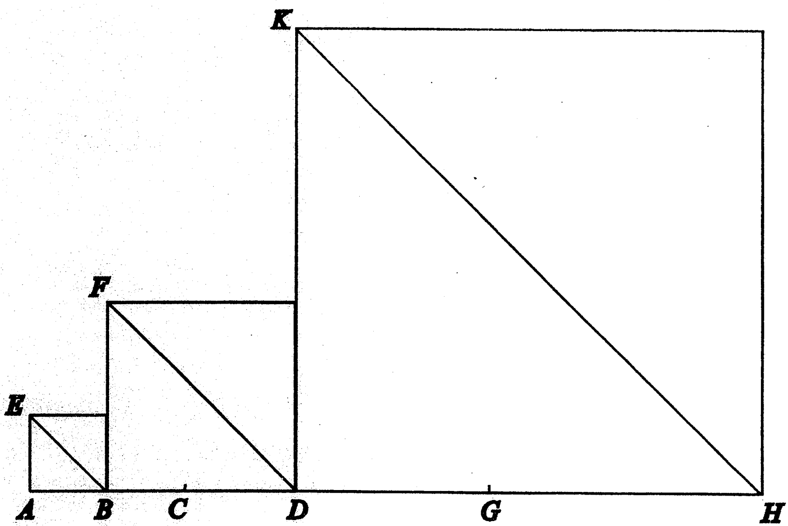
Function

A School Mathematics Journal

Volume 23 Part 2 April 1999



Department of Mathematics & Statistics – Monash University

Reg. by Aust. Post Publ. No. PP338685/0015

Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Function is published five times a year, appearing in February, April, June, August, and October. Price for five issues (including postage): \$20.00* ; single issues \$5.00. Payments should be sent to: The Business Manager, Function, Department of Mathematics & Statistics, Monash University, Clayton VIC 3168, AUSTRALIA; cheques and money orders should be made payable to Monash University.

For more information about *Function* see the journal home page at `http://www.maths.monash.edu.au/~cristina/function.html`.

* \$10 for *bona fide* secondary or tertiary students.

EDITORIAL

We welcome our readers to this issue of *Function*, which we hope brings something for everyone.

Although articles about fractals abound, this is an area of mathematics which attracts many mathematicians and non-mathematicians alike. Anthony Sofo contributes to another view of fractals, and analyses the well known *Koch snowflake* curve, its perimeter, the area it encloses, and its fractal dimension.

What is the intersection of two spheres in four-dimensional space? You may wish to join Michael Englefield in a tour through one-, two-, and three dimensional space to address this question.

We received another letter from Kim Dean about the recent discovery by the eccentric Welsh mathematician and physicist Dai Fwls ap Rhyll. This time he came up with a rather controversial result which contradicts the well accepted Pythagoras' Theorem. We include in this issue his working out, for your evaluation.

In the regular *History of Mathematics* you will find several proofs—some of them not so known—of another classic in mathematics: the irrationality of the square root of 2. The diagram on the front cover supports one of the proofs included.

Thirty-four years ago computing was a very modern subject; you will find in our *Computers and Computing* column a summary of a publication from that time similar to *Function* ... reading articles written in those days about this matter is a rather amusing exercise.

As usual, there are a few new problems in our *Problem Corner*. You may wish to send us their solutions; we will publish them if they reach us by July 1, 1999.

Happy reading and problem solving!

* * * * *

FRACTAL SHAPES

Anthony Sofo, Victoria University of Technology

1. Introduction

For about 15 years between the late 1950's and early 1970's Benoit Mandelbrot developed his ideas on a branch of mathematics known as Fractal Geometry. Fractal Geometry can be used to describe and analyse the irregularity of the natural world. Fractals are shapes that look more or less the same on all, or many, scales of magnification. Consider a coastline, of Australia for example, the most obvious example of a fractal in nature. Maps of coastlines drawn on different scales all seem to a show a similar distribution of bays and headlands. Whether in the Joseph Bonaparte Gulf near Darwin or the Spencer Gulf near Adelaide, each bay appears to have its own smaller bays and headlands almost ad infinitum. The same general structures appear on all coastlines. It seems that coastlines are crinkly however close we get. Any tiny piece of coastline magnified any number of times still looks like a coastline and it is this phenomenon which is called *self similarity*. Self similarity is widely exhibited by shapes both in nature, or designed, that are fractals. We will not indulge in stating a precise mathematical definition. Another way of thinking about some, but not all, fractal shapes is to view them as being regularly irregular, in that they have a boundary that moves all over the place in either a predictable or unpredictable manner, and yet there are certain characteristic patterns that appear and repeat themselves at different scales of viewing, although they may not occur in exactly the same way. This self similarity property of fractals is not shared by shapes that are defined by Euclidean Geometry; they lose their structure, or self similarity property, when magnified. For example, when viewed from afar the surface of the earth looks like a ball; however when viewed up close it appears almost flat. This appearance of 'flatness' may have been one major reason why many people used to think that travelling too far on the high seas would have meant going over the 'edge'.

Traditional geometry has to ignore the crinkles of the real world because they are irregular and so these shapes do not submit readily to standard mathematical formulae. On the other hand the notion of self similarity in fractals allows us to see a sort of order in the apparent chaos of these shapes. A question we may wish to ask about the coastline of Australia, or any other land mass, is, "How long is it?" We could certainly look at a map and then according to its scale 'measure' the coastline. Each scale would produce a different answer because the greater the scale, then the greater the detail that is shown on the map and more and more crinkles appear on the coastline. We could perhaps, get a long string and actually measure the perimeter, but again all the crinkles in the coastline would make it appear that it was never ending. Is the coastline really infinite as we measure more and more of its detail or is it just very long but finite? We can of course ask the same questions about the surface area and volume of a country. To consider these questions, we construct mathematical shapes which free us from the physical world and allow us to proceed further into this intriguing world of fractals.

2. Curves

In 1904, the Swedish mathematician Helge Von Koch produced an interesting geometric construction which is now called the 'Koch snowflake'. Take an equilateral triangle with sides of length one unit. One-third of the way along each side place a triangle one-third of the size of the original so that its apex points outwards, and erase the line where the old and new triangle join. This construction gives us a Star of David shape shown in Figure 1.

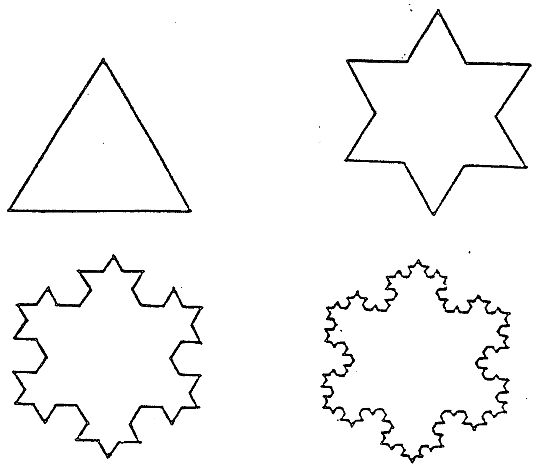

Figure 1. The construction of the Koch snowflake curve.

This new shape, construction 2, has four times as many sides as the original construction 1, and each of these new triangles is one-ninth of area of the original unit sides triangle. This process can be repeated indefinitely. The Koch snowflake curve is therefore crinkly. Now let us investigate the question of its perimeter and area.

