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Department of Mathematics & Statistics – Monash University

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Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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EDITORIAL

Welcome to our readers!

The calculation on the front cover involving a 2x2 determinant is printed on one of the glass doors in the Department of Mathematics at the University of Erlangen, Germany. The calculation is correct independently of the side of the door you view it. Unfortunately we had to reproduce it on paper, but you can check it yourself by using a mirror.

This issue starts with an account of the discovery of a new mathematical constant which arises from modifying the Fibonacci sequence by introducing random factors in its recursive formula. We recommend you to read it, as the result is quite surprising.

Ken Evans looks at a simple application of mathematics to a real-estate related question: Is there a block of land such that if the length is decreased and the width is increased by the same amount, the areas would be equal? Perhaps you could try answering the question before reading the article …

In the *History of Mathematics* column, Michael Deakin carries further the story in the last issue about the very early attempts at sorting out the basis of probability theory. For the readers not so familiar with the computational power of spreadsheets, the article in the *Computers and Computing* column gives examples of the use of this kind of software in solving mathematical problems which involve recursion.

The Australian team had another outstanding performance at the last International Mathematical Olympiad. Our correspondent sent us this news and the problems presented to the participants. For the less ambitious, we include the regular *Problem Corner*, with solutions from our readers to previously published problems, and new problems for you to attempt.

Finally, you will also find in this issue a review of a book about Paul Erdös— one of the greatest mathematicians of the 20th century. A "must read" for anybody interested in mathematics.

* * * * *

A NEW MATHEMATICAL CONSTANT

Malcolm Clark, Monash University

Most readers of *Function* will be aware of various famous mathematical constants: π, the ratio of the circumference to the diameter of a circle, e, the base of natural logarithms, and φ, the Golden Ratio so beloved by ancient Greek artists and mathematicians, given by the formula $\varphi = (1+\sqrt{5})/2 = 1.61803$ approximately.

A recent mathematical result by Divakar Viswanath, a young Indian-born computer scientist now at the University of Chicago, has led to the discovery of a new mathematical constant, and a surprising connection with the well-known Fibonacci sequence.

The story begins in the early 13th century, when the great Italian mathematician Fibonacci posed the following simple problem:

> A man puts a pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year, if the nature of these rabbits is such that every month each pair bears a new pair which from the second month on becomes productive?

Ignoring such realistic happenings such as death, escape, infertility or whatever, it can be shown that the number of pairs of rabbits in the garden at the end of each month are given by the numbers in the *Fibonacci sequence*:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots .$$

This sequence is defined by the initial conditions

$$f_1 = 1, f_2 = 1$$

together with the *recurrence relation*

$$f_{n+2} = f_n + f_{n+1} \quad \text{for all positive integers } n. \qquad (1)$$

Most popular books on mathematics mention the Fibonacci sequence, and observe that the Fibonacci numbers arise frequently in nature. For example, if you count the number of petals on a flower, the answer is often a Fibonacci number, much more frequently than you would get by chance. Furthermore, many plants, such as sunflowers, cauliflowers, pineapples and pine cones, exhibit distinctive spiral patterns. There are usually two systems of florets, seeds, twigs, petals, or whatever, going in opposite directions, and the number of spirals in these systems are generally consecutive Fibonacci numbers.

As well as its connections with the natural world, the Fibonacci sequence has a number of curious mathematical properties. Perhaps the most surprising is that as you proceed along the sequence, the ratios of successive terms get closer and closer to the Golden Ratio φ (see Robyn Arianrhod's article in *Function*, Vol 16, Part 4, 1992). Furthermore, the successive terms of the sequence become approximately proportional to the n-th power of φ.

To be more precise,

$$f_n \approx \frac{\varphi^n}{\sqrt{5}} \quad \text{for large } n.$$

In other words, to compute the n-th term in the Fibonacci sequence, raise the Golden Ratio to the power n, divide by $\sqrt{5}$, and round off to the nearest integer.

This procedure works remarkably well, even for $n = 3$! In that case, $\varphi^3 / \sqrt{5} = 1.8944...$, which rounds off correctly to give $f_3 = 2$.

This procedure gives an easy way of answering Fibonacci's original question, assuming that unlike Fibonacci himself we have access to a scientific calculator. Since Fibonacci assumed that each pair of rabbits did not become productive until after two months, the answer to his original question is f_{14}, the 14th term in the sequence. The 14 arises from the 12 months in the year plus the lag of two months before the first pair of rabbits start to reproduce.

You can easily verify that $\varphi^{14} / \sqrt{5} = 376.987...$, which estimates $f_{14} = 377$ to an accuracy of 0.0034%!

We now ask a "What if?" question typical of mathematicians. Suppose that when you generate the Fibonacci sequence, you flip two coins at each stage. If the first coin comes up heads, you multiply f_n in the recurrence relation (1) by +1, but if it comes up tails, you multiply it by -1. Similarly, the multiplier of f_{n+1} is either $+1$ or -1, depending on whether the other coin comes up heads or tails.

For example, one possible sequence you could get in this way is:

1, 1, (HH) 2, (HT) 1, (TT) −3, (TH) 4, (HT) 7, (HH) 11, ...

Another is:

1, 1, (HT) 0, (HT) −1, (TT) 1, (HT) 2, (HH) 3, (TT) −5, ...

The random sign changes can lead to sequences that suddenly switch from large positive numbers to large negative numbers, such as:

1, 1, 2, 3, 5, 8, 13, 21, −8, 13, −21, ...

as well as sequences that cycle endlessly through a particular pattern, such as:

1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, ...

or

1, 1, 0, 1, −1, 0, −1, 1, 0, 1, −1, ...

If your coins keep coming up heads every time, you can even generate the original Fibonacci sequence:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Note that the recurrence relation for these *random* Fibonacci sequences can be expressed as:

$$f_{n+2} = X_n f_n + Y_n f_{n+1}$$

where each X_n and Y_n is equally likely to take the value $+1$ or -1, independently of all the other random X_n's and Y_n's.

With these random sequences showing such a variety of behaviour, it's not obvious that such random sequences follow the nice kind of growth pattern of the Fibonacci sequence.

But they do. In 1999, Viswanath showed that, ignoring special cases, the absolute value of the n-th number of any random Fibonacci sequence generated as described is approximately equal to the n-th power of the number $K = 1.13198824$.

Actually, we need to be more precise. Because the sequences are generated randomly, there are infinitely many possibilities. Some of them will clearly not have this property. For example, the sequence that cycles endlessly through 1, 1, 0 does not have this property, and neither does the original Fibonacci sequence. But those are special cases. What Viswanath showed was that if you start to generate one of these random sequences, then with probability 1, the sequence you get will have the 1.13198824 property. In other words, you can safely bet your life that for your sequence, the bigger n is, the closer the absolute value of the n-th number gets to the n-th power of 1.13198824.

Viswanath's result brings to an end a puzzle that originated in a 1960 paper that showed that for a *general* class of random-sequence generating processes that includes the random Fibonacci sequence, the absolute value of the n-th member of the sequence will, with probability 1, get closer to the n-th power of some fixed number. Since this general result applied to the random Fibonacci sequence we have described, it followed that, with probability 1, the absolute value of the n-th term in such a sequence will get closer and closer to the n-th power of some number K. But no one knew the value of the number K, or even how to calculate it.

