in 1887 and Part I of *Symbolic Logic* in 1896 (a Part II was incomplete when he died).

When Dover Publications decided to reprint an omnibus volume of Carroll's logical writings, they put out a two-volume work, of which the first volume contained *Symbolic Logic* (Part I) and *The Game of Logic*, and the second combined *Pillow Problems* and *A Tangled Tale*. Dover used the name Carroll for all the works they reprinted, but the story just told tells us that Carroll (as I shall continue to call him) actually saw *Pillow Problems* as a serious contribution to Mathematics (but the others not). Carroll's error is thus perhaps the more serious on this account. *Pillow Problems* is a collection of 72 worked problems, and our story concerns the last of them.

Recently, Robin Turner, formerly of the Monash University department of Physics, but now retired and back in his native England, raised the question of Problem 72 in *Pillow Problems*. Carroll's answer is absurd, and Turner thought Carroll must have seen this. Personally, I'm a little dubious; I'm inclined to think he was fooled by his own specious logic.

Here's the problem.

> "A bag contains 2 counters, as to which nothing is known except that each is either black or white. Ascertain their colours without taking them out of the bag."

The task set is clearly impossible, but Carroll goes on to give the answer:

> "One is black, and the other white."

Now this is nonsense. Nothing in the data prevents the possibility of the counters both being black (or both white).

However, Carroll argues his case.

> "We know that, if a bag contained 3 counters, 2 being black and 1 white, the chance of drawing a black one would be $\frac{2}{3}$; and that any *other* state of things would *not* give this chance.
>
> "Now the chances, that the given bag contains ($\alpha$) *BB*, ($\beta$) *BW*, ($\gamma$) *WW*, are respectively $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.
>
> "Add a black counter.
>
> "Then the chances, that it contains ($\alpha$) *BBB*, ($\beta$) *BWB*, ($\gamma$) *WWB*, are as before, $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.
>
> "Hence the chance, of now drawing a black one,

$$=\frac{1}{4}\cdot 1+\frac{1}{2}\cdot\frac{2}{3}+\frac{1}{4}\cdot\frac{1}{3}=\frac{2}{3}\ .$$

"Hence the bag now contains *BBW* (since any *other* state of things would *not* give this chance).

"Hence before the black counter was added, it contained *BW*, one black counter and one white."

It all sounds so logical! But, as we saw the answer is not only wrong but also quite absurd. Where is the mistake?

Now Carroll has told us that we know nothing about the counters "except that each is either black or white". So the probabilities of BB, BW, WW are respectively $p$, $q$, $r$, where $p \geq 0, q \geq 0, r \geq 0, p+q+r=1$.

[This is all that can be said in general. However, if it is further known that the counters are drawn at random from a (large) population in such a way as to make $B$ and $W$ equally likely, then we can go further and put $p=\frac{1}{4}$, $q=\frac{1}{2}$, $r=\frac{1}{4}$, and, if you look carefully at Carroll's answer, you'll see that he actually does this. This is his first error, because he later "deduces" the equivalent of $p=r=0$, $q=1$, and so he produces a self-contradiction.]

But, for now, follow his argument and suppose a black counter to be added to the bag. This counter, "the ring-in" let us call it, is necessarily black. As to the others, all we can say of each is that "nothing is known except that it is either black or white".

One can agree with Carroll that the chances of the bag now containing ($\alpha$) BBB, ($\beta$) BWB, ($\gamma$) WWB are as before, but this is to say they are $p$, $q$, $r$ respectively.

So now consider the chance of drawing a black counter in one dip from the modified bag. This can now happen in various ways. Either:

(a) we draw out the ring-in, which is necessarily black, *or*
(b) we draw an original from a BB situation, *or*
(c) we draw an original from a BW situation *and* we happen to select B.

These are three mutually exclusive events, so we will need to add the three corresponding probabilities. The probability of (a) is $\frac{1}{3}$. To get the probability of (b) we need to multiply the probabilities of the two events required to bring it about: the probability of drawing an original counter is $\frac{2}{3}$, and the probability of *BB* is *p*; so the overall probability of (b) is $\frac{2}{3}\cdot p$. In the same way we can compute the probability of (c), which turns out to be $\frac{2}{3}\cdot q\cdot\frac{1}{2}$. So the total probability of selecting a black counter is $\frac{1}{3}+\frac{2p}{3}+\frac{q}{3}$.

If, with Carroll, we set $p=\frac{1}{4}$, $q=\frac{1}{2}$, we now get the answer $\frac{2}{3}$, which is what Carroll needs, but other choices of $p$ and $q$ do not necessarily give this result. Essentially, Carroll has forgotten the different status of the ring-in. Remember that this is known to be black; whereas, of the other two counters, "nothing is known except that [each] is either black or white". His $\frac{2}{3}$ applies only if two of the counters are *known* to be black, the other white. This is his really big blunder.

But now we can take the analysis further. From the result Carroll wants, we deduce that he gets his way if and only if $\frac{1}{3}+\frac{2p}{3}+\frac{q}{3}=\frac{2}{3}$, which simplifies to the condition $2p+q=1$ and because $p+q+r=1$, we must have the general solution $p=r=\frac{1}{2}(1-q)$. So if his argument is to work this is the condition that must apply. [We may also find this result by interchanging the words "black" and "white" throughout to find $2r+q=1$.]

Carroll's solution may be written $p=r=0$; $q=1$, which is only a special case of this general solution, and see how many errors he made in reaching it!