3. Area

The area of the original triangle is $\frac{\sqrt{3}}{4}$. After the first iteration the area is $\frac{\sqrt{3}}{4} + 3 \times \frac{1}{9} \times \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{3 \times 4}$. One of the smaller triangles has an area of $\frac{1}{9}$ of the area of the first triangle. After the second iteration,

$$\text{Area} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{3 \times 4} + \frac{12\sqrt{3}}{4 \times 3^4}$$

After n iterations,

$$\begin{aligned}
\text{Area} &= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{4 \times 3^2} + \frac{4 \times 3\sqrt{3}}{4 \times 3^4} + \frac{4^2 \times 3\sqrt{3}}{4 \times 3^6} + \ldots + \frac{4^{n-1} \times 3\sqrt{3}}{4 \times 3^{2n}} \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\left(\frac{1}{3} + \frac{2^2}{3^3} + \frac{2^4}{3^5} + \ldots\right) \\
&= \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4}\left(\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^4 + \ldots + \left(\frac{2}{3}\right)^{2n}\right)\right].
\end{aligned}$$

This expression contains a geometric series with common ratio $\frac{4}{9}$ and first term $\frac{4}{9}$. As $n \to \infty$ the area approaches a limit and the limit is

$$A_\infty = \frac{\sqrt{3}}{4}\left[1 + \frac{3}{4}\left(\frac{4/9}{1-4/9}\right)\right] = \frac{\sqrt{3}}{4}\left(1 + \frac{3}{5}\right) = \frac{2\sqrt{3}}{5}.$$

This area is $\frac{8}{5}$ times the original area of the triangle. The Koch snowflake area is finite.

4. Perimeter

Similarly we can consider the length of the curve. The perimeter of the first curve is $P_0 = 3$. After the first iteration $P_1 = 3\times\left(\frac{4}{3}\right) = 4$. After the second iteration $P_2 = 3\times\left(\frac{4}{3}\right)^2 = \frac{16}{3}$, and after n iterations $P_n = 3\times\left(\frac{4}{3}\right)^n$. As $n \to \infty$, $P_n \to \infty$, the perimeter increases without bound. The Koch snowflake has a finite area and an infinite perimeter.

It is possible to construct other shapes, for example the Sierpinski carpet which exhibit similar properties as the Koch snowflake.

5. Fractal dimensions

Few of us have any difficulty in considering a line as having one dimension, this sheet of paper as having two dimensions and a cube as having three dimensions. Is it possible to have an object that is non-integral in dimension? The answer is yet, and it was Mandelbrot who put the idea of fractional dimension on a firm mathematical basis. Roughly speaking we think of dimension as given by the number of independent directions or degrees of freedom. We need to find a characterisation of dimension that will permit generalisation. To do this we consider the effects of dimension on the measure of similar geometric shapes. Consider a line segment of unit length. If we quadruple its length, that is, expand it by a scaling factor of four, we get a line segment of length four. This line segment, of course, contains four congruent components. Consider a unit square. If we expand the square by a scaling factor of four, that is, we quadruple its sides, we get a square, (see Figure 2), whose area is sixteen times as great. Equivalently, this means that the expanded square consists of sixteen congruent components. Observe that $16 = 4^2$. Finally, consider a unit cube. If we expand it by a scaling factor of four, we get a cube consisting of sixty four congruent components, see Figure 2. In this case $64 = 4^3$.

Observe that in each of the three cases illustrated in Figure 2, we have a dimension d, a scaling factor s and a number of components N which satisfies the equation $N = s^d$. This equation may be written as

$$\begin{aligned} \log N &= \log s^d \\ &= d \log s \\ \text{so } d &= \frac{\log N}{\log s} \end{aligned}$$

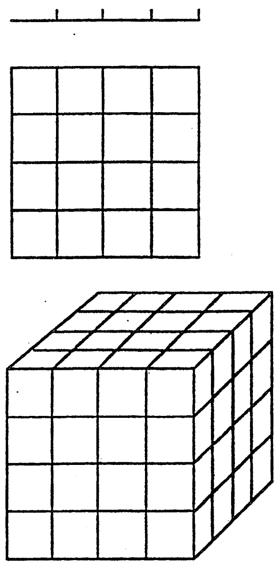

Figure 2. The line (dimension 1), square (dimension 2) and cube (dimension 3).

For the present, we have been careful to confine our attention to objects which are *self similar*; that is, the expanded objects could be dissected into congruent components similar to the original objects. Circles and cones are simple examples of objects which are not self similar. Nevertheless, the relation which links measure, area, volume, etc., to dimension and scale continues to hold: quadrupling the radius of a circle increases its area by a factor of $16 = 4^2$, even though there is no way to cut the expanded circle into sixteen circles congruent to the original. Similarly, quadrupling the radius and height of a cone increases its volume by a factor of $64 = 4^3$. All the familiar geometric shapes seem to fall into this pattern.

However, neither N nor s need necessarily be an integer for the equation $N = s^d$ to be valid. This is not particularly surprising because the idea of geometric similarity is easy to accept for non integral scaling factors. Dimension, however, is another matter; we expect d to be an integer. Let us see if it is possible to produce a geometrical object whose expansion by a factor s can be dissected into N components such that $d = \dfrac{\log N}{\log s}$ is *not* an integer. Let us return to the snowflake curve of Von Koch. To use the above formula for the 'self similarity dimension' it is necessary to take a closer look at a portion of the snowflake curve that is similar. Figure 3 shows the construction for the snowflake curve confined to a line segment, at each stage, we replace line segments with broken lines which are 4/3 as long. We see that each segment consists of $N = 4$ components with a scaling factor $s = 3$. Therefore, by our formula, this curve has self similarity dimension

$$d = \frac{\log 4}{\log 3} \approx 1.2619.$$

Therefore the snowflake fractal has a fractional dimension of about 1.26; the coastline of Britain, it has been worked out, has a dimension approximately 1.25.

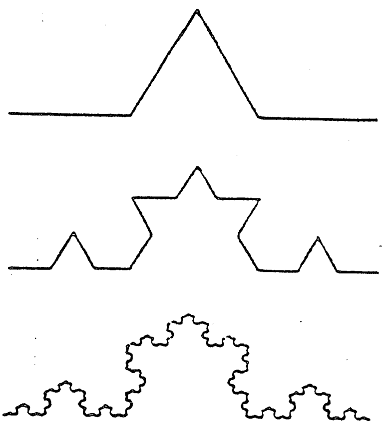

Figure 3. The first three iterations of the Van Koch curve.

Roughly speaking, if an object has fractal dimension more than one but less than two it is better at filling up space than is an ordinary one dimensional object, but not quite so good as a two dimensional one. A crinkly line of say, dimension 1.34 is better at filling up space than a one dimensional straight line because you need more ink to draw the crinkle than you do to draw a straight line. A line of dimension 1.42 is even crinklier and needs more ink. Some fractal curves are so wiggly and detailed that they fill up nearly all of the surface they are drawn on. The *Peano space filling curve* has the remarkable property in that it is a line (and hence might be thought to be one dimensional), and yet it passes through every point in a square, that is, it fills a two dimensional plane, and its dimension is two.

Many other fractal shapes may be viewed in Paul Bourke's Brain Dynamics Centre at `http//www.mhri.edu.au/~pdb/fractals/index.html`.

[*Ed: For more about fractals see the following issues of Function:* 5(5), 11(2), 13(4), 14(3), 18(2), 18(4), 19(3), 19(5).]

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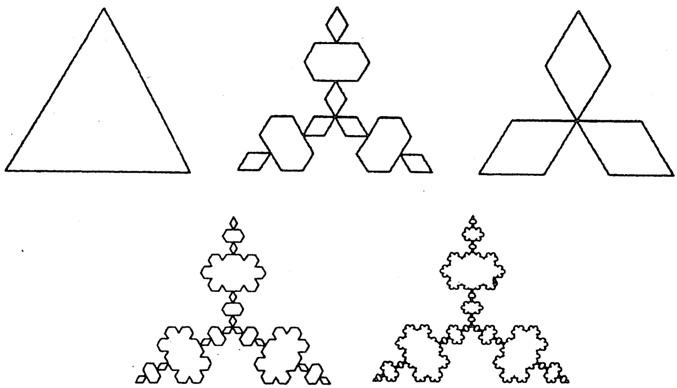
The Anti-Snowflake Curve

If the triangles in the construction of the VonKoch curve described above are pointed inward, the anti-snowflake curve is generated. Here are the first few steps towards its construction.



We leave to the readers to determine the perimeter of this curve and the area it encloses.