What Viswanath did was find a way to compute K, or at least its first 8 decimal places. Almost certainly, K is irrational, and so cannot be computed exactly. Since there was no known algorithm for computing K, Viswanath had to adopt a rather indirect route, involving large doses of advanced mathematics and some heavy computing. Since his computation made use of "floating-point arithmetic" – which is not exact – he had to conduct a detailed mathematical analysis to obtain an upper limit on any possible errors in the computation.

With that computation, mathematics has a new constant, with a surprising connection to Fibonacci and his pair of rabbits.

Note: This article is based partly on a simplified account by Keith Angle on the web pages of the Mathematical Association of America. To read this, first go to the web address:

`http://www.maa.org/`

then select <u>Columns</u>, then <u>Devlin's Angle</u>, then <u>Past Columns</u>, and then look for Devlin's March 1999 "column". (Incidentally, there is a host of interesting material on this web site.)

Viswanath's result has now been published in the journal *Mathematics of Computation*, Vol 69 (2000), pp. 1131–1155. To find out more about Viswanath and this result, look at his web page:

`http://www.cd.uchicago.edu/~divakar/`

For loads of interesting information about Fibonacci sequences and related topics, visit the web page

`http://www.ee.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html`

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It may come as no surprise that my ability to communicate in English was severely limited when I came to Baltimore. The hardest problem was ordering meals, since I invariably got something that didn't even vaguely resemble what I thought I ought to be getting. I finally solved the problem of lunches, which I ate in a drugstore, a block from where I lived . I learned to say reliably, "Cream cheese sandwich and coffee." Unfortunately, the young man who served me would always respond : "On toast?" and all I could do was to smile inanely. The result was satisfactory since I liked what I got, and my smiles were taken as signs of acquiescence. I looked up "toast" in the somewhat inadequate Polish-English, English-Polish pocket dictionary I carried and it gave it just one meaning: "Gentlemen, the King!" Having been logically conditioned, I assumed that "on toast" must be some kind of salutation and I proceeded on these assumptions. For a period of about two weeks the following ritual took place at lunch:

I: "Cream cheese sandwich and coffee."

Waiter: "On toast?"

I (bowing slightly and smiling): "On toast!"

Somehow it dawned on me that something was not quite right and I finally asked van Kampen. He laughed a little longer than I thought was kind, and finally said: "Why didn't you at least ask once in the negative? You would have soon known what 'on toast' meant". "I didn't want to risk to be impolite", I replied and he laughed again.

—Marc Kac in *Enigmas of Chance*
1985, New York: Harper and Row

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DIFFERENCE AND RATIO

K McR Evans

A friend of mine recently asked me for the dimensions of my rectangular block of land. I was able to say, "22m frontage (width) and 45m depth (length). He said, "That's interesting because my block has a frontage of 20m and a depth of 47m; mine is 2m longer than yours and 2m less in width, so the areas of our blocks will be the same." "Not true!", I said, to which she replied, "please explain."

I began, "The perimeters of our blocks are the same, but the area of mine is $45 \times 22\text{m}^2 = 990\text{m}^2$, and the area of yours is $47 \times 20\text{m}^2 = 940\text{m}^2$, so there is a difference of 50m^2. My friend understood this but didn't find it satisfying. He said, "Is there a block such that, if I decreased the width/length and increased the length/width by the same amount, the areas would be equal?" "Good question!", I said, "We will have to do some algebra to find out."

Consider two blocks of land with dimensions as shown in Figure 1.

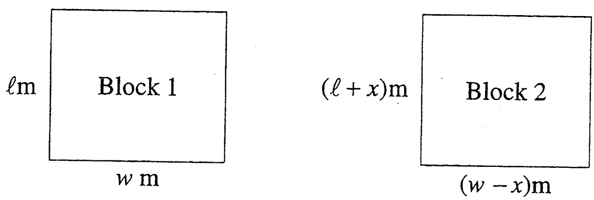

Figure 1

First notice that the perimeter of block 1, $2(\ell + w)\text{m}$, is equal to that of block 2. Next we look at possible values of the variables. Clearly $\ell > 0$ and $w > 0$, but x may be negative or zero as well as positive. However there are some restrictions on x: since $\ell + x > 0$, $x > -\ell$ and since $w - x > 0$, $x < w$. Thus x is between $-\ell$ and w, which is written

$$-\ell < x < w \qquad (1)$$

Quantities are often compared by their *difference* (eg Ian Thorpe won the 200m race by 10m) or by their *ratio* (eg you ate 3 times as much as your younger brother). In this case, let $y\,\mathrm{m}^2$ be the difference in the area of the two blocks. Thus

$$\begin{aligned} y &= \left| \ell w - (\ell + x)(w - x) \right| \\ &= \left| \ell w - (\ell w - \ell x + wx - x^2 \right| \\ &= \left| \ell x - wx + x^2 \right| \\ &= \left| x(x + (\ell - w)) \right| \end{aligned} \tag{2}$$

The two blocks have the same area if and only if $y = 0$. From (2), $y = 0$ if and only if $x = 0$ or $x = -(\ell - w) = w - \ell$. If $x = 0$, block 2 has the same dimensions as block 1. If $x = w - \ell, (\ell \neq w)$, block 2 (see Figure 1) has width $(w - (w - \ell))\mathrm{m} = \ell \mathrm{m}$ and length $(\ell + w - \ell)m = wm$ so block 2 has the same dimensions as block 1, but with length and width interchanged. If $\ell = w$, $y = 0$ if and only if $x = 0$ so the blocks have the same dimensions.

Further light can be thrown on how y varies with x by sketching the graph of equation (2) with the restriction (1). Note that, as x approaches w, y approaches $\left| w(\ell - w + w) \right| = \ell w$, and similarly as x approaches $-\ell$, y approaches ℓw. (The area of block 2 approaches zero, and the difference in areas approaches the area of block 1). With the restriction (1), the graph of the quadratic equation

$$y = x(x + (\ell - w)) \tag{3}$$

is a parabolic arc shown in Fig. 2: in 2(a) and 2(c) part of the parabolic arc is dotted. The graph of (2) coincides with that of (3) wherever that of (3) is on or above the x-axis, and is the reflection of the graph of (3) in the x-axis where the graph of (3) is below the x-axis. Hence the graph of (2) is is the union of three parabolic arcs if $\ell > w$ or if $\ell < w$, and is one parabolic arc if $\ell = w$. (See Figure 2)

The graphs in Figure 2 suggest another problem. At first glance it might appear that y cannot be greater than (or equal to) ℓw, ie it might appear that the difference in areas of blocks 1 and 2 cannot be greater than or equal to the area of block 1. This is true if $\ell = w$, (Figure 2(b)), but in the other cases it might be possible for the y-coordinate of the turning point of the "middle" arc to be greater than or equal to ℓw. The condition for this to occur turns our to depend on the *ratio* ℓ / w.