Now look at a somewhat simpler problem. Suppose the bag contains just one counter, either black or white, and that we try the same approach and add a black counter. I leave this as an exercise to the reader, but it will be seen readily enough that the argument falls in a heap. A similar problem occurs with the case of three counters being in the bag initially. (This was first pointed out by Eperson in 1933, and his conclusion was mentioned in the earlier *Function* articles.)

More generally, consider a bag containing $n$ counters, of which $s$ are black and $n - s$ white. Then Carroll's first calculation would go: " We know that, if a bag contained $n + 1$ counters, $s + 1$ being black and $n - s$ white, the chance of drawing a black one would be $\frac{s+1}{n+1}$; and that ... ". His second calculation is harder in the general case; let us see how we can do it. Recall that Carroll had (in the case of 2 counters inside the bag) 3 possible states of affairs, with respective probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$.

But we saw that Carroll had no right to assume this. We need a much more general approach. So let the bag contain $n$ counters, of which $s$ (an unknown quantity) are black. Suppose the probability of this being the case is $p_s$ for each value of $s$. Then, for all values of $s$, $p_s \geq 0$ and also $p_0 + p_1 + p_2 + .... + p_n = 1$. If we now add a black counter to the bag, then the probability of drawing out a black one from the modified bag is

$$\frac{1}{n+1} + \frac{n}{n+1}\left[\frac{0p_0 + 1p_1 + 2p_2 + .... + np_n}{n}\right]$$

in complete analogy with the case $n = 2$, discussed earlier. Carroll wants this to be the same as $\frac{s+1}{n+1}$, again as we saw previously.

So now what Carroll wants is that these two expressions be equal. The condition for this may easily be reduced to

$$1p_1 + 2p_2 + .... + np_n = s, \qquad (*)$$

which tells us that the "expected number", $s$, of black counters in the bag is given by the left-hand side of this last equation. [Our earlier equation

$2p+q=1$ was a special case of Equation (*).] Now Carroll chose values for the various probabilities $p_s$. In fact he had a special case of

$$p_s = \frac{1}{2^n}\binom{n}{s}$$

because he tacitly assumed that the counters were drawn randomly from a very large population in which black and white counters were equally numerous. If we now substitute these values into the equation above, and simplify, we find $s = \frac{n}{2}$. [A proof is given below.]

But this is inconsistent with the assumed probability density, because it tells us that

$$p_s = \begin{cases} 0 \text{ if } & s \neq n/2 \\ 1 \text{ if } & s = n/2 \end{cases}$$

and this is a *different probability density from the one Carroll assumed.*

Oddly enough this comes to the same thing as another well-known fallacy in probabilistic reasoning. The "expected value" is the value that we would achieve by averaging over a great many trials. It is not the same thing as the value we would have any right to "expect" on any single trial!

The only way we could get the same probability density by the two approaches would be to assume a probability density of a form like this last at the very outset, and this of course leads to consistency. The form of the assumed distribution would have to be

$$p_s = \begin{cases} 0 \text{ if } & s \neq S \\ 1 \text{ if } & s = S \end{cases}$$

where $S$ is some *particular* value of $s$.

But this assumption is the same as knowing beforehand how many black counters were in the bag! ($S$ of them!)

It remains to show that Carroll's assumption $p_s := \frac{1}{2^n}\binom{n}{s}$ leads to the inconsistent conclusion $s = \frac{n}{2}$. For this choice of the $p_s$, Equation (*) becomes

$$0 \cdot \frac{1}{2^n} + 1 \cdot \frac{n}{2^n} + 2 \cdot \frac{\binom{n}{2}}{2^n} + \ldots + (n-1) \cdot \frac{n}{2^n} + n \cdot \frac{1}{2^n} = s.$$

Now the left-hand side of this expression is equal to

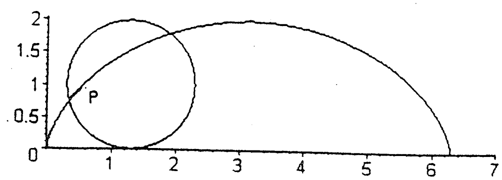
$$\frac{n}{2^n}\left\{1 + (n-1) + \binom{n-1}{2} + \ldots + (n-1) + 1\right\}$$

a sum that has a known value, which you may have seen (although possibly in disguise). That value is $2^{n-1}$, so our equation becomes $s = \frac{n}{2}$ as claimed.

That is to say, the Carroll-type argument "works" only if there are exactly as many white counters as black. No wonder it failed when $n$ equalled 1 or 3. No wonder, also, that it is most seductive in the case $n = 2$.

## A Correction

In our previous issue, Figure 1 of Marko Razpet's article on Cycloids and their relatives was misprinted. Here is the correct version. As the circle rolls along the $x$-axis, the point $P$ traces out the cycloid.



## Letter to the Editor

I enjoyed the magic square ambigram on the cover of *Function, Vol* 24, *Part* 3. Here is another example of this type of square with the added property of remaining magic when held up to a mirror! I created this square in 1976 when I was a student at McMaster University. The sum of the four numbers in any row, column or diagonal is always equal to 19,998. There are a number of other combinations of four numbers that sum to this magic constant as well (the four corner numbers, the four in the centre, etc). I created this square by first making a standard 4×4 square using the numbers from 1 to 16. I then subtracted 1 from each number, converted all the numbers to base 2, and then changed all the 0s to 8s (leaving the 1s intact).

| | | | |
|---|---|---|---|
| 8888 | 1118 | 1181 | 8811 |
| 1811 | 8181 | 8118 | 1888 |
| 8111 | 1881 | 1818 | 8188 |
| 1188 | 8818 | 8881 | 1111 |

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# APPROXIMATING PI

The number $\pi$ is often defined as the ratio of a circle's circumference to its diameter, but actually it is also a fundamental constant of the number system. Euler's formula $e^{i\pi} = -1$ connects the fundamental constant $\pi$ with another fundamental constant, $e$, the base of natural logarithms and also $i$, the square root of $-1$.