FOUR-DIMENSIONAL SPHERES

Michael J Englefield, Monash University

The aim of this article is to give a physical realisation of 4-dimensional spheres and their intersections. This will be obtained by extending the corresponding results in 2 and 3 dimensions, which will therefore be considered first.

In 3 dimensions, the results concern the intersection of two spheres, and are shown geometrically in Figure 1:

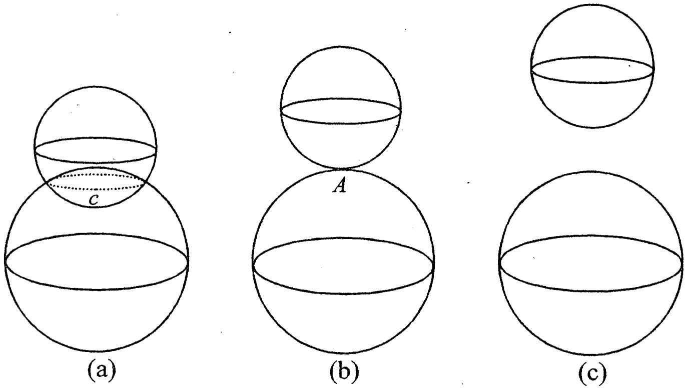


Figure 1: (a) spheres intersecting in a circle c; (b) spheres intersecting in a point A; and (c) spheres with no intersection.

In 2 dimensions, the results concern the intersection of two circles, and are obvious geometrically (Figure 2):

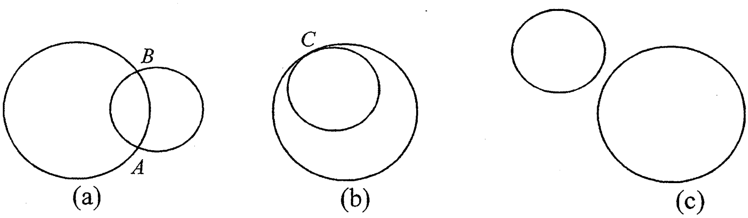


Figure 2: (a) 2 circles intersecting in 2 points A, B; (b) 2 circles intersecting in a point C; (c) 2 circles with no intersection.

In 2 dimensions a circle of radius r may be represented by the equation

$$x^2 + y^2 = r^2 \tag{1}$$

taking the origin of spatial coordinates (x, y) at the centre of the circle. The equation expresses the fact that the circle consists of all points at distance r from $(0,0)$; the point $P(x, y)$ is at distance $OP = \sqrt{x^2 + y^2}$ from the origin (Figure 3), using Pythagoras's theorem.

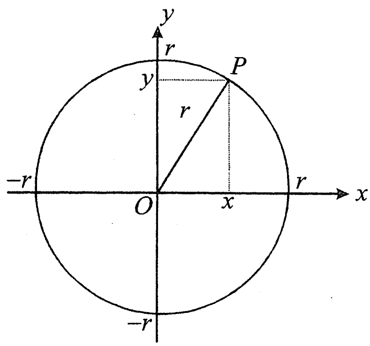


Figure 3 : $x^2 + y^2 = r^2$

In general, the equation $(x-a)^2 + (y-b)^2 = r^2$ represents a circle with centre at (a, b).

In 3 dimensions the spatial coordinates are (x, y, z), and the distance of $P = (x, y, z)$ from the origin is $\sqrt{x^2 + y^2 + z^2}$. This follows from a double application of Pythagoras's theorem (Figure 4). The points satisfying

$$x^2 + y^2 + z^2 = r^2 \tag{2}$$

are those at distance r from the origin, so this equation represents a sphere of radius r.

A concise statement of the results in Figures 1 and 2 can be given as follows. If a point is regarded as a circle of zero radius, then two (different) spheres either have no intersection, or the intersection is a circle. If a circle is regarded as a 2-dimensional sphere, then two 3-dimensional spheres intersect (if at all) in a 2-dimensional sphere. Noting that $2 = 3 - 1$ leads to the conjecture that the intersection of 2-dimensional spheres (shown in Figure 2) is a "(2 – 1)-dimensional sphere". But what does this mean?

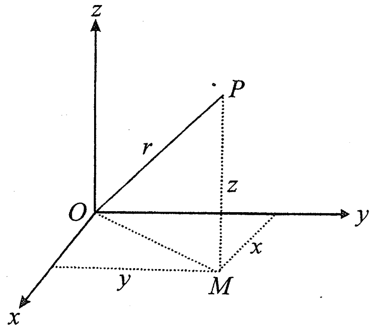


Figure 4: $OM^2 = x^2 + y^2$, $OP^2 = OM^2 + MP^2 = x^2 + y^2 + z^2$

Comparing equations (1) and (2) suggests that an example of a "1-dimensional sphere" is represented by the equation $x^2 = r^2$. This is just the two points with coordinates $x = \pm r$ on a one-dimensional line. Thus, in general, a 1-dimensional sphere is a pair of points. By allowing zero "radius" $r = 0$, a single point is included. Then Figure 2 illustrates that the two 2-dimensional spheres intersect (if at all) in a 1-dimensional sphere.

Thus the result in Figure 1 can be taken *down* 1 dimension, giving the result in Figure 2. Can the result in Figure 1 be taken *up* 1 dimension? A formal statement is easy: two different "4-dimensional spheres" intersect (if at all) in a 3-dimensional sphere. But does this mean anything? What illustration generalizes Figures 1, 2 and 4?

In Figures 3 and 4, the coordinates (x, y) or (x, y, z) represent space dimensions. Physical applications that consider a 4th dimension often take this to represent time, using a 4th coordinate t: (x, y, z, t) means the space point (x, y, z) at time t. Then the extension of equations (1) and (2) is clearly

$$x^2 + y^2 + z^2 + t^2 = r^2 \tag{3}$$

To work up an illustration of equation (3), consider first adding the time dimension to one space dimension, and then adding time to two space dimensions. In the first case, (3) becomes $x^2+t^2=r^2$, and an analogue of Figure 3 can be used:

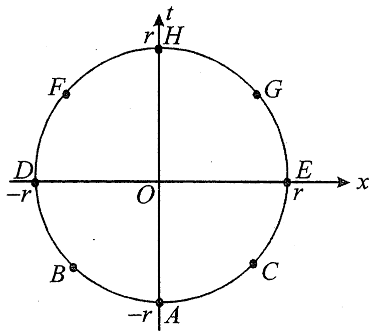


Figure 5(a)

Remember that the space-time point (x, t) means the space point x at time t. Take $r = 2$ as a specific example of Figure 5(a). At time $t = -2$ the equation $(x^2 + t^2 = 4)$ gives $x = 0$; at $t = -1, x = \pm\sqrt{3}$; at $t = 0$, $x = \pm 2$; at $t = 1$, $x = \pm\sqrt{3}$; and at time $t = 2$, $x = 0$ again. A 1-dimensional space illustration is possible.

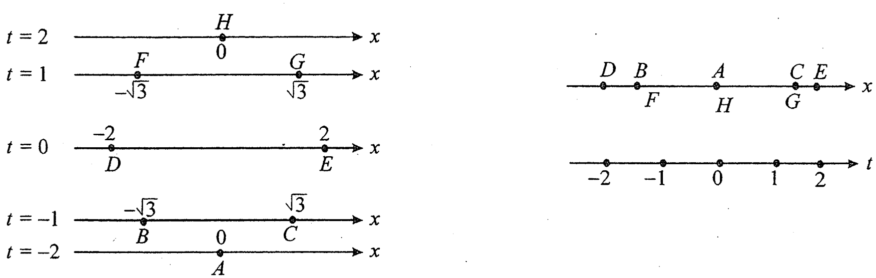


Figure 5(b) and (c)

In dynamics this could represent two particles appearing (or being created) at $x = 0$ at time $t = -2$, then moving apart until $t = 0$, then moving together until they coincide at $t = 2$, when they disappear (Figures 5(b) and 5(c)).