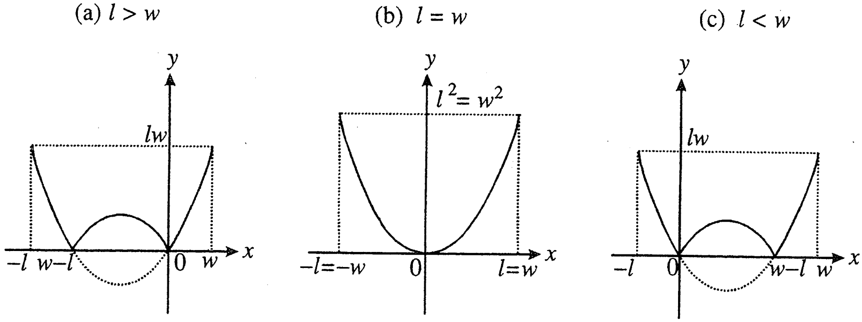

Figure 2

Firstly we determine the coordinates of the turning point of the graph of (2). By symmetry the x-coordinate is $\dfrac{w-\ell}{2}$. Hence from (2), the y coordinate is

$$\begin{aligned}\left|\frac{w-\ell}{2}\left(\frac{w-\ell}{2}+\ell-w\right)\right| &= \left|\frac{w-\ell}{2}\times\frac{\ell-w}{2}\right| \\ &= \left|-\frac{(\ell-w)^2}{4}\right| \\ &= \frac{(\ell-w)^2}{4}\end{aligned}$$

The condition for the y-coordinate to be greater than or equal to ℓw is

$$\frac{(\ell-w)^2}{4} \ge \ell w$$

$$\begin{aligned}&\Leftrightarrow \quad \ell^2 - 2w\ell + w^2 \ge 4\ell w \\ &\Leftrightarrow \quad \ell^2 - 6w\ell + w^2 \ge 0 \\ &\Leftrightarrow \quad (\ell-3w)^2 - 8w^2 \ge 0 \\ &\Leftrightarrow \quad (\ell - (3-2\sqrt{2})w)(\ell-(3+2\sqrt{2})w) \ge 0\end{aligned}$$

$$\Leftrightarrow \quad \ell \geq (3+2\sqrt{2})w \text{ or } \ell \leq (3-2\sqrt{2})w$$

$$\Leftrightarrow \quad \begin{array}{l} \ell / w \geq 3+2\sqrt{2} \approx 5.838 \text{ or} \\ \ell / w \leq 3-2\sqrt{2} \approx 0.172 \end{array} \tag{4}$$

As a check, consider the case $\ell = 7w > (3+2\sqrt{2})w$. The y-coordinate of the turning point is $\dfrac{(7w-w)^2}{4} = 9 \cdot w^2$ while $\ell w = 7w^2$. Thus the difference in areas of the two blocks is greater than the area of block 1. The area of block 2 must be $16w^2$ m^2.

Exercises

1. (a) If $\ell = 7w$ and $x = \dfrac{w-\ell}{2}$, find, in terms of w, the area of each block, and show that block 2 is a square.

   (b) Show that if $x = \dfrac{w-\ell}{2}$, then block 2 is always square.

2. Show that in (4), the critical values of ℓ / w, viz

$$3 + 2\sqrt{2} \text{ and } 3 - 2\sqrt{2}$$

are reciprocals of each other. Why might you expect this to be the case?

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> [Wolfgang Bolyai] was extremely modest. No monument, said he, should stand over his grave, only an apple-tree in memory of the three apples: the two of Eve and Paris, which made hell out of earth, and that of Newton, which elevated the earth again into the circle of the heavenly bodies.
>
> —F Cajori in *History of Elementary Mathematics*
> 1910, New York

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Book Review: ***The Man who loved only Numbers: The story of Paul Erdös and the search for Mathematical Truth*, by Paul Hoffman** (New York: Hyperion, 1998, ISBN 0-7868-8406-1, 318 pp. Incl. Index).

Paul Erdös (pronounced "air-dish"), who died in September 1996 at the age of 83, was one of the most famous and prolific mathematicians of the twentieth century. He wrote or co-authored almost 1500 academic research papers, most of them of great significance.

Erdös was a mathematical prodigy, who at the age of 3 could multiply 3-digit numbers in his head. For the last 25 years of his life, he worked on mathematical problems for 19 hours or so per day: "a mathematician", he would say, "is a machine for converting coffee into theorems."

As Hoffman describes it,

> Erdös structured his life to maximise the amount of time he had for mathematics. He had no wife or children, no job, no hobbies, not even a home to tie him down. He lived out of a shabby suitcase and a drab orange plastic bag from Centrum Aruhaz ("Central Warehouse"), a large department store in Budapest. In a never-ending search for good mathematical problems and fresh mathematical talent, Erdös crisscrossed four continents at a frenzied pace, moving from one university or research centre to the next. His modus operandi was to show up on the doorstep of a fellow mathematician, declare, "My brain is open," work with his host for a day or two, until he was bored or his host was run down, and then move on to another home.

Hoffman uses a large number of sources to give a fascinating picture of Erdös: his intense curiosity, his desire to seek out and encourage new mathematical talent, his sense of humour and quirky sayings ("SF", "The Book", "bosses", "slaves", "epsilons"), and his encouragement of collaboration amongst mathematicians.

The mathematical problems covered in this book include the search for prime numbers, patterns amongst prime numbers, and combinatorial algorithms. All are described clearly and succinctly, and Hoffman conveys both the importance of the problems and the excitement amongst mathematicians when a solution is found.

The book contains eleven chapters, numbered 0, 1, 2, e, 3, π, 4, 5, 6, 7, ∞, a sequence Erdös would have approved of. There is a 14-page index, a 10-page bibliography, and 10 pages of acknowledgments and source notes.

Hoffman tells the story of Paul Erdös in a style which is always readable and absorbing, and by turns funny, moving and exciting. This book is a "must-read" for anybody interested in mathematics.

Reviewed by
Malcolm Clark,
Monash University

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Why did Hungary produce so many mathematicians? Hungary was a small country (it is even smaller today) not much industrialised, and it produced a disproportionately large number of mathematicians, several of whom were active in this country. Why was that so? ... a good part of the answer can be found in [Lipót] Fejér's personality: He attracted many people to mathematics by the success of his own work and by his personal charm. He sat in a coffee house with young people who could not help loving him and trying to imitate him as he wrote formulas on the menus and alternately spoke about mathematics and told stories about mathematicians. In fact, almost all Hungarian mathematicians who were his contemporaries or somewhat younger were personally influenced by him, and several started their mathematical career by working on his problems.

—George Polya in "Some Mathematicians I have known"
Amer Math Monthly 76, pp. 749

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Paul [Erdős] has the theory that God has a book containing all the theorems of mathematics with their absolutely most beautiful proofs, and when he wants to express particular appreciation of a proof he exclaims, "This one is from the book!"

—Ross Honsberger in *Mathematical Morsels*
DC: Math Assoc of America, 1978

HISTORY OF MATHEMATICS

More on the Basis of Statistics

Michael A B Deakin

In my last column, I looked at some of the very early attempts at sorting out the basis of Probability Theory, and concentrated on how we assign probabilities to events. If we are told what the probability is that some event or other will occur, then we can make computations on this basis, but in real life we often have to find out for ourselves what that probability is, and to discover this from limited data. In this column, I want to carry this story further, but again I will have to restrict consideration to a very limited part of the whole, in part because of space, but also in part because of the inherent difficulty of much of the subject.

Broadly speaking, theories about the meaning of probability fall into two main categories. On the one hand, there are those referred to as "frequentist" or "classical", and these are the ones mostly taught in our schools and universities. On the other hand are those theories that are classed as "subjective", "Bayesian" or "neo-Bayesian". These last two descriptions derive from the name of Thomas Bayes, who featured in the previous article.

In order to make the distinction clearer, I will again consider the case of a coin being tossed. On the classical account, the coin will have an inherent probability of landing "tails", and this probability is a fixed property of the coin itself. Call it p, say; then p will be a number between 0 and 1. Whatever toss we make, there will be a probability p that the coin will land "tails" and this probability will be independent of what went on in any previous tosses. The tosses are referred to as being "independent" and the coin is said to have "no memory".