Euler's formula has been referred to as "the most remarkable formula in the whole of Mathematics". It may be written in various ways. One nice one is due to Paul Leyland, and is noticed at the website listed below. It goes:

$$\pi = \ln(2-3)/\sqrt{2-3}\,.$$

(This formula is *exact* but it needs a good knowledge of complex numbers to interpret it correctly.)

$\pi$ is known to be transcendental, which means that there is no polynomial equation with rational coefficients whose solution is $\pi$. It follows from this that there is no ruler and compass construction that can yield a length $\pi$ from a unit length.

The task we have just discussed is known as "squaring the circle" and it is one of three famous problems that the ancient Greeks tried and failed to achieve. (The other two are the "duplication of the cube" and the trisection of a general angle. These two were discussed in *Function*, *Vol* **23**, *Part* **4**.) All three tasks are now known to be impossible. The older version of "squaring the circle" in fact asked for the construction of a square with an area equal to that of a given circle. This is equivalent to constructing a length of $\sqrt{\pi}$ , a task which is also known to be impossible. (However, see the last *Function* cover for a bit of harmless fun!)

Nonetheless, if we can accept close approximations to $\pi$, then there are a number of ruler and compass constructions that do very well. A commonly used approximation for $\pi$ is $\frac{22}{7}$. This is used because it is a rather simple number, but it is only a rough approximation. We have

$$\pi = 3.141592653\ldots\ldots ,$$

whereas

$$\frac{22}{7} = 3.142\ldots\ldots. .$$

A much better rational approximation is $\pi \approx \frac{355}{113}$. This is an easily memorised fraction because the digits 1, 3, 5 each occur twice in the pattern of the letter S (reading from bottom left to top right). It is also much better as an approximation:

$$\frac{355}{113} = 3.1415929\ldots. .$$

This approximation was the basis for a discussion in *Function*, *Vol* **21**, *Part* **2**. It also appeared in an earlier issue of *Function*. *Volume* **1**, *Part* **3** contained an article on the mathematician Srinivasa Ramanujan (1887-1920). Ramanujan showed an early and remarkable talent for Mathematics, and he achieved much in spite of many disadvantages. He was very largely self-taught (but not quite to the extent that was once believed, e.g. at the time of the earlier *Function* article).

He specialised in number theory and his output runs to several volumes despite his death at the age of 33. Among his interests was the approximation of $\pi$. He published a ruler and compass construction of $\sqrt{\frac{355}{113}}$ which we reproduced in the earlier issue and also reprint here. It first appeared in the *Journal of the Indian Mathematical Society*, *Vol* **5** (1913), p. 132. He also came up with the formula

$$\pi \approx \frac{63}{25} \times \frac{17 + 15\sqrt{5}}{7 + 15\sqrt{5}} = 3.14159265380\ldots\ldots ,$$

which is correct to nine decimal places (and is construcible by ruler and compass). Another of his formulae is:

$$\pi \approx \left(9^2 + \frac{19^2}{22}\right)^{1/4} = 3.141592652.....,$$

which is correct to eight decimal places and is also constructible.

There are many other approximate formulae for $\pi$. Ramanujan gave other such approximations. Many, though more complicated, were even more accurate. However, we will not pursue these here. Instead we note a number of other remarkable approximations. You may find a good list at

http://www.primepuzzles.net/puzz_050.htm

Among the approximations they mention are

$$\pi \approx \frac{2^9}{163} = 3.1411......$$

and

$$\pi \approx 43^{7/23} = 3.141539.....$$

(The first of these is constructible with ruler and compass; the second is not.)

Perhaps the most surprising is

$$\pi \approx 4\sqrt{\tau}\,,$$

where $\tau$ is the golden ratio $\frac{\sqrt{5}-1}{2}$. This is not particularly accurate; its value is 3.144... . But it is very simple and also interesting in that it relates to another constant of the number system. (It is also the focus of a strange crank, who claims that this is the exact value of $\pi$ and that all the world's mathematicians have got the matter wrong! Some of this stuff appears on the internet, but we will not encourage error by giving the address.) The right-hand expression also describes a number constructible with ruler and compass.

# SQUARING THE CIRCLE

**S Ramanujan**

Let $PQR$ be a circle with centre $O$, of which the diameter is $PR$. Bisect $PO$ at $H$ and let $T$ be the point of trisection of $OR$ nearer to $R$. Draw $TQ$ perpendicular to $PR$ and place the chord $RS = TQ$.

Join $PS$, and draw $OM$ and $TN$ parallel to $RS$. Place a chord $PK = PM$, and draw the tangent $PL = MN$. Join $RL$, $RK$ and $KL$. Cut off $RC = RH$. Draw $CD$ parallel to $KL$, meeting $RL$ at $D$.

Then the square on $RD$ will be equal to the circle $PQR$ approximately.

For $RS^2 = \frac{5}{36}d^2$, where $d$ is the diameter of the circle.

Therefore $PS^2 = \frac{31}{36}d^2$.

But $PL$ and $PK$ are equal to $MN$ and $PM$ respectively.