For the case $x^2+y^2+t^2=r^2$, an analogue of Figure 4 can be used:

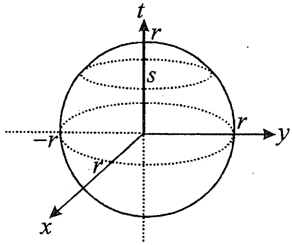


Figure 6(a)

Remember that (x, y, t) means the space point (x, y) at time t. At time $t=-r$ the equation becomes $x^2+y^2=0$ which gives $x=y=0$; at time $t=0$, the equation gives the circle $x^2+y^2=r^2$; at time $t=s$ in Figure 6(a) the equation gives the circle $x^2+y^2=r^2-s^2$, assuming $s^2<r^2$; at time $t=r$ the equation gives the point (0,0) again. The analogue of Figure 5(b) is a ring expanding from zero radius to radius r as time increases from $t=-r$ to $t=0$, and contracting to zero radius from time $t=0$ to time $t=r$, when it disappears (Figures 6(b) and 6(c)).

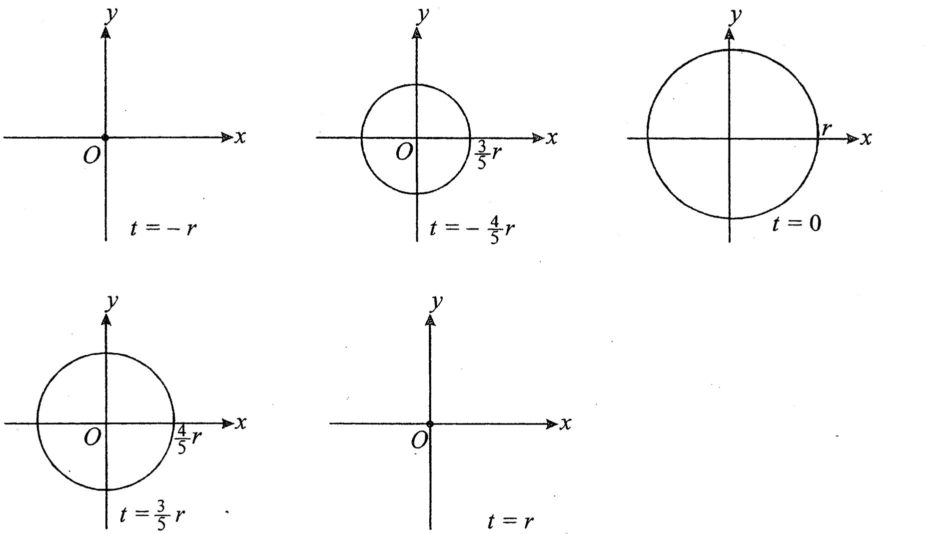


Figure 6(b)

To summarise these realisations when one of the dimensions is time: a (3-dimensional) sphere is a space circle (2-dimensional sphere) which appears as a (central) point at some initial time, expands to a maximum radius, then contracts to the central point at some final time and disappears; a 2-dimensional sphere is a pair of points that appear, coincident, at some initial time, then move apart to maximum separation, then move together to coincide at some final time, when they disappear.

A realisation of the 4-dimensional sphere represented by equation (3) is evidently a space sphere which appears as the space point (0,0,0) at initial time $t = -r$, expands to a maximum radius r at time $t = 0$, then contracts to the central point (0,0,0) at time $t = r$ when it disappears. For a picture, add a third space dimension in Figure 6(b), and change the circles to spheres (Figure 7). A physical example is an enlarging, then shrinking soap bubble. The general 4-dimensional sphere is represented by an equation like (3) with the position of the centre changed from $x = y = z = t = 0$ to another point (x_0, y_0, z_0, t_0). The interpretation is the same, with the expanding and contracting space sphere centred on (x_0, y_0, z_0), and existing between the times $t = t_0 - r$ and $t = t_0 + r$.

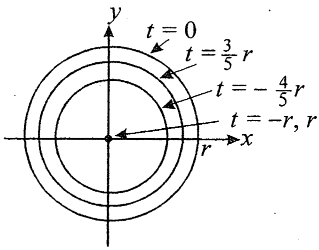


Figure 6(c)

Finally consider the intersections, starting with two 3-dimensional spheres where time is one dimension. Each is represented as in Figure 6(a) or Figure 6(b), with the centres generally at different points. The circles exist only in finite time intervals, and an intersection is possible only if these time intervals overlap. At any particular time in this overlap, two circles exist, and the representation is one of diagrams in Figure 2. Each circle is either expanding or contracting about its centre. There are various cases: when one circle appears, the other circle may already exist, expanding or contracting, or may appear later before the first circle disappears. In all cases where there is an intersection, the illustrations in Figure 2 appear in the order (c), (b), (a), (b), (c), possibly with (a) omitted. In both (c) and (b) either one circle is inside the other, or one circle is outside the other. The picture of the intersection is like Figure 5(b).

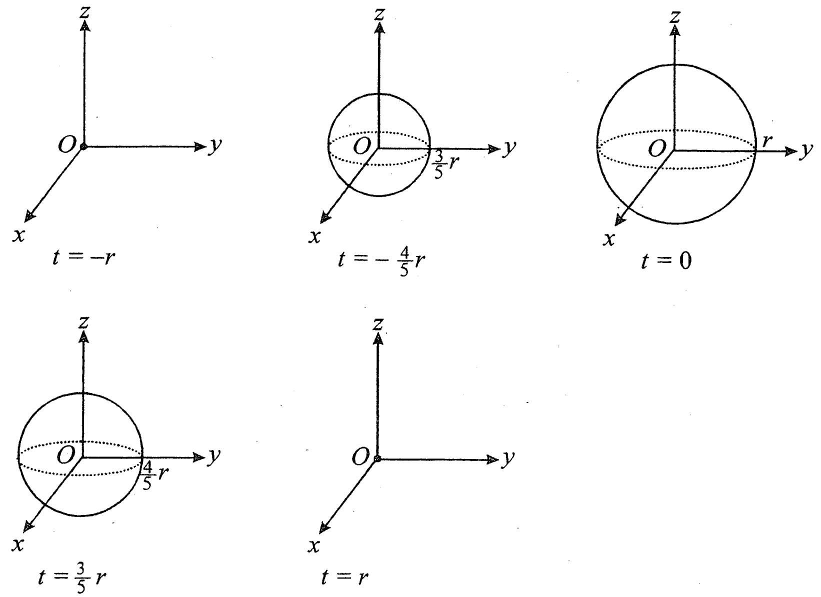


Figure 7: The 4-dimensional sphere represented by equation (3).

The realisation shows that the result that the intersection of two spheres is a circle remains true when one of the three dimensions is time. If the problem is approached algebraically, this remark is trivial, since the only change is a substitution of the symbol 't' for the symbol 'z' in sphere equations such as (2). However the interpretation is different, and this interpretation extends easily into 4 dimensions. The expected result has already been stated: two different 4-dimensional spheres intersect (if at all) in a 3-dimensional sphere. This can now be verified as for 3-dimensional spheres.

The two 4-dimensional spheres are each represented as in Figure 7, by a space sphere expanding from a central point to a maximum radius then contracting to the point, existing only in a finite time interval. The two time intervals must overlap for an intersection to exist, and then at any particular time in the overlap the representation is one of the diagrams in Figure 1. Each sphere is either expanding or contracting about its centre. For an intersection the cases illustrated in Figure 1 must appear in the order (c), (b), (a), (b), (c), perhaps with (a) omitted. In Figure 1(a) the spheres are expanding or contracting, so the intersection circle is also expanding or contracting. The intersection evolves as in Figure 6(b) or 6(a), and is therefore a 3-dimensional sphere.

I apologise to readers who find this article a long and complicated discussion of a simple result. Mathematicians, comfortable with the concept of an n-dimensional space, may give their favourite definition of distance in this space, and a sphere is the set of points at constant distance (radius) from a fixed point (centre). The intersection of two spheres is then an $(n-1)$ dimensional sphere; this follows from simple geometry, which may be approached algebraically after introducing coordinates and the extension of equations (1) and (2). Using time as an extraordinary dimension then complicates matters, but may help to convince non-mathematical friends the result has some meaning.