This way of looking at things is most commonly associated with the name of R A Fisher (1890–1962), and it is the viewpoint I used when I discussed the estimate we would make of the value of p in the event of our observing 19 tails in 20 tosses. Fisher's methodology involves the "doubting Thomas" whom we seek to convince and the levels of evidence that I rather disrespectfully referred to as "magic numbers".

There is however another way to look at matters. This is associated with a number of modern thinkers as well as with the names of Bayes and Laplace, who appeared in the previous column. There are various possible approaches, but the one I shall describe is most associated with the name of the Italian mathematician

Bruno de Finetti (1906–1985). According to de Finetti, probability is not something that resides in the coin itself, but rather measures *the strength of our own belief.* For example, if I think an event is virtually impossible, then I will assign to it a probability only just above zero, while if I thought the event to be almost certain, I would put a probability of almost 1 upon it.

The difficulty in assigning probabilities does not appear if we have a very large data-bank to draw upon. So if a coin showed "tails" 95% of the time in (say) 1 000 000 tosses, then Fisher would assign to p a value very close to 0.95. So, although from a different point of view, would de Finetti. For Fisher, whatever "magic number" we picked, we would see a very small range of probable values of p and these would all lie very close to 0.95. For de Finetti, one would be very silly to hold to a belief other than this, because the evidence should have shown us that the coin had a strong bias toward "tails", which came up 95% of the time in the course of a long experiment.

The difference between the two outlooks however becomes very marked if the data-set is small. John Venn (1834–1923) put this very well.

> "Let a [fair] penny be tossed up a very great many times; we may then be supposed to know for certain this fact (amongst many others) that in the long run head and tail will occur about equally often. But suppose we consider only a moderate number of throws, or fewer still, and so continue down to three or two, or even one? We have, as extreme cases, certainty or something undistinguishably near it, and utter uncertainty. Have we not, between these two extremes, all gradations of belief?"

[John Venn, incidentally, is the same Venn after whom Venn diagrams are named.]

For de Finetti, there was no concept of independence of different tosses. This is not because de Finetti assumed that the outcome of one toss affected the probable outcome of another. Rather he looked at matters like this. Suppose, for example, I toss a coin n times and I see (to make matters simple) n tails. (Remember that this was the situation considered by Laplace in his "Law of Succession".) Then my *belief* that I am seeing an experiment with a strongly biassed coin will grow stronger with each successive tail that comes up. We aren't talking about the coin as such; rather about *my belief* about the coin.

In what follows, I shall attempt to show some of the flavour of de Finetti's work. However, I have reconstructed it from various sources that you may not consider very reliable. I first learned of this body of theory over 15 years ago, and I got my information from a lecture by Dennis Lindley (who wrote an article for *Function, Vol* 9, *Part* 3). It is hard to find the main result (Equation (5) below) given explicitly in de Finetti's writings, and after much searching I reluctantly gave up. (de Finetti, even in English translation, is an extremely difficult author and he also wrote an awful lot!)

De Finetti replaced the notion of "independence" with that of "exchangeability". To illustrate what is involved, suppose that a coin is tossed three times. Then we can get 8 different results of which (with obvious notation) TTH, THT and HTT are three possibilities. What these three possibilities have in common is that two tails and one head have appeared in all of them. If we say that the probability of 2 tails and 1 head (which can occur in three different ways) is $\omega_2^{(3)}$, then the average (mean) of the possibly different probabilities is $\frac{1}{3}\omega_2^{(3)}$. Now, not because he says that the three different events need actually have the same probability, but rather because we restrict information to the number of tails and of heads in any experiment, he takes this value as the probability of two tails and one head. The three results are "exchangeable" because *we* do not distinguish them.

More generally the probability of r tails and s heads (where $r + s = n$) will be written as $\omega_r^{(n)}\Big/\binom{n}{r}$ where $\binom{n}{r}$ is the number of ways in which r objects may be chosen from a set of n such objects. To make for ease of expression, I will write $\omega_r^{(n)}\Big/\binom{n}{r} = f(r,s)$, where s is a shorthand for $n - r$. Then $f(r,s)$ will be the probability we attach to any string of T's and H's, just as long as there are r of the former and s of the latter. Thus $f(r,s)$ is the probability assigned to any specific sequence involving r tails and s heads, but it makes no difference *which* such sequence we use; any one may be *exchanged* for any other. The different sequences give differing *histories* of the state of my belief, but the end result is the same.

Now let us suppose that we have tossed the coin n times and that r tails have appeared. We ask what is the probability that the next toss will yield a tail. (This is the same sort of question that Laplace asked in relation to his Law of Succession, but in that case, we had $r = n$.) de Finetti calls this probability $p_r^{(n)}$, where

$$p_r^{(n)} = f(r+1,s)/f(r,s) \tag{1}$$

because of Bayes' Theorem. The number $p_r^{(n)}$ is the conditional probability of a tail on the $(n+1)$-th toss, given what went on in the first n.

We now set out to discover formulae for $f(r, s)$ and $p_r^{(n)}$. For the moment, note that $f(0,0)=1$. (If we haven't tossed the coin at all, then we can be quite certain that there will have been no tails and no heads either!) Now put

$$f(1,0) = \frac{a}{a+b} \quad \text{and} \quad f(0,1) = \frac{b}{a+b} \tag{2}$$

where a and b are positive numbers, or (at the extreme) one or the other (but not both) could be zero. We note that

$$f(1,0) + f(0,1) = 1$$

so that it makes sense to talk of the two terms on the left as "probabilities".

This last equation is an example of a more general result that de Finetti proved. Suppose that in a sequence of n coin-tosses, we have observed r tails and s heads. On the next, the $(n+1)$-th toss, we extend either the number of tails or else the number of heads (but not both). These are the only possibilities, which means that we must add the relevant probabilities. Thus

$$f(r,s) = f(r+1,s) + f(r,s+1). \tag{3}$$

Unfortunately, equations (1, 2, 3) are not sufficient to determine the values of $f(r,s)$ and $p_r^{(n)}$. We need some extra information. This can be supplied in the form of a highly technical result proved by de Finetti, but this would take us way beyond what readers of *Function* are expected to follow. Here is my own attempt to fill the gap with a piece of plausible reasoning that is however by no means a strict proof. Consider $p_n^{(n)}$. This is the probability that after n successive tails, we will see yet another. Remember that this is the probability Laplace set out to calculate in his "Law of Succession". Remember too that for de Finetti $p_n^{(n)}$ is the probability *we assign* to the event that n tails will be followed by an $(n+1)$st. When n = 0, the value is $a/(a+b)$ as previously agreed. When n is very large, the probability approaches 1 as n gets bigger and bigger. This is because, as we accumulate more evidence on the matter, our belief changes and we come closer to the view that we have here a coin with tails on both sides.