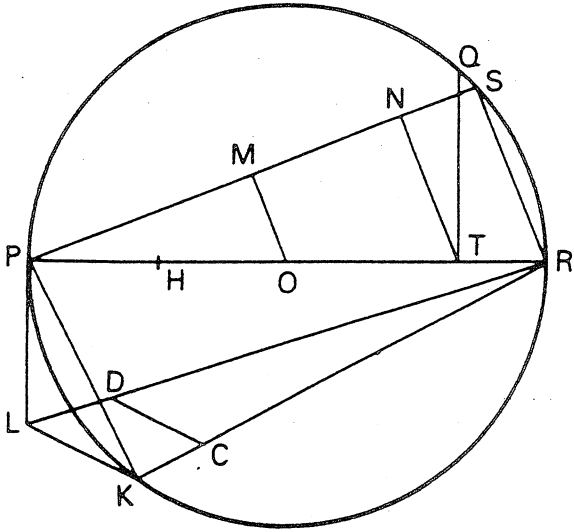
Therefore $PK^2 = \frac{31}{144}d^2$ and $PL^2 = \frac{31}{324}d^2$.

Hence $RK^2 = PB^2 - PK^2 = \frac{113}{144}d^2$ and $RL^2 = PR^2 + PL^2 = \frac{355}{324}d^2$.

But $\frac{RK}{RL} = \frac{RC}{RD} = \frac{3}{2}\sqrt{\frac{113}{355}}$ and $RD = \frac{3}{4}d$.

Therefore $RD = \frac{d}{2}\sqrt{\frac{355}{113}} = r\sqrt{\pi}$ very nearly.

*Note*. If the area of the circle be 140 000 square miles, then $RD$ is greater than the true length by about an inch.




## HISTORY OF MATHEMATICS

### The Good Oil on some Mathematical Words

**Michael A B Deakin**

From time to time, this column strays beyond the bounds of what might strictly be called the History of Mathematics to deal with topics on the fringe of that study, properly so called. Most recently, my last column, in its second half, discussed matters that had perhaps more to do with Philosophy than with Mathematics as such.

This time I will once more stray outside the strict limits, to deal with a number of problem words and their usage. Before I do so, however, I deal briefly with the matter of how words are used and what we mean by "correct" usage.

I am still surprised at how many of my colleagues and other well-educated people fail to appreciate this point. There is a belief in the community, perhaps even especially in the educated community, that some turns of phrase are "right" and others "wrong". But this isn't how matters lie; things are more complicated.

On the one hand, it's perverse and indeed useless to insist on usages that no-one else adopts. Lewis Carroll (in real life the mathematician Charles Lutwidge Dodgson) gently satirised this approach in *Through the Looking Glass*: "'When *I* use a word,' Humpty Dumpty said in a rather scornful tone, 'it means just what I choose it to mean … neither more nor less.'" (He may well have had in mind the tendency of mathematicians to define specialist meanings when words represent technical mathematical concepts.) However, words *do* have agreed meanings, even if those agreed meanings change over time.

Here is a commonly remarked example of such change. When I was in secondary school the word "disinterested" meant that the person so described had no financial, emotional or other such "interest" in the outcome of (e.g.) a sporting context. It was expected that the umpire, for example, would be "disinterested". This was not, however, to say that he or she should not take a keen interest in the progress of the game. Umpires indeed should take such an interest and pay attention to the passages of play. We don't want umpires to be "uninterested". This, once standard, distinction has now almost completely disappeared. Today the words "disinterested" and "uninterested" are very often used interchangeably. We may regret this, because a useful distinction has been lost; but there is nothing we can *do* about it!

There is at least one other way in which "correct" speech is differentiated from "incorrect". When I was in primary school, almost all of my classmates would have said, instead of "I'm not going", "I ain't going". Our teachers laboured mightily to eradicate this habit. Of course the students were acting like the majority of their peers, so that if correctness were decided merely by frequency of use, then "ain't" would have been correct. What the teachers had in mind, however, was that, if in later life one of these students used such a turn of phrase in the context of (say) a job interview, then the job application would be unlikely to succeed.

So correctness depends on at least two factors: (1) general acceptance of a usage, (2) the social status of that usage. With these things in mind, let us look at some usages with a mathematical flavour to them. After all, the way we use mathematical words shows us much about how we approach the underlying concepts.

## A. "Less/Fewer"

"Express Lane: 8 items or less" says my local supermarket, and I find it hard not to want to correct their grammar to "8 items or fewer". The general rule has been: "*fewer* means *a smaller number of*, while *less* means *a smaller quantity of*". This is from Murray-Smith's *Right Words: A Guide to English Usage in Australia* ". He goes on to say that in general "*fewer* should be used with plurals, and *less* with singulars". We could also put the first of his distinctions as: *fewer* goes with discrete quantities, those that we count using the natural numbers, while *less* is to be used when the quantities involved are continuous.

On both stories, we should say "less than ten kilograms", not "fewer than ...." (although *kilograms* is a plural word). Murray-Smith explains that here we have a shorthand for *less than a weight of ten kilograms* (and the implied word *weight* is singular); the other distinction would have it that the number of kilograms need not be a positive integer, but could well be (say) 0.391, as we may readily check at the meat or the cheese counter of the same supermarket.

But when we come to "items", the word is plural and the measure in terms of natural numbers. So *fewer* is to be preferred. However, the case is not entirely clear-cut. We mathematicians use *less than* almost all the time, and *fewer than* almost never! "The result [The Riemann Hypothesis] has been checked for all cases less than 1.5 billion", for example. So we might see the supermarket advice as a shorthand for "You may use this lane if the number of items in your trolley is 8 or less". In this form of words, *fewer* sounds a trifle pedantic; *less* is more natural.