* * * * *

Truth lasts throughout eternity
When once the stupid world its light discerns:
The theorem, coupled with Pythagoras' name,
Holds true today, as it did in olden times.

A splendid sacrifice Pythagoras brought
The gods, who blessed him with this ray divine;
A great burnt offering of a hundred kine,
Proclaimed afar the sage's gratitude.

Now since that day, all cattle [blockheads] when they scent
New truth about to see the light of day,
In frightful bellowings manifest their dismay;

Pythagoras fill them all with terror;
And powerless to shut out light by error, In sheer despair they shut their eyes and tremble.

– Adelbert von Chamiso in *Gedichte*, 1835

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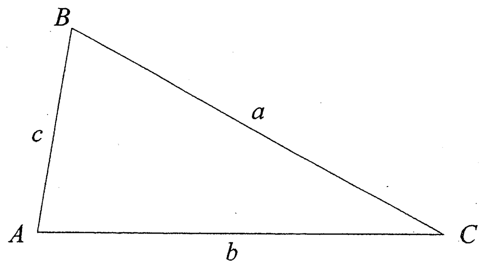
LETTER TO THE EDITOR

Kim Dean, from Union College Windsor, wrote:

Each year I try to acquaint readers with some aspect of the work of the eccentric Welsh mathematician and physicist, Dai Fwls ap Rhyll. He first came to my notice back in 1980 when he won the Prix Le Bon for his ground-breaking theory of gravity. Since my humble efforts to popularise his work way back then, he has written to me from time to time to announce yet more amazing discoveries. He is not always the most faithful of correspondents, however, so that 1998 was only one of many years in which he failed to write.

Common to all his work is a deep questioning of accepted results and methods in mathematics and its applications. He has, I'm pleased to say, written this year, and his letter is in the same vein. It has to do with trigonometry, and its incompatibility with arithmetic and geometry. As usual, his arguments are reasonably accessible, and I have no qualms about *Function* readers being able to follow him.

He begins with a triangle ABC and according to the usual convention, the side opposite angle A is called a, and this symbol also stands for the length of that side, as A likewise stands for the (radian) measure of the angle designated A. Similarly for the other sides and angles. See the diagram.



Dr Fwls supposes, without loss of generality, that $a > b$. (We may always arrange for this except in the special case of an equilateral triangle to be discussed below.) Then $a\cos C > b\cos C$ and by a standard formula (easy to prove if you don't know it already; alternatively consult any standard text)

$$a = b\cos C + c\cos B, \quad \text{so that} \quad b\cos C = a - c\cos B.$$

Similarly

$$b = a\cos C + c\cos A, \quad \text{so that} \quad a\cos C = b - c\cos A.$$

It follows that $b - c\cos A > a - c\cos B$, and so now we have

$$c\cos B - c\cos A > a - b.$$

Multiply both sides of this inequality by $2ab$, and apply the cosine rule to both terms on the left. This gives

$$b\left(a^2 + c^2 - b^2\right) - a\left(b^2 + c^2 - a^2\right) > 2ab(a-b).$$

From this result, some relatively simple algebra shows that

$$a^3 - b^3 - a^2 b + ab^2 > c^2(a-b).$$

But now, the left-hand side may be factored to give

$$(a-b)\left(a^2 + b^2\right) > c^2(a-b).$$

Thus

$$a^2 + b^2 > c^2. \qquad (*)$$

Dr Fwls was at pains to point out that this last step does *not* rely on the old chestnut of dividing by zero. It was clearly stated that $a > b$, and so $a - b \neq 0$. However, the theorem is always true, because in the only case *not* so far not considered (that of the equilateral triangle), equation (*) is clearly satisfied.

Equation (*) has several disturbing consequences. For a start, it is quite at odds with Pythagoras's Theorem, in particular it tells us that $4^2 + 3^2 > 5^2$, i.e. $25 > 25$, which is a bit puzzling. The specific example Dr Fwls supplied was even more challenging. He set $a = 4, b = 3, c = 6$, and so came up with $25 > 36$, by means of quite impeccable arguments.

So trigonometry is incompatible with arithmetic and with geometry as well!

HISTORY OF MATHEMATICS

The Best Proof?

Michael A B Deakin

I want to revisit a topic I discussed earlier (in my column of October 1993): that of *irrational* numbers, or as I put it then "unreasonable numbers".

An *irrational* number is one that can't be expressed as a *ratio* of two whole numbers, or integers. The best-known such number is $\sqrt{2}$, and it is widely thought that this is the number whose irrationality was first proved.

The usual story has it that Pythagoras or one of his followers (round about 500 BC) found that the diagonal of a square was *incommensurable* with its side. This is to say that we are *necessarily* unable to find a length that goes some exact (integral) number of times into both the length of the side and the length of the diagonal. It will readily be understood that this is the same as saying that:

There are no integers p, q such that $\sqrt{2} = \dfrac{p}{q}$.

I gave in my earlier account some of the background into this result, whose historical origins are somewhat blurred. The first reliable piece of writing (which is where history proper begins) comes from Aristotle, who was born about 100 years after Pythagoras died. There is some dispute over quite what Aristotle meant, but it is clear that he is referring to an argument that shows some geometrically derived number to be irrational, or more precisely that the diagonal (of something) is incommensurable with that something's side. The 'something' is usually presumed to be a square, and some translations (among them the one I used when last I wrote) actually incorporate this understanding into their English version.

It is possible that the 'something' was a regular pentagon (as a few brave souls have argued) or even a regular hexagon (as a few even braver souls have suggested). But most probably it was a square. In any case, I shall proceed for the rest of this column to investigate the case of $\sqrt{2}$. For the *mathematical* point involved, this is the simplest assumption to make. I shall consider several proofs of the statement given in italics above. Until I began to prepare this article, I had not known that so many existed!

The principal passage from Aristotle occurs in his *Prior Analytics*, which is a textbook on logic and on the construction of valid arguments. The relevant passage is listed as I, 23, 41a in the usual way in which such references are quoted. This system relies on a now out-of-date edition of the work, and it better suits the needs of specialist scholars than those of the general reader. A recent (and good) translation is that of Robin Smith, published by the Hackett Publishing Company of Indianapolis in 1989. In this edition, you will find the passage towards the bottom of page 37. Here it is.

"For all those [deductions] which come to a conclusion through an impossibility *deduce* the falsehood, but *prove* the original from an assumption when something impossible results when its contradiction is supposed, <proving,> for example, that the diagonal is incommensurable because if it is put as commensurable, then odd numbers become equal to even ones. It *deduces* that odd numbers become equal to even ones, then, but it *proves* the diagonal to be incommensurable from an assumption since a falsehood results by means of its contradiction."[1]

The passage continues but without further significant mention of incommensurables. It will be clear immediately that Aristotle's main point is *not* to give the proof of incommensurability. Rather he is analysing the structure of an argument with which he supposes his readers to be already familiar. "If we assume the diagonal and the side to be commensurable, then we can deduce, *by means of a completely valid argument*, that odd numbers would equal even ones." The fault must lie with the initial assumption, *precisely because the chain of deduction is sound*, although its conclusion is a nonsense.

It is generally thought (but not by everyone) that Aristotle had in mind this proof (which I also put into my earlier article).

Suppose $\sqrt{2} = \frac{p}{q}$ where p and q are integers and where the fraction is expressed in its lowest terms (that is to say, p and q have no common factor other than 1). It then follows that

$$p^2 = 2q^2 \qquad (*)$$

[1] Notice a little scholarly device in this passage. The square brackets [...] indicate that *I* have interpolated a word (in order to explain the context, as established by previous sentences in the text); the angle brackets <...> indicate that *Smith* has inserted a word into the translation (in order to make the sense clearer in the English). The italics are Smith's.

and therefore p^2 is an even number. But if p^2 is even, then p itself is also even because of a known theorem (listed as IX.12 inEuclid's *Elements* and discussed in my column of June 1993). Because p is even, then q, having no factor (other than 1) in common with p must therefore be odd.