We are thus looking for a number dependent on n, that lies between $a/(a+b)$ on the one hand and 1 on the other, approaching the latter ever more closely as n increases. There is a very obvious candidate for such a number. It is $(a+n)/(a+b+n)$, and so we are led to put

$$p_n^{(n)} = \frac{a+n}{a+b+n} \tag{4}$$

Equations (1–4) are now sufficient to determine $f(r, s)$ and $p_r^{(n)}$. We find

$$p_r^{(n)} = \frac{a+r}{a+b+n} = \frac{a+r}{(a+r)+(b+s)} \tag{5}$$

We may also determine $f(r,s)$, but to do this we will need some notation that will not be familiar to most readers. The symbol $(x)_k$ is known as Pochhammer's symbol, after Leo Pochhammer (1841–1920), a German mathematician. It is defined as

$$(x)_k = x(x+1)(x+2)\ldots(x+k-1).$$

[Currently there is something of a reaction to this usage; see

`http://mathworld.wolfram.com/PochhammerSymbol.html`

However, in my view, Pochhammer's symbol still holds sway, and so I will continue to use it what follows.]

Using Pochhammer's symbol, we can write out a very elegant form for $f(r,s)$. We find

$$f(r,s) = \frac{(a)_r(b)_s}{(a+b)_{r+s}} \tag{6}$$

It is instructive to compare equation (6) with the result a frequentist would reach. We earlier set $a/(a+b)$ as the probability that the first toss would yield tails. For the frequentist, assuming independence of the tosses, this must also be the probability of a tail on each subsequent toss. Likewise the probability of a head on any given toss will be $b/(a+b)$ and so the probability of any given sequence of r tails and s heads must be

$$F(r,s) = \frac{a^r b^s}{(a+b)^{r+s}} \tag{7}$$

Note that although both formulae (6) and (7) claim to give the probability of the same sequence, they are not in direct conflict. They interpret the underlying concept in different ways.

For the Bayesian, formulae (5) and (6) now give us all the information we need about the coin we are tossing and our estimate of the likelihood of its landing tails up. Or rather, they would do if we knew the values of a and b. This is the point at which many of the classicists find difficulty with the arguments of the subjectivists. The simple truth is that *there is no way to give a formula for the values of these numbers* ! Indeed, different people, with different degrees of belief, will likely come up with different answers! For the frequentists (classicists) this is a major criticism. They say that on their analysis, probability is well defined, being the observed ratio of (in this case) tails to total tosses. If we say to them: "What about the case of small numbers?", they reply: "So what? We can't answer such questions very precisely because we don't have enough evidence."

However, de Finetti and the subjectivists would point out that in real life we often *do* have to answer such questions when we don't have as much evidence as we would like. If anything, they say, it's more honest and accurate to acknowledge this up front. Take the obvious case of a bookmaker setting odds that (say) Collingwood will beat Carlton in tomorrow's match at the Melbourne Cricket Ground. It's no use saying that we would need to play this same match (under exactly the same conditions) over and over until we had a big enough data-set to give a precise figure. That can't happen! What bookmakers do is use their knowledge of the game, the players and the conditions to guess the likely outcome. It *is* a guess that they make, but it is an *informed* guess. They may even have little rules and formulae they use in making their guesses, but these will be closely guarded trade secrets!

Similar points may be made in respect to medical diagnosis, weather prediction, oil prospecting, commercial forecasts and the like.

We may not have a foolproof means of finding values for a and b, but we *do* have an interpretation of what they mean. If we think of equation (5) from the point of view of a dyed-in-the-wool frequentist, then we say that for such a person, if r tails turn up in n tosses, then r/n is our best estimate of the probability of tails. Such a person, looking at equation (5), sees $a + b$ *virtual* tosses, of which a have yielded *virtual* tails. So the subjectivist, as looked upon by the frequentist, is claiming some further special knowledge: knowledge that applies even before the coin is ever actually tossed. [This analysis makes better sense if a and b are thought of as integers, but this restriction is not forced on us.]

The *prior* frequency is $a/(a+b)$, and this may be thought of as the initial probability of "tails" in our experiment. A bookmaker setting odds for the Collingwood-Carlton match might think at first of the probability of a Collingwood victory as 1/4 and set the odds accordingly. In the same vein, $a + b$ may be seen as a measure of the tenacity with which the initial belief is held. It directly affects the number of tosses we would need to see if we were to change our mind.

To explore this question further, I used the Excel Spreadsheet to simulate coin-tossing with different degrees of bias. If I set a (classical) probability p that the simulated coin will land "tails", then I can store the value of p in (say) the cell `a1`. The formula `=INT(RAND()-$a$1)+1` will then return a value 1 if a random number generated by the spreadsheet exceeds p, but 0 if it doesn't. Each 1 will correspond to a tail and each 0 to a head. We can get the computer to keep track of the total number of tails generated, and so produce a measured probability along classical lines. Likewise, we can program in formula (5), and compare the two.

As a trial, I set $p = 0.95$, as outlined in my previous column, but I assumed a prior probability of 0.5. That is to say, I was simulating an experimenter who started out believing the simulated coin was fair, whereas in fact it was highly biased. I ran the program many times with a sequence of 1000 simulated tosses and setting various values for $a + b$. It's an exercise you may like to try for yourself.

One typical run had $a + b = 50$. Now after 1000 simulated tosses, we expect 950 or so tails. (We can use classical theory here because we have told the computer in advance what the probability actually is; in real life, we don't have this luxury.) So we expect, from equation (5), that the estimate will still be somewhat biased by our initial estimate. The predicted probability of heads, incorporating the "wrong" prior probability 0.5 is (950+25)/(1000+50), or 0.929, as opposed to 0.950. The strongly held, if erroneous, belief that the coin is fair still exerts some influence even after 1000 tosses. One of my computer simulations produced 963 tails in the 1000 tosses (i.e. a measured probability of 0.963, within acceptable error of the expected value of 0.950) and gave a de

Finetti probability of 0.943. But after only 50 tosses, the corresponding results were 0.94 and 0.72, so that the 50-toss experiment had not done nearly so much to shake the initial belief.

We may now revisit Laplace's Law of Succession. This corresponds to the case of $a = b = 1$ in equation (4). (Which implies a very weak initial conviction that the coin is fair.) de Finetti extends this to equation (5) and so has, for this special case, the probability of an $(r+1)$st tail after a history of r tails from n tosses as $(r+1)/(n+2)$.

There is much more to be said on this interesting but difficult subject. But I hope that I have given something of the flavour of the discussion. For more, see the reading list below.

Further Reading

The collection *Studies in Subjective Probability* (Ed. H E Kyburg and H E Smokler) has originals of many important papers, including the English translation of one of de Finetti's major investigations, on which much of my present account is based. This same collection includes the source of the quote from Venn's work. It also contains an excellent introductory summary by the editors. Another useful work (among many) is J R Lucas's *The Concept of Probability*, especially its Appendix II.

Note

That memory may be an unreliable guide to history is strikingly borne out by the possible need for a correction to my previous column. There I asserted that a US 1cent coin, spun on its edge, nearly always landed tails up. This was my memory from my graduate student days in Chicago. Either this memory is wrong, or else the coin has been changed since then.

I recently obtained a US 1cent coin (dated 1997) and repeated the experiment. In 40 spins, it landed *heads up* on 30 of the trials!

* * * * *

Like the crest of a peacock so is mathematics at the head of all knowledge.

[An old Indian saying]

COMPUTERS AND COMPUTING

It is so Easy With Spreadsheets!

Cristina Varsavsky

Although spreadsheets were primarily designed for simple accounting related calculations such as adding, subtracting, multiplying or dividing numbers layed out in columns and rows, they are a very useful tool for solving a range of mathematical problems. For example, I often see students using spreadsheets for plotting a function: they simply generate a table with variable and function values corresponding to points on the graph of the function, and then use the graphing facility joining all points with segments. The curve obtained can look very smooth, provided the points generated are sufficiently close on the graph of the function.