The case is one of slightly uncertain usage. Murray-Smith notes that the April 1985 *Newsletter* of *The Society of Editors* (surely an authoritative source) has "In publishing women get less perks than men"; they don't say "fewer perks", although Murray-Smith says they "should know better"!

All in all, if you follow the manuals, you're unlikely to sound silly, but their advice is not always rigidly binding. I leave readers to explore for themselves the various usages of "more" and "greater".

## B. Proved/Proven

If, like me, you sometimes spend idle moments with your computer and play the games that come with recent versions of Windows, you will likely encounter the game of Freecell. Its Help file states: "The object of the game is to move all the cards to the home cells, using the free cells as placeholders. To win, you make four stacks of cards on the home cells: one for each suit, stacked in order of rank, from lowest to highest." After this it offers the information that "It is believed (although not proven) that every game is winnable". [This belief, incidentally, is now known to be incorrect.]

Here I would tend to say "proved" rather than "proven", and I think that most Australians of my generation would do the same. This is a case in which two words, originally with somewhat different meanings, have merged (as we saw with "disinterested" and "uninterested"). "Proven", which tended to be pronounced *PROAVEN* rather than *PROOVEN*, had a precise meaning in Law, especially in Scotland, where "not proven" is an allowable verdict, intermediate between "guilty" and "not guilty".

[The alleged poisoner, Madeleine Smith, was discharged with a "not proven" verdict, which was widely interpreted as the jury's way of saying "We think she did it, but the crown have not proved their case beyond reasonable doubt".]

This same usage is common with, for example, oil reserves. "Proven oil reserves" are those oil reserves that have been subject to test drilling and assessed as likely to be commercially attractive.

More recently, "proven" has come to mean simply "proved", and is pronounced as *PROOVEN*. The Merriam-Webster Dictionary of 1994 has this to say on usage: "The past participle *proven*, orig[inally] the past participle of *preve*, a Middle English variant of *prove* that survived in Scotland, has gradually worked its way into standard English over the past three and a half centuries. It seems to have first become established in legal use and to have come only slowly into literary use. Tennyson was one of its earliest frequent users, prob[ably] for metrical reasons. It was disapproved by 19th century grammarians, one of whom included it in a list of "words that are not words." Surveys made some 30 or 40 years ago indicated that *proved* was about four times as frequent as *proven*. But our evidence from the last 10 or 15 years shows this no longer to be the case. As a past participle *proven* is now about as frequent as *proved* in all contexts. As an attributive adjective <proved or proven gas reserves> *proven* is much more common than *proved*."

Merriam-Webster is an American Dictionary. The Oxford, which is English, is somewhat different in its approach, and rather less friendly to "proven". In 1933, they had *proven* as a variant of *proved* and noted its connection to Scottish law; by 1982 they had accepted the "oil reserves" usage, and noted the meaning "tested, approved, shown to be successful".

Nonetheless, there are two considerations here. The first is that young people tend to say and to write *proven*, rather than *proved*. Thus the former is likely to win out, if only by natural selection. The second is that international English now follows the American, rather than the British, standard. (After all, there are a lot more Americans than Britishers!)

**C. Oval**

Among several other related meanings, an oval may be either a geometric figure or else something possessing this shape, as for example a football field (in Australian Rules). The word itself means "egg-shaped" in its earliest origins, but much of the time it means "elliptical". For the mathematician today, every ellipse is an oval, but it is not true that every oval is an ellipse.

There is probably no fixed consensus among mathematicians as to which curves are ovals. Certainly, an oval is usually a simple closed curve; one that joins up at its two ends, and does not intersect itself, have two separate parts or breaks in its line or involve any other such complications. I would also claim that in many usages, the word "oval" implies that the curve is convex, which is to say that (in lay language) it "bulges out" all around its length, or perhaps (at very least) nowhere "bulges inward". (So that Flemington Racecourse could be described as "oval", although its perimeter includes straight sections.)

However, there are many exceptions to this rule. One is to be found on the cover of a previous issue of *Function* (*Vol* **10**, *Part* **5**). This showed a set of curves called "the ovals of Descartes", and those shown on the front cover were all strict ovals in the sense given. On the back cover, however, there were shown other members of this same family, and these did not have all the properties just listed.

*The Oxford Dictionary* lists many meanings for the word, but those describing plane figures are "A plane figure resembling the longitudinal section of an egg; a closed curve having the chief axis considerably longer than the one at right angles to it, and curvature greatest at each end; strictly with one end more pointed than the other, as in most eggs, though popularly also applied to a regular ellipse; in *Mod*[ern] *geom*[etry] applied also to any closed curve (other than the circle or ellipse) esp[ecially] one without a node or a cusp".

This deserves a few comments. Most hen's eggs possess the property mentioned (one end being pointier than the other), but this is not necessarily true for the eggs of other species. Nodes are self-intersections; cusps are points at which the slope is discontinuous. One of the ovals of Descartes in fact does have a cusp, but we don't discriminate against it! It is a strange turn of phrase that uses the term "oval" of the US president's office (which is elliptical in shape) when on another usage, "oval" can apply to almost anything except an ellipse! Another set of curves (Cassini's Ovals) in fact includes members that fall into two parts or else self-intersect.

All this may explain why the use of the term "oval" is now not widespread in Mathematics; so many different meanings have been attached to it!