But if p is even then $p = 2r$ for some number r. If we now substitute this into equation (*), we find

$$q^2 = 2r^2$$

and so we find (as we did before with p) that q is even, when we have just deduced that it is odd! Thus $\sqrt{2}$ is irrational.

Another widely quoted proof I learned in my own schooldays, from Volume II of Barnard and Child's *A New Algebra,* a very popular textbook that went through many printings from its initial appearance in 1912. The proof is itself much older than this. I'm not entirely sure how old. This proof writes equation (*) in the form

$$2 = \frac{p^2}{q^2}$$

and notes that no cancellation is possible between the top and the bottom on the right-hand side of this equation. This is because there are no common factors (other than 1) between the p and the q. Thus the right-hand side must be an integer and so $q^2 = 1$ and in consequence $q = 1$. This means that $\sqrt{2}$ is an integer, which is clearly impossible (as $1 < \sqrt{2} < 2$).

This second proof is sometimes seen as more advanced than the first, because it makes implied reference to the "prime factor theorem" that states that

Every integer greater than 1 may be decomposed into a product of primes in precisely one way (if we ignore the order of the factors and do not count 1 as a prime).

It seems to me that this theorem is rather obvious to the student and that it requires considerable mathematical maturity to see that it is something that needs proof. This refinement could easily be delayed, to my way of thinking, but tastes differ in this matter. (The proof of the prime factor theorem is actually rather subtle.)

However, long before this proof became fashionable, another was published. It occurs in Euclid's *Elements* as Proposition 9 of Book X. It is usually attributed to

Thaetetus, a contemporary of Plato's and probably one of Aristotle's teachers, so it may well be this proof that Aristotle had in mind when he wrote his passage in the *Prior Analytics*. The theorem to be proved is more general than the irrationality of the single number $\sqrt{2}$. It states:

> *Unless a number is the square of an* integer, *then its square root is incommensurable with the unit (i.e. the square root will be irrational).*

(It is not difficult to modify either of the two previous proofs to prove the same generalisation. I leave this to the reader as an exercise.)

The Euclid-Thaetetus proof proceeds as follows. Suppose that two lengths A and B are commensurable. Then there is some third length C, say, such that $A = pC$ and $B = qC$, where p and q are integers. If now we form the squares on sides A and B, then the ratio of their areas will be $\frac{p^2}{q^2}$, which is the ratio of two square numbers. Now suppose that the two lengths A and B are incommensurable. Euclid then asserts that the ratio of their squares cannot be the ratio of two square integers $\frac{p^2}{q^2}$, because if it were then (by taking square roots on both sides) we could find integers p and q that would make A and B commensurable after all. We thus have the following pairings: A and B commensurable, ratio of squares as $\frac{p^2}{q^2}$; A and B *in*commensurable, ratio of squares *not* as $\frac{p^2}{q^2}$. No other possibilities exist.

In particular take a square of side $\sqrt{2}$. If we take this to be A and we set $B = 1$, then their squares are in the ratio $\frac{2}{1}$ and the numerator of this is not a square number. Thus $\sqrt{2}$ and 1 are incommensurable, or in other words $\sqrt{2}$ is irrational.

This proof was analysed in great detail in 1935 by Oskar Becker, a German scholar with expertise in both formal logic and in the history of mathematics. Euclid's concept of commensurability includes the implicit assumption that the "fractions" we derive are expressed in their lowest terms. In other words, if the ratio of squares is (say) $\frac{8}{2}$, then it would automatically be reduced to $\frac{4}{1}$. The analysis of this (not supplied by Euclid) now looks very like that of our second proof, depending on the prime factor theorem. Becker showed that *if* we accept the prime factor theorem, then the proof is fine (so that Euclid-Thaetetus may in one sense be regarded as the original authors of my second proof above), but that we *may*, if we so desire, get by with a somewhat weaker result. The details (which I won't go into here) make the proof look much more like a version of the first proof. If indeed Thaetetus taught Aristotle, then this would make sense of the passage from the *Prior Analytics*. There may well have been details available to Aristotle that didn't make it into Euclid's text!

Because Euclid is remembered principally as a geometer, we might have thought that a more "geometric-looking" proof would be more to his taste. Such a proof is known. It comes in part from a later Greek mathematician, Proclus (410-485 AD), who wrote a Commentary on Book I of Euclid's *Elements*. Look at Figure 1, which first appeared in a discussion of Proclus written by a historian of Mathematics called Fridericus Hultsch.[2]

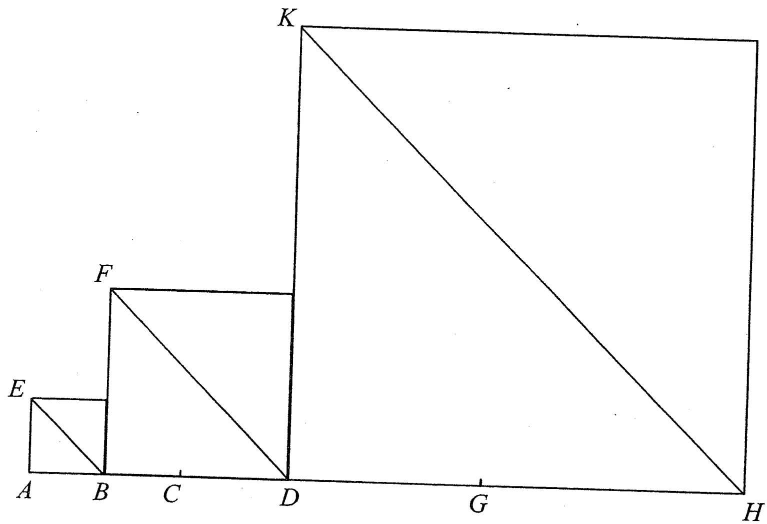


Figure 1

The left-hand square has side AB and diagonal EB. Although this is not Hultsch's original notation, we will let $EB = p$, $AB = q$. The next square is so constructed that its base BD has length $p+q$. From Pythagoras' theorem, we may deduce that its diagonal DF has length $2q+p$. This means that the

[2] Hultsch also turned up in an earlier History of Mathematics column. See Winifred Frost's account of the early woman mathematician Pandrosion in *Function, Vol* 16, *Part* 3.

diagonal FD of the larger square is equal to the sum of the bases AB and BD, i.e. the length AD. This pattern persists as we move to the larger square to the right, and so on. This is not in itself a full proof of the irrationality of $\sqrt{2}$, however it may be used to construct one and I will do this later.

Hultsch's diagram appeared in 1901, and found its way into Sir Thomas Heath's edition of Euclid's *Elements*. The first edition of this work appeared in 1908, and a second revised edition in 1925. It is this second edition that has remained the standard English version of Euclid's *Elements* to this day.

The first *full* geometric proof of the irrationality of $\sqrt{2}$ however appeared as recently as last century and that in a book of Algebra! The book in question was Chrystal's *Algebra*, which was first published in 1886. It used Figure 2, which I give here in a slightly modified form.

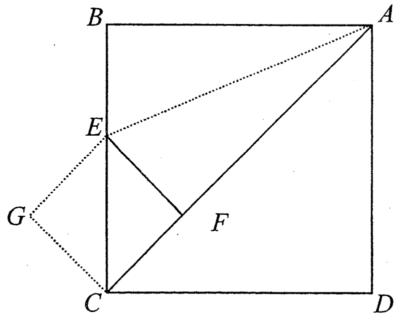


Figure 2

$ABCD$ is a square of which CD is a side and AC a diagonal. The point F lies on AC and is so chosen that $AB = BC = AF$. At F, draw a perpendicular FE to the diagonal AC and meeting BC in E. Join AE. (This is the step Chrystal himself omits, trusting his readers to provide the details.) In triangles ABE, AFE: angles ABE, AFE are equal to one another and are both right angles, the length AB equals the length AF (by construction) and the side AE is common. Thus the triangles are congruent and, in particular, $BE = EF$. With just a little more work, we may establish (by similar means) that angle CEF equals angle ECF (both are equal to $\frac{\pi}{4}$ radians, or $45°$). It then follows that $FC = EF$.