But the really powerful feature of a spreadsheet is the ease with which recursive calculations can be made. We have already included some articles in *Function* which involved recursion, the most recent is in *Vol* 23 *Part* 4 where we say that recursion is a process of defining an object in terms of itself. In that article, and also in this issue of *Function*, we have a typical example of a sequence defined by recursion, the Fibonacci sequence

$$f_1 = 1,\ f_2 = 1 \text{ and } f_{n+2} = f_{n+1} + f_n \qquad n \geq 1$$

To generate this sequence using a spreadsheet[1], we use column **A** for the position, n, of the terms within the sequence and column **B** for the actual terms. Put a 1 in cells **A1**, **B1** and **B2** and a 2 in cell **A2**. Now put the formula

=A2+1 (1)

in cell **A3**, and the formula

=B1+B2 (2)

in cell **B3**.

Next, we copy the formulae in **A3** and **B3** to as many consecutive cells below these two as we wish. You can do this in two ways:

[1] I use *Excel*, but there are other spreadsheet software around. Another popular one is *Lotus*.

(i) highlight the two cells **A3** and **B3**, select **COPY** (from the **EDIT** menu), then highlight the cells where you would like to copy to, and select **PASTE** (from the **EDIT** menu); or

(ii) highlight the two cells **A3** and **B3**, and drag the fill handle on the bottom-right corner of the selection as far down as you wish.

Your worksheet should look like the one shown in Figure 1, with more or less rows filled in, depending on how far down you copied the formulae.

	A	B
1	1	1
2	2	1
3	3	2
4	4	3
5	5	5
6	6	8
7	7	13
8	8	21
9	9	34
10	10	55
11	11	89
12	12	144
13	13	233
14	14	377
15	15	610
16	16	987
17	17	1597
18	18	2584

Figure 1

The very convenient "copy" feature of any spreadsheet is that formulae involving cells are copied relative to the cells[2]. So the formula (1) we placed in **A3**, should not be read in absolute terms; it really means

one more than the cell above.

Similarly, formula (2) means

the sum of the two cells above this one.

2 If you need to refer to the absolute address of a cell, then you have to add a dollar sign in front of the row and column; eg. C12.

So when we place formula (2) in the cell **B3**, the spreadsheet software calculates the sum of **B1** and **B2**, but when we copy it in cell **B5**, it means the sum of **B3** and **B4**.

Now that you have the first few terms of the sequence in a spreadsheet column, you can verify that the ratio of two consecutive terms becomes approaches the golden ratio as n becomes larger. To do this, we put the formula of the first ratio

=B2/B1 (3)

in cell **C2**, and copy this formula down to the last term of our sequence. The result is shown in Figure 2, where you can see that the ratio between the 18^{th} and 17^{th} terms already shares the first six decimal digits of $\varphi = (1+\sqrt{5})/2 \approx 1.618033989$.

	A	B	C
1	1	1	
2	2	1	1
3	3	2	2
4	4	3	1.5
5	5	5	1.67
6	6	8	1.6
7	7	13	1.625
8	8	21	1.615384615
9	9	34	1.619047619
10	10	55	1.617647059
11	11	89	1.618181818
12	12	144	1.617977528
12	13	233	1.618055556
14	14	377	1.618025751
15	15	610	1.618037135
16	16	987	1.618032787
17	17	1597	1.618034448
18	18	2584	1.618033813

Figure 2

You could also check the result mentioned by Malcolm Clark in this issue on page 107, namely that $f_n \approx \varphi^n / \sqrt{5}$ for large n. If you put the formula

=((1+SQRT(5))/2)^A1/SQRT(5)

in cell **D1** and copy it down and then format column **D** with 0 decimal places, you will see that that the terms of this new sequence are identical to the terms of the

Fibonacci sequence. To see the accuracy with which the above formula approximates each term, place in **`E1`** the formula

`=(C1-B1)/B1*100`

and copy it down the column. You will see that by the 10^{th} term the error in the approximation is already as good as 0.007%.

I leave to the readers the task to investigate the "random" Fibonacci sequence discussed by Malcolm Clark and the new constant identified by Viswanath. To generate this sequence you will need to use the random function to obtain two sequences of randomly generated 1's and -1's. In *Excel*, you could do this with the function **`RANDBETWEEN(0,1)`**. This will output either 0 or 1, and you will then have to replace the 0's with a -1.

This spreadsheet capability of performing recursive calculations is also very useful when taking numerical—rather than exact—approaches to problem solving. One such area is the modelling of physical situations with differential equations. Take for example the equation modelling inhibited growth of a population of some kind of creatures. This model supposes that the population cannot exceed a limited value L and assumes that the growth is proportional to both the existing population and the remaining room for growth. The equation describing this is

$$\frac{dx}{dt} = k\,x\,(L-x) \tag{4}$$

where $x(t)$ represents the population at time t.

If we take a small period of time Δt, this will imply a growth Δx in the number of creatures, and we can approximate equation (4) as follows

$$\frac{\Delta x}{\Delta t} \approx k\,x\,(L-x)$$

or

$$x(t+\Delta t) - x(t) = k\,x(t)\ (L-x(t))$$

which we rewrite as

$$x(t+\Delta t) = x(t) + k\,x(t)\ (L-x(t))\,\Delta t \tag{5}$$

The above is a recursive definition. It expresses the value of x at a point of time in terms of its value a fraction of time Δt before. So using a spreadsheet we could find out the number of creatures at time intervals Δt, and find out, for example, how long it will take for the population to grow to its maximum size.

Let us suppose that the whole population L is 1000, that the initial number of creatures is 10, and that $k = 0.02$. Also assume that time is measured in days and that $\Delta t = 0.01$. We create a spreadsheet with the time in column **A**, and the value of x in column **B**. We put 10 in **B2**, and in **B3** the formula

```
=B2+0.02*B2*(1000-B2)*0.01,
```

which we copy down column **B**. Figure 3 shows the first 11 approximations modelling the growth of this population

	A	B
1	t	x
2	0	10.00
3	0.01	11.98
4	0.02	14.35
5	0.03	17.18
6	0.04	20.55
7	0.05	24.58
8	0.06	29.37
9	0.07	35.07
10	0.08	41.84
11	0.09	49.86
12	0.1	59.34

Figure 3

If you copy down more rows, you will see that the population reaches its maximum possible size of 1000 when $t \approx 0.6$. Of course, you could also plot the data and have a visual model of the growth of this population.

As you can see in the examples included here, you do not need to be an expert programmer to solve problems which involve heavy computational procedures. In many cases, the spreadsheet software you have in your computer is an excellent and easy-to-use tool to assist you in your task.

* * * * *

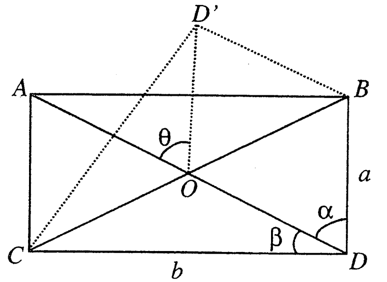
Plato said that God geometrizes continually.

—Plutarch in *Convivialium,* liber 8, 2

PROBLEM CORNER

PROBLEM 24.2.1 (H C Bolton, Melbourne)

The figure below shows a rectangle $ABCD$ and its corner D folded about the diagonal BC into the position D'. The rectangle has sides of length b, a with $b > a$. Express $\tan\theta$ in terms of a, b.