## Oblong

When I was in primary school, we were taught the term "oval" when "ellipse" was really meant. It was only later that I was introduced to the word "ellipse". And so it was with "rectangle"; the word given to this figure in the primary classroom was "oblong". I later came to wonder whether this usage wasn't merely a local curiosity, and perhaps unknown outside rural Tasmania, but *The Oxford Dictionary* assures me otherwise. Indeed they note a usage in Geometry: "Rectangular with adjacent sides unequal". In other words, a rectangle that is not also a square.

I doubt if modern geometers use the word in this sense, or indeed at all! The basic meaning is (again from the *Oxford* ): "Elongated in one direction (usually a deviation from an exact square or circle; having the chief axis considerably longer than the transverse diameter".

However, it seems to me that if "oval" is now on its way out of Mathematics, then "oblong" has already left.

**There is/There are**

In *Function's* early days, we needed to publish a correction to an article that in fact contained two errors. I wrote "There are a couple of errors in ...", but the then Chief Editor, Gordon Preston, altered this to "There *is* a couple of errors in ... ". Had the sentence been slightly recast, then we would both have agreed on "There are two errors in ... ". My way of looking at things was that "a couple of" meant *two*, Gordon's was that "couple" was a singular word: there was only one couple. Presumably Gordon would have said: "A couple of men *has* come to see you", while I would say : "A couple of men *have* come to see you".

But, even if we disagreed on this particular issue, we in fact agreed on the more basic principle that one should say "There *is* one ... ", as against "There *are* two (or more) ... ". Currently there is something of a blurring of this distinction. It is now by no means uncommon, perhaps even more common than not, to hear politicians or broadcasters say such things as : "There is two objections to the GST [for example]".

Such usages have been with us for some time, but I would see them as becoming more common in the last few years. One authority, Eric Partridge, in his *Usage and Abusage*, after quoting several examples going back to 1777 of the "there is [many]" usage, nonetheless condemns it: "**there is many** is incorrect for *there are many*". However, he somewhat weakens his stance by stating that "There is many a [slip 'twixt the cup and the lip] is correct".

Perhaps this last example shows that, here as elsewhere, day-to-say grammar does not always follow strict rules of logic. There are other related cases. J E Littlewood, in *A Mathematician's Miscellany*, notes (p 40) the usages 'more than one is' and 'fewer than two are'. In both these examples, the "feel" of an adjacent word overrides the dictates of strict logic.

These cases are not like the disinterested/uninterested one seen earlier. No real change of meaning is involved between "There is ... " and "There are ... ". For the moment, the case is closer to that of "ain't". It sounds uneducated to say "There is ... ", where a more careful speaker would say "There are ... ". But if enough politicians and broadcasters continue with the former, then this perception will fade away.

**Average/Ordinary**

The words "average" and "ordinary" have come, quite recently, to mean "poor" or "below expectation". If we think about it, this seems silly. I suppose it could be said that the football commentator who says "[X] put in a very average performance" means that X, playing at the elite level involved, is expected to put in an A-grade effort at all times. This, however, does not explain many popular usages of these words. Everyone has to be the best! (However silly this sounds when we put it like that!)

The word "average" has taken on this pejorative (i.e. derogatory) meaning quite recently. It is not noted even by Merriam-Webster. However, the word "ordinary" took this path somewhat earlier. In this case, Merriam-Webster note the meaning: "deficient in quality: poor, inferior".

We may also notice that this is the *second* time that the word "ordinary" has been downgraded in meaning. Early last century, with the altered spelling "ornery", it appeared, probably in America, with a meaning somewhat akin to its pejorative meaning today. However, this version of the word rapidly slid further down the slope of derogation and today it has only one meaning "cantankerous": the only meaning Merriam-Webster now allow, although *The Oxford* lists others, now obsolete.

Nor is "ordinary" the only word to slide from a meaning more or less approximating "normal" to another, less complimentary one. The word "mediocre" originally referred to a hill of medium height, as to a medium-size bar on a bar-chart. "Mediocre" still retains a meaning of "in the middle of the range", but this is rarely the sense nowadays. Like the modern senses of "average" and "ordinary", it has come to manifest a predominantly pejorative flavour.

# COMPUTERS AND COMPUTING

## Maths Treasure in Legal Trouble

**Cristina Varsavsky**

In Function *Vol* 23, Part 2 we started the new practice of referencing mathematics resources on the Web; these were well received by some of our readers who wrote about the usefulness of those resources. One of the listed mathematics gems on the Web was Eric Weisstein's *World of Mathematics* located at

http://mathworld.wolfram.com/.

The World of Mathematics Web site, also known as *Eric's Treasure Trove of Maths*, is the result of resources Weisstein collected and built over his whole life as a mathematician and later extended with contributions from the numerous Web site visitors. Even though his main mathematical interest is planetary astronomy, the Web site contains thousands of notes, cross-references, figures, algorithms and Java applets spanning over the whole spectrum of mathematical topics and applications.

If you haven't yet seen this site, then do not rush to typing in the url in your browser as you might well be disappointed to find out that the site has been shut down by court order while pending trial.

The court order was obtained by CRC Press, a publishing company known for technical reference books. It was with this company that Weisstein signed a book deal three years ago in which he had agreed to turn the then existing Web resources into an encyclopaedia. This 1969-page book, titled *Concise Encyclopaedia of Mathematics*, turned out to be a best seller sold both in hardcover and CD-ROM format, and it is now in its second printing.