So $CF = AC - AB$ and $CE = CB - CF$. Now if AB and AC were each whole number multiples of some unit length, then so would CF be a whole number multiple of that same unit, and so too would be CE. But now complete the figure by drawing CG perpendicular to CF and EG perpendicular to EF, and so form a square $CFEG$, of which CF is the side and CE the diagonal. We have just shown that these would each have to be multiples of some common unit.

Now we may repeat this process to find a smaller square with the same property, and *that* square would give rise to an even smaller square, and so on, with the square becoming arbitrarily small. However small the supposed unit is taken to be, we can make a square whose side and diagonal are supposedly integral multiples of that unit even smaller than the unit itself. And this clearly is a nonsense.

So, following Aristotle's pattern of argument, we have shown the premiss (of commensurability) to be false.

Chrystal's argument is somewhat clumsy. It can be tidied up by redrawing the diagram. Look again at Figure 1.

This time, start at the *right* of the diagram, with the largest square, whose diagonal is HK and whose base is HD. Slightly alter our previous notation to make $p = HK, q = HD$ for subsequent discussion. Whereas Proclus and Hultsch read the diagram from left to right, we will go backwards, from right to left. The extended baseline HB is so constructed that $HB = HK$, and the second square erected on the base BD. Etc. The gist of the argument that follows is then the same as Chrystal's, but the details are much simpler.

These two versions of the geometric proof are very much in a tradition of other such proofs. In my earlier paper, I gave an account of the late Chris Ash's proof along similar lines of the irrationality of the golden ratio, $\left(1+\sqrt{5}\right)/2$.

In more recent times, other proofs have appeared, although it may be queried how "new" they really are. One such is attributed to Ivan Niven, an American mathematician, whose work has been described in*Function* before.[3] Niven's proof is very elegant.

Suppose $\sqrt{2} = \frac{p}{q}$ and suppose also that q is the *smallest* integer for which this is so. Then $q\sqrt{2}$ is an integer and q is the smallest integer that achieves this. Now form $q^* = q\sqrt{2} - q$. Clearly q^* is an integer, because $q\sqrt{2} = p$, which is an integer. Clearly also $q^* < q$, which we may easily prove from the inequality

[3] See for example my column in June 1996, and an article by Niven himself in the issue for February 1984.

$\sqrt{2} < 2$. But now $q^* \sqrt{2}$ is seen to be an integer smaller than q, and so we have a contradiction, as q was assumed to be the *smallest* integer that did the job.

This is really extremely elegant. But how new is it? Well it depends on how you count things. It's really the "Proclus-Hultsch" proof in disguise. Go back to Figure 1. We supposed that *HK* and *HB* were both integral multiples of some unit, and in terms of that unit, we took *HK* to have length p and *HB* length q. Thus *BD* has length $p-q$. Because of geometric similarity, *DF* has length

$$(p-q)\sqrt{2} = (p-q)\frac{p}{q} = \frac{p^2 - pq}{q} = \frac{2q^2 - pq}{q} = 2q - p.$$

[These are the same equations as Proclus and Hultsch gave, but this time in the amended notation.]

But now we may recast this equation as $\sqrt{2} = \dfrac{2q-p}{p-q}$ and we see that we have a fraction with the denominator $p-q$. But now look at Niven's denominator q^*. We have

$$q^* = q\sqrt{2} - q = q\frac{p}{q} - q = p - q.$$

So all that has happened is that a geometric proof has been expressed in algebraic form and reduced to its bare essentials (Niven didn't bother with the numerator; there was no need. But it can be shown very easily that had he done so, he would have found $2q-p$, which for later reference, I will call p^*. You may care to look into this yourself.)

Another modern proof is that appearing in *The Book of Numbers* by John Conway and Richard Guy. Conway recently wrote that he's "sure it's not new", but this may be modesty. No-one has yet pointed to an earlier account in quite these terms. Here's how the proof goes.

Suppose $\sqrt{2} = \dfrac{p}{q}$ and suppose that p, q are the *smallest* integers that satisfy this equation. Then because $1 < \sqrt{2} < 2$, we have $\sqrt{2} = 1 + \dfrac{Q}{q}$, where $0 < Q < q$.

In other words, $\dfrac{Q}{q}$ is a proper fraction. But $\dfrac{2q}{p}$ (which would also have to equal

$\sqrt{2}$) may also be expressed in such fractional terms. Indeed we also have $1 < \frac{2q}{p} < 2$, so that $\sqrt{2} = 1 + \frac{P}{p}$, where $0 < P < p$.

But now we must have $\frac{Q}{q} = \frac{P}{p}$ and this means that $\frac{p}{q} = \frac{P}{Q}$. But we just found that $P < p$ and $Q < q$, so it follows that p, q couldn't have been the smallest integers after all. Again we have a contradiction.

Actually Conway and Guy give this proof for the general case of an integer other than a perfect square (not just for 2). I modified it to this special case. However, I did so for a reason.

In the course of my account of the Conway-Guy proof, I did not evaluate either P or Q. However it is not difficult to do so and, when we do, we find:

$$Q = p - q = q^*, \text{ and (with a little more difficulty) } P = 2q - p = p^*.$$

It thus follows that the Conway-Guy proof is fundamentally the same as the Niven proof, which in its turn was a simplified version of the "Proclus-Hultsch" argument, which in *its* turn was a simple improvement to Chrystal's proof. [It should however be pointed out that both the Niven proof, like the Conway-Guy proof, was initially given in a more general form and proved that $\sqrt{N}$ is irrational unless N is a perfect square. This generalisation takes a little more work to achieve in the case of the two geometric versions.]

Thus the last four proofs I have outlined of the irrationality of $\sqrt{2}$ are *mathematically* equivalent. On the other hand, they *look* very different. *Psychologically* they are not the same. To me this makes them *different* proofs, because I believe that the key to mathematics is the increase it brings to our understanding; the *psychological aspect* is paramount. Not everyone shares this view. Perhaps I will expand on it in a later column.

But, however we look at things, we will surely agree that the irrationality of $\sqrt{2}$ is well and truly established!

* * * * *

COMPUTERS AND COMPUTING

Thirty-Four Years Ago

Peter Grossman

Recently I came across an old copy of *Matrix*, dated 1965. *Matrix* was a magazine produced by the Melbourne University Mathematical Society, and it carried similar kinds of articles to those that appear in *Function*. Indeed, some of the names in that issue of *Matrix* will be known to *Function* readers. One of the editors was Malcolm Clark, now an editor of *Function*, and one of the articles was written by G A Watterson, who has also served on the editorial board of *Function*. I was curious to see what the similarities and differences were between mathematics magazines then and now.

Many – in fact most – of the articles in the 1965 issue of *Matrix* would be equally at home in a recent issue of *Function*. There were articles on magic squares, mathematics in biology (including the Hardy-Weinberg law, which also appeared recently in *Function*; see M Deakin's article on G H Hardy in *Vol* 19 *Part* 3, pp 82–8), number theory, relativity, noughts and crosses, voting systems (another topic that has appeared more than once in *Function*), mathematics in Babylon, Fibonacci numbers, and several other topics. You might be tempted to conclude that mathematics has remained stagnant over the past thirty years, but of course that is not the case at all. *Function* has also dealt with topics that could not have been written about in 1965, because the mathematics had not been developed then; fractals is one such example. A more sensible conclusion to draw is that mathematics has an enduring quality; mathematical results obtained in the past remain not only true (obviously!) but often also useful and relevant.

Not surprisingly, the greatest change between then and now is in the area of computing. An article in *Matrix* by Graham Leary provides an introduction to the programming language ALGOL 60. The article begins by referring to the (then) popular term "electronic brain" for a computer. That expression has completely died out, probably because computers are now so familiar to us that we all know they don't behave very much like a brain at all. In the 1960s, however, a computer was a large machine that was typically found only in a university or a large organisation, and as such it was a mystery to most people. As a child at that time, I recall that one of the highlights of attending a university open day with my father was the chance to see a computer! (Another article in *Matrix* contains the forecast:

"The day is rapidly approaching when every large business will have its computer, with staff to attend it.")