SOLUTION (Carlos Victor)

From the similar triangles $OD'B$ and ODB we have $\angle D'OB = \angle DOB = 2\beta$. From this it follows $\theta = \pi - 4\beta$, so that $\tan\theta = -\tan 4\beta$. Since

$$\tan\beta = \frac{a}{b}, \ \tan 2\beta = \frac{2\frac{a}{b}}{1-\left(\frac{a}{b}\right)^2} = \frac{2ab}{b^2-a^2}.$$

From this it follows that

$$\tan 4\beta = \frac{4ab\,(b^2-a^2)}{(b^2-a^2)^2 - 4a^2b^2}, \ \text{so} \ \tan\theta = \frac{4ab\,(b^2-a^2)}{4a^2b^2-(b^2-a^2)^2}.$$

Solutions were also received from Keith Anker and the proposer. Julius Guest submitted a formula for the angle θ in terms of the side lengths a, b.

PROBLEM 24.2.2 (from Crux Mathematicorum with Mathematical Mayhem)

In a quadrilateral $P_1 P_2 P_3 P_4$ suppose that the diagonals intersect at the point $M \neq P_i$ and points P_2, P_4 lie on opposite sides of the line P_1P_3. Let the angles MP_1P_4, MP_3P_4, MP_1P_2 and MP_3P_2 be $\alpha_1, \alpha_2, \beta_1$ and β_2 respectively.

Prove that

$$\left|\frac{P_1M}{MP_3}\right| = \frac{\cot\alpha_1 \pm \cot\beta_1}{\cot\alpha_2 \pm \cot\beta_2}$$

where the +(–) sign holds if the line segment P_1P_3 is located inside (outside) the quadrilateral.

Comment: A complete solution would have to consider the four cases corresponding to the possibilities that M can lie between P_1 and P_3 or not, and between P_2 and P_4 or not.

SOLUTION

Assume M is a point of the line P_1P_3 except P_1, P_3, with P_2 and P_4 on opposite sides of M. Let $\omega = \angle P_1MP_4$. Then

$$\sin\angle P_1P_4M = \sin(\pi - (\alpha_1 + \omega)),$$

while $\sin\angle P_3P_4M = \sin(\omega \pm \alpha_2)$ where '+' corresponds to the subcase where M is outside the segment P_1P_3, while '–' corresponds to where M is between P_1 and P_3. From the sine rule applied to triangles MP_1P_4 and MP_3P_4 we have

$$P_1M\,\frac{\sin\alpha_1}{\sin(\alpha_1 + \omega)} = MP_4 = MP_3\,\frac{\sin\alpha_2}{\sin(\omega \pm \alpha_2)}.$$

Hence

$$\begin{aligned}\frac{P_1M}{MP_3} &= \frac{\sin\alpha_2\sin(\alpha_1+\omega)}{\sin(\omega\pm\alpha_2)\sin\alpha_1} \\ &= \frac{\sin\alpha_2\sin\alpha_1\cos\omega + \sin\alpha_2\cos\alpha_1\sin\omega}{\sin\omega\cos\alpha_2\sin\alpha_1 \pm \cos\omega\sin\alpha_2\sin\alpha_1} \\ &= \frac{\cot\omega + \cot\alpha_1}{\cot a_2 \pm \cot\omega}\end{aligned}$$

(1)

(where we divided both the numerator and the denominator by the product of three sines to get the last equality).

Similarly from triangles MP_1P_2 and MP_3P_2 we have

$$\frac{P_1M}{MP_3} = \frac{\sin\beta_2 \sin(\omega - \beta_1)}{\sin\beta_1 \sin(\omega \mp \beta_2)} = \frac{\cot\beta_1 + \cot\omega}{\cot\beta_2 \mp \cot\omega} \tag{2}$$

The result follows by combining (1) and (2) using the following property of proportions: if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ then $\frac{a}{b} = \frac{(c+e)}{(d+f)}$.

A solution was received from Carlos Victor.

PROBLEM 24.2.3 (from Crux Mathematicorum with Mathematical Mayhem)

Find all solutions to the inequality

$$n^2 + n - 5 < \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n+1}{3} \right\rfloor + \left\lfloor \frac{n+2}{3} \right\rfloor < n^2 + 2n - 2.$$

where n is a natural number and $\lfloor x \rfloor$ denotes the greatest integer not exceeding the real number x.

SOLUTION (Claudio Archoncher, Brazil)

We consider the three cases $n = 3k,\ 3k+1,\ 3k+2$ where k is a natural number.

(i) $n = 3k$ gives $\left\lfloor \frac{n}{3} \right\rfloor = \left\lfloor \frac{n+1}{3} \right\rfloor = \left\lfloor \frac{n+2}{3} \right\rfloor = k$, so that the inequality is

$$9k^2 + 3k - 5 < 3k < 9k^2 + 6k - 2 .$$

The first inequality implies $k = 0$, but the second inequality does not hold for this case.

(ii) $n = 3k + 1$ gives us

$$(3k+1)^2 + (3k+1) - 5 < 3k+1 < (3k+1)^2 + 2(3k+1) - 2$$

and once again the first inequality gives $k = 0$ but this does not satisfy the second inequality.

(iii) $n = 3k + 2$ yields

$$(3k+2)^2 + (3k+2) - 5 < 3k+2 < (3k+2)^2 + 2(3k+2) - 2$$

and this time we see that $k = 0$ does satisfy the inequalities, and furthermore it is the only solution.

Hence the solution is $n = 2$.

Solutions were also submitted by Carlos Victor and Julius Guest.

PROBLEM 24.2.4 (from Mathematical Spectrum)

Evaluate the improper integral

$$\int_a^b \left(\frac{x-a}{b-x} \right)^{\frac{1}{2}} dx$$

SOLUTION

The identity

$$(x-a) + (b-x) \equiv b - a$$

allows us to find, for each x in $[a, b]$, a value θ such that

$$x - a = (b-a)\sin^2\theta\,,\ b - x = (b-a)\cos^2\theta\,.$$

From either of these expressions we find .

$$\frac{dx}{d\theta} = 2(b-a)\sin\theta\cos\theta$$

Now by change of variable we see

$$\begin{aligned} I &= \int_a^b \left(\frac{x-a}{b-x}\right)^{\frac{1}{2}} dx = 2\int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}(b-a)\sin\theta\cos\theta\, d\theta \\ &= (b-a)\int_0^{\pi/2} (1-\cos 2\theta)d\theta \\ &= (b-a)\frac{\pi}{2} \end{aligned}$$

Solutions were received from K. Anker, J.A. Deakin, and Carlos Victor.

PROBLEM 24.2.5 (from Mathematics and Informatics Quarterly)

In decimal notation, suppose that $\overline{cabadb} = (\overline{ab})(\overline{acbba})$ where *a, b, c, d* are all different natural numbers. Find *a, b, c, d*. [Note: $\overline{abc}$ denotes the number made up with the digits *a, b,* and *c*.]

SOLUTION

We have $\overline{cabadb} : \overline{ab} = \overline{acbba}$. Since $c < 10$, we must have $a \le 3$. We also have $a \times b \equiv b \pmod{10}$.

Suppose $a = 3$. Then $b = 5$ or 0. Since $3 \times 35 = 105$, $b = 5$ is impossible. If $b = 0$, then $c = 9$ and we have the impossible situation.

$$\begin{array}{l} \overline{9303d0} : 30 = 39003. \\ \underline{90} \\ \ \ 30 \end{array}$$

Hence $a \neq 3$.