The agreement signed between the two parties was not very clear about who has the copyright of the Web site which formed the basis of the book. Eric's understanding was that he sold CRC the right to print the book and that he kept the right to keep up his Web site. Moreover, after the deal with CRC, *Eric's Treasure Trove of Math's* continued expanding and evolving from its original version, while the encyclopedia remained the same.

At first, CRC saw the Web site as a good promotional aid to the sales of the book and did not claim any copyright over it. So why the sudden change of mind? Why has the relationship between the author and the publisher turned sour? The heart of the matter is the latest (naive?) move from Weisstein to hand the Web site sponsorship to Wolfram Research Institute where he is now a full time employee. *Eric's Treasure Trove of Mathematics*, came to be called *The World of Mathematics* when moved to the Wolfram Web site. It is this association between the author and Wolfram Research Institute that sparked the legal action taken by CRC. Wolfram Research Institute, a company known for making the computer algebra software *Mathematica*, happens to be a direct competitor of CRC.

The central issue of the dispute is the extent of what CRC acquired when signing the contract with Weisstein. The contract says that the author granted CRC the full and exclusive rights to the book, including "without limitation, the right to reproduce, publish, sell, and distribute copies of the Work [book], selections therefrom, and translations and other derivative Works based upon the Work [book], in print, audio-visual, electronic, or by any and all media now or hereafter known or devised, and the right to license or authorize others to do any or all of the foregoing throughout the world." So the key issue is whether the Web site is a derivative of the book or not. CRC claims it is, while Wolfram Research institute claims that the book is an authorised derivative work and the Web site is the original work. Web publishing is causing some serious headaches to lawyers and judges when trying to apply the definition of publishing to media introduced after the relevant law was written.

Internet has led to changes in the way we access and interact with information spread all over the world. We bookmark these resources, and we expect to be able to access them when needed; they are like books on the shelves of the world wide library. But beware! Books can simply evaporate from the shelves.

While the legal battle goes on, the Mathematics community is being deprived of these valuable resources. We hope this legal dispute is resolved promptly, and that by the time you receive this issue of *Function* the Web-based Mathematics trove is available again to teachers, students and lovers of mathematics. If not, you could visit the Web site to learn more about the intricacies involved in the legal process. But you will probably find more interesting to visit the trouble-free science treasure trove, also a creation of Eric Weisstein. Given the popularity and public support of the maths treasure cove, Weisstein extended it to cover several areas of science. This is located at

http://www.treasure-troves.com

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## PROBLEMS AND SOLUTIONS

### SOLUTION TO PROBLEM 24.3.3

A solution was submitted by Zdravko F Starc of Vršac, Yugoslavia, but reached us too late for acknowledgement in the previous issue.

**PROBLEM 24.4.1** (from *Mathematical Spectrum* ) read:

Let $ABC$ be an acute-angled triangle and let $D$ and $E$ be the points on $BC$ such that angle $ADB$ is a right angle and angle $DAB$ = angle $EAC$. Prove that:

$$(\text{area } \Delta\, EAC) > (\text{area } \Delta\, DAB) \text{ if and only if } AC > AB.$$

### SOLUTION

Because angle $ADB$ is a right angle, $AD < AE$. There are two cases to consider: (1) $AC > AB$, (2) $AC < AB$.

See the diagram overleaf.

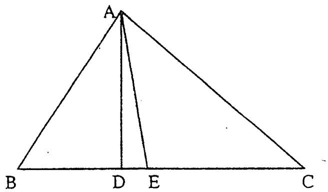


**Figure 1**

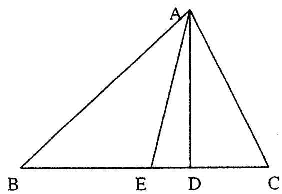


**Figure 2**

In the first case,

$$\begin{aligned}\text{area } \Delta DAB &= \frac{1}{2} AD.AB \sin \angle BAD \\ &= \frac{1}{2} AD.AB \sin \angle EFC < \frac{1}{2} AE.AC \sin \angle EFC = \text{area } \Delta EAC.\end{aligned}$$

In this scenario, the point $E$ lies in the interval $DC$, and outside the interval $BD$. In the other case, Case (2) above, the point $E$ lies in the interval $BD$, and outside the interval $DC$.

Then angle $DAB$ = angle $EAB$ + angle $EAD$,
and angle $EAC$ = angle $DAC$ + angle $EAD$.

Thus angle $EAB$ = angle $DAC$. The given condition is precisely that with used previously, but with the rôles of $B$ and $C$ interchanged. We thus have

$$\text{area } \Delta DAC < \text{area } \Delta EAB.$$

The result follows by adding to both sides the area of the triangle $EAD$.

[This solution combines one supplied by John Jeavons, previously editor of this section, and another submitted by Carlos Victor. Another solution, using a different argument, was sent in by Julius Guest.]

**PROBLEM 24.4.2** (also from *Mathematical Spectrum* ) read:

The Smarandache function is defined by $\eta(n) =$ the smallest positive integer $m$ such that $n$ divides $m!$.

(a) Calculate $\eta(p^{p+1})$, where $p$ is a prime.
(b) Find all positive integers $n$ such that $\eta(n) = 10$.
(c) Prove that, for every real number $k$, there is a positive integer $n$ such that

$$\frac{n}{\eta(n)} > k.$$

Does $\frac{n}{\eta(n)} \to \infty$ as $n \to \infty$?

**SOLUTION**

(a) The integers: $p$, $2p$, $3p$, ... , $p.p$ are all to be factors of the number $m!$ so the smallest positive integer available is $m = p^2$, since $p$ is prime.

Thus $\eta(p^{p+1}) = p^2$.

(b) Because we can write $10!$ as $2^8.3^4.5^2.7$ and $9!$ as $2^7.3^4.5.7$, $n$ must have the form $2^a.3^b.5^c.7^d$, where $0 \le a \le 8$, $0 \le b \le 4$, $0 \le c \le 2$, $0 \le d \le 1$. And so for $n$ to divide the first number and not the second, we need also to satisfy either $a = 8$ or $c = 2$ (or perhaps both).

(c) If $p$ is prime, then from Part(a) we have $\frac{p^{p+1}}{\eta(p^{p+1})} = p^{p-1} \ge p^2$ if $p \ge 3$, and this will be greater than $k$ for all primes greater than $\sqrt{k}$. The limit, however, is not infinity (in fact does not exist) since for all primes, no matter how large, $\frac{p}{\eta(p)} = 1$.

[This solution combines those supplied by John Jeavons, previously editor of this section, and Carlos Victor.]

**PROBLEM 24.4.3** (from *Crux Mathematicorum* ) read:

Find all real numbers $x$ such that

$$x=\left(x-\frac{1}{x}\right)^{1/2}+\left(1-\frac{1}{x}\right)^{1/2}$$

**SOLUTION**

Let $u=\left(x-\frac{1}{x}\right)^{1/2}$ and $v=\left(1-\frac{1}{x}\right)^{1/2}$ so that $u^2=x-\frac{1}{x}$ and $v^2=1-\frac{1}{x}$ .

Then $u + v = x$ and $u^2-v^2=x-1$. Since $x\neq 0$, we have by division $u-v=1-\frac{1}{x}$ . By addition, we have $2u=x+1+\frac{1}{x}$ . This may be written as $2u=u^2+1$, which reduces to $(u-1)^2=0$. Thus $u = 1$.

So $x-\frac{1}{x}=1$, and this equation has the solutions $x=\frac{1\pm\sqrt{5}}{2}$ . Substitution into the original equation shows that only $x=\frac{1+\sqrt{5}}{2}$ fulfils all the requirements.

[This solution was sent in by J A Deakin. The problem appeared in *Mathematical Mayhem* as well as in *Crux Mathematicorum*, as *Function* indicated last year. Mr Deakin notes that it previously appeared in C W Trigg's book *Mathematical Quickies* (Dover, 1985), where a solution is given equivalent to that reproduced above. Other solutions were submitted by Julius Guest, John Jeavons and Carlos Victor.]

**PROBLEM 24.4.4** (from *Parabola*) read:

In a triangle with sides $a$, $b$, $c$ the angle opposite side $a$ is twice the angle opposite side $b$. Prove that $a^2=b(b+c)$.

## SOLUTION

Since $\angle A = 2\times\angle B$, we may write $\angle C = \pi - 3\times\angle B$, or more succinctly $A = 2B$ and $C = \pi - 3B$. Now use the sine rule to find:

$$a = 2R\sin(2B), \quad b = 2R\sin B, \quad c = 2R\sin(3B),$$

where $R$ is the circumradius.

Then

$$b(b+c) = 4R^2 \sin B(\sin B + \sin(3B)) = 16R^2 \sin^2 B\cos^2 B = (2R\sin(2B))^2 = a^2.$$

[This solution came from Julius Guest. Solutions were also received from J A Deakin, John Jeavons and Carlos Victor.]

Here are some new problems.

**PROBLEM 25.1.1** (Submitted by T Trotter from an article in *Journal of Recreational Mathematics* : available online at

http://www.geocities.com/ttrotter3/ )

The *triangular numbers* are those numbers that "count the dots" in triangular arrays:

```
*        *        *        *
         * *      * *      * *
                  * * *    * * *
                           * * * * ,      etc.
```

Denote the first triangular number by $T_1$, the second by $T_2$, etc. Then the first few triangular numbers are $T_1 = 1,\ T_2 = 3,\ T_3 = 6,\ T_4 = 10,$ etc.

Prove that $T_n + T_{n-1}T_{n+1} = (T_n)^2$ .

[There is quite a lot written on the triangular numbers and many nice formulae are known for them. Mr Trotter discovered this one in 1973. He wonders if there was any previous discovery.]

**PROBLEM 25.1.2** (Submitted by Julius Guest)

$ABC$ is a triangle and $a$ is the length of the side $BC$, $b$ the length of the side $CA$, and $c$ the length of the side $AB$. Let $h_A$ be the length of the perpendicular drawn from $A$ to the side $BC$. Suppose that the magnitude of the angle $A$ is known, as are the length $h_A$ and the difference $b - c$.

Determine $a$, $b$ and $c$.

**PROBLEM 25.1.3** (Submitted by Julius Guest)

$ABC$ is a triangle and $a$ is the length of the side $BC$, $b$ the length of the side $CA$, and $c$ the length of the side $AB$. Let $h_A$ be the length of the perpendicular drawn from $A$ to the side $BC$. Suppose that the magnitude of the angle $A$ is known, as are the length $a$ and the difference $b^2 - c^2$.

Determine $a$, $b$ and $c$.

**PROBLEM 25.1.4** (Submitted by J A Deakin)

Evaluate $\int_0^\pi \frac{xdx}{1+\cos^2 x}$.

"The mathematician seeks a new logical relationship, a new proof of an old relationship, or a new synthesis of many relationships."

P R Halmos

Published by Department of Mathematics & Statistics, Monash University