For readers who are curious to know what an ALGOL program looks like, here is one reproduced from Graham Leary's article.

```
begin real a,b,c,x,Ans;
Read (a,b,c);
if a = 0 then stop;
x:=(c-b)/a;
if x > 0 then begin Ans:=sqrt(x); go to Output end;
Ans:=sqrt(-x);
Output: Print (a,b,c,x,Ans);
end;
```

I will leave it to readers to work out what the program does; it is simple enough to follow. Note in particular the "go to" statement, common in programs of that era but now made virtually obsolete by the introduction of other programming constructs. (Java doesn't even have a "go to" construct.)

What happened to ALGOL? Some other programming languages that were around at the time – FORTRAN[1] and COBOL – are still in use today, in updated form. ALGOL, however, is no longer used. Did it just die? Actually, the situation is more complicated, as new programming languages often draw on features of earlier ones. Pascal, which was developed in the 1970s, incorporates some of the features of ALGOL (the use of **begin** and **end** to form compound statements, seen in the above code, for example), and the more recently developed languages C and Java also share some of these features.

Returning to *Matrix*, perhaps some of the most interesting parts to look at now are the advertisements. They include advertisements by potential employers of mathematics graduates, some of which, such as the Bureau of Meteorology, still employ mathematicians today. In 1965 the Bureau was offering starting salaries of around 1500 pounds for graduates during training as meteorologists, increasing to about 3900 pounds at the top of the salary scale. At the changeover to decimal currency, which occurred in the following year, one pound converted to two dollars. By looking at the changes in the CPI between then and now,[2] it can be

[1] At that time, most computers had only upper case letters available for programming, and for this reason it was considered appropriate to write the names of programming languages entirely in upper case also. The latest Fortran standard – Fortran 90 – dispenses with this convention.

[2] Source: Australian Bureau of Statistics.

found that prices have increased by a factor of about 8.1, so 1500 pounds is equivalent to about $24300 today in real terms.

One of the greatest changes apparent between 1965 and today is in social attitudes. While one article in *Matrix* acknowledged that women "successfully occupy positions at many levels in many kinds of mathematical activity in schools and universities and elsewhere", an advertisement on the back cover by a major mining company presented a different picture that would be quite unacceptable today. "A big company can offer big opportunities for ambitious young men", it announced, going to suggest that the company could offer a professional career path "if you are a university or technical college man yet to make a decision on your career".

Finally, the following quote from *Matrix* is too good not to pass on. It is attributed to an unidentified computer engineer:

"As far as I know, this computer has never made an undetected error."

* * * * *

> I came to Göttingen as a country lad of eighteen, having chosen that university mainly because the director of the high school happened to be a cousin of Hilbert's and had given me a letter of recommendation to him. In the fullness of my innocence and ignorance, I made bold to take the course Hilbert had announced for that term, on the notion of number and the quadrature of the circle. Most of it went straight over my head. But the new world swung open for me, and I had not sat long at Hilbert's feet before the resolution formed itself in my young heart that I must by all means read and study whatever this man had written. And after the first year I went home with Hilbert's *Zahlbericht* under my arm, and during the summer vacation I worked my way through it—without any previous knowledge of elementary number theory or Galois theory. These were the happiest months of my life, whose shine, across years burdened with our common share of doubt and failure, still comforts my soul.

–Hermann Weyl in "David Hilbert and His Mathematical Work"
Bull of the Am Math Soc 50

PROBLEM CORNER

SOLUTIONS

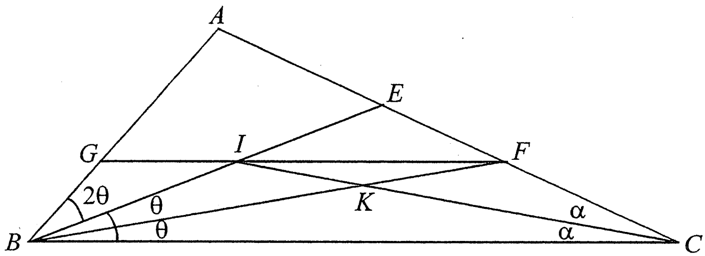
PROBLEM 22.5.1 (K R S Sastry, Bangalore, India)

Let I be the incentre of the triangle ABC. Let E and F be points on $\overline{AC}$ such that $\overline{BE}$ bisects $\angle ABC$ and $\overline{BF}$ bisects $\angle EBC$.

(a) Prove that $\overline{FI}$ is parallel to $\overline{BC}$ if and only if $\angle ABC = 2\angle ACB$.

(b) Let G be the point where $\overline{FI}$ extended meets $\overline{AB}$. Prove that $\overline{GE}$ is parallel to $\overline{BF}$ if and only if $\angle ABC = 2\angle ACB$.

SOLUTION (Carlos Victor, Rio de Janeiro, Brazil)



(a) Suppose that FI is parallel to BC then $\angle FIC = \alpha$ and $IF = FC$. Similarly, in triangle BIF we have $IF = BI$ and therefore $BI = FC$ and $BIFC$ forms an isosceles trapezium. Hence $\angle EBC = \angle ECB$ so that $\angle ABC = 2\angle ACB$.

For the converse suppose that $\angle ABC = 2\angle ACB$, so $\theta = \alpha$. If K is the point of intersection of BF and CI then $BK = CK$ and triangles BIK and FCK are congruent with $IK = FK$. Consequently $\angle IFB = \angle CIF$, and since $\angle FKC = \theta + \alpha = 2\theta$, we see that $\angle IFB = \theta$ so that FI is parallel to BC.

(b) Suppose that $\angle ABC = 2\angle ACB$ so that from part (a) FI is parallel to BC and $FC = FI = BI$ and also $\angle EIF = 2\theta$. It now follows that triangles BGI

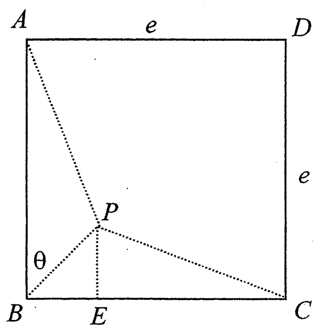
and IEF are congruent and consequently $GI = IE$ and in triangle GEI $\angle IEG = \angle EGI = \frac{1}{2}\angle EIF = \theta$, so that BF is parallel to GE.

For the converse let BF be parallel to GE, we then have $\dfrac{EF}{GB} = \dfrac{AE}{AG}$ and $\dfrac{AF}{AG} = \dfrac{IF}{IG}$ where the first equality follows from the fact that triangles AGE and ABF are similar and the second equality holds because AI is the bisector of $\angle BAC$. Now applying the theorem of Menelaus to triangle AGF with transversal BE we have $\dfrac{EF}{AE} \cdot \dfrac{AB}{BG} \cdot \dfrac{IG}{IF} = 1$, and together with the above equalities we see that $AB = AF$. Hence $\angle ABF = \angle AFB$ so $2\alpha + \theta = 3\theta$ and then $\alpha = \theta$, so that $\angle ABC = 2\angle ACB$.

PROBLEM 22.2.2 (K R S Sastry, Bangalore, India)

Let P be an interior point of the square $ABCD$. Prove that P lies on the diagonal $\overline{AC}$ if and only if, in that order, PA^2, PB^2, PC^2 are in arithmetic progression.

SOLUTION (proposer)



Let $\angle ABP = \theta$ so that $\angle PBC = 90° - \theta$. Let E be a point on BC such that PE is perpendicular to BC. By the cosine rule $PA^2 = e^2 + PB^2 - 2ePB\cos\theta$ and $PC^2 = e^2 + PB^2 - 2ePB\sin\theta$, so that

$$PA^2 + PC^2 = 2PB^2 + 2e(e - PB\cos\theta - PB\sin\theta).$$

Now PA^2, PB