Suppose $a = 2$. Then $b = 0$ and so $c = 4$ or 5.

$$\begin{array}{l} \overline{4202d0} : 20 = 24002, \\ \underline{40} \\ \ \ 20 \end{array} \qquad \begin{array}{l} \overline{5202d0} : 20 = 25002. \\ \underline{40} \\ \ \ 120 \end{array}$$

Both situations are impossible. So $a \neq 2$.

Suppose $a = 1$. Since $c \neq 1$, we must have $c = 2$ or 3. With $c = 3$, we have

$$\overline{31b1db} : \overline{1b} = \overline{13bb1}.$$

This gives $6 \leq b \leq 9$. But $b = 9$ gives

$$\begin{array}{l} \overline{3191d9} : 19 = 13991. \\ \underline{19} \\ 129 \end{array}$$

which is impossible. Clearly $b = 8, 7, 6$ are also impossible. Thus $c = 2$ and we have

$$\overline{21b1db} : \overline{1b} = \overline{12bb1}.$$

which gives $b = 7$ or 8. With $b = 8$ we have

$$\begin{array}{l} \overline{2181d8} : 18 = 12881. \\ \underline{18} \\ \ \ 38 \\ \ \ \underline{36} \\ \ \ \ \ 21 \end{array}$$

which is impossible. This gives $b = 7$ and

$$
\begin{array}{l}
\overline{2171d7} : 17 = 12771 \\
\underline{17} \\
\;\;47 \\
\;\;\underline{34} \\
\;\;131 \\
\;\;\underline{119} \\
\;\;\;\;12d \\
\;\;\;\;\underline{119}
\end{array}
$$

This is satisfied with $d = 0$. Thus the unique solution is given by $a = 1$, $b = 7$, $c = 2$, $d = 0$.

A solution was received from Carlos Victor.

Correction to Problem 24.3.1

The example given was incorrect. It should read

For example, when $n = 4$,

$$\frac{1}{1\cdot 4} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 1} + \frac{1}{4\cdot 3} = 1$$

PROBLEMS

PROBLEM 24.4.1 (from Mathematical Spectrum)

Let ABC be an acute angled triangle and let D and E be the points on BC such that angle ADB is a right angle and angle DAB = angle EAC. Prove that

$$(\text{area } \Delta\, EAC) > (\text{area } \Delta\, DAB) \text{ if and only if } AC > AB.$$

PROBLEM 24.4.2 (from Mathematical Spectrum)

The Smarandache function is defined by $\eta(n)$ = the smallest positive integer m such that n divides $m!$.

(a) Calculate $\eta(p^{p+1})$, where p is a prime.

(b) Find all positive integers n such that $\eta(n) = 10$.

(c) Prove that, for every real number k, there is a positive integer n such that

$$\frac{n}{\eta(n)} > k$$

Does $\dfrac{n}{\eta(n)} \to \infty$ as n $\to \infty$?

PROBLEM 24.4.3 (from Crux Mathematicorum with Mathematical Mayhem)

Find all real numbers x such that

$$x = \left(x - \frac{1}{x}\right)^{\frac{1}{2}} + \left(1 - \frac{1}{x}\right)^{\frac{1}{2}}$$

PROBLEM 24.4.4 (from Parabola)

In a triangle with sides a, b, c the angle opposite side a is twice the angle opposite side b. Prove that $a^2 = b(b+c)$

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Mathematicians are like Frenchmen: whatever you say to them they translate it into their own language and forthwith it is something entirely different.

—Goethe in *Maximen und Reflexionen, Sechste Abteilung*

To call in the statistician after the experiment is done may be no more than asking him to perform a post-mortem examination: he may be able to say what the experiment died of.

— Fisher, Ronald Aylmer at *Indian Statistical Congress*, Sankhya, ca 1938.

OLYMPIAD NEWS

The XLI International Mathematical Olympiad

Australian ahead of all European Union countries and Canada

Between 13 and 24 July, the Korean city of Taejŏ was the venue for this year's IMO. Never before have no many countries and such a large number of students participated in this competition: 461 students from 82 countries. As usual, they had to contend with six problems during nine hours spread equally over two days in succession.

Here are the problems:

Problem 1

Two circles Γ_1 and Γ_2 intersect at M and N.

Let l be the common tangent to Γ_1 and Γ_2 so that M is closer to l than N is. Let l touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to l meet the circle Γ_1 again at C and the circle Γ_2 again at D.

Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q.

Show that $EP = EQ$.

Problem 2

Let a, b, c be a positive real numbers such that $abc = 1$. Prove that

$$\left(a-1+\frac{1}{b}\right)\left(b-1+\frac{1}{c}\right)\left(c-1+\frac{1}{a}\right) \leq 1.$$

Problem 3

Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at the same point.

For a positive real number λ, define a *move* as follows:

> Choose any two fleas, at points A and B, with A to the left of B; let the flea at A jump to the point c on the line to the right of B with $BC/AB = \lambda$.

Determine all values of λ such that, for any point M on the line and any initial positions of the n fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of M.

Problem 4

A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card.

A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen.

How many ways are there to put all the cards into the boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

Problem 5

Determine whether or not there exists a positive integer n such that n is divisible by exactly 2000 different prime numbers, and

$$2^n + 1 \text{ is divisible by } n.$$

Problem 6

Let AH_1, BH_2, CH_3 be the altitudes of an acute-angled triangle ABC. The incircle of the triangle ABC touches the sides BC, CA, AB at T_1, T_2, T_3, respectively. Let the lines l_1, l_2, l_3, be the reflections of the lines H_2H_3, H_3H_1, H_1H_2 in the lines T_2T_3, T_3T_1, T_1T_2, respectively.

Prove that l_1, l_2, l_3, determine a triangle whose vertices lie on the incircle of the triangle ABC.

Australia came sixteenth with 122 points out of possible 252. Ahead of Australia were the teams from China (218 points), Russia (215), USA (184), Korea (172), Hungary (171), Vietnam (169), Bulgaria (167), Belarus (165), Iran (160), Chinese Taiwan (164), Israel (139), Romania (139), Ukraine (135), India (128), and Japan (124).

The other countries of the Asian Pacific Mathematics Olympiad scored as follows:

Canada (112), Argentina (88), South Africa (81), Hong Kong (80), Thailand (78), Singapore (76), Mexico (75), Colombia (61), Indonesia (54), Trinidad & Tobago (40), New Zealand (34), Malaysia (32), Peru (32), the Philippines (23), Sri Lanka (21), Ecuador (19).

The award distribution for the Australian team was:

- Peter McNamara, Western Australia, gold medal
- Geoffrey Chu, Victoria, silver medal
- Thomas Sewell, New South Wales, silver medal
- Allan Sky, ACT, silver medal
- Thomas Xia, New South Wales, bronze medal.

Congratulations to our excellent team!

* * * * *

Einstein's words

If my theory of relativity is proven successful, Germany will claim me as a German and France will declare that I am a citizen of the world. Should my theory prove untrue, France will say that I am a German and Germany will declare that I am a Jew.

—Address at the Sorbonne, Paris.

These thoughts did not come in any verbal formulation. I rarely think in words at all. A thought comes, and I may try to express it in words afterward.

—In H. Eves Mathematical Circles Adieu,
Boston: Prindle, Weber and Schmidt, 1977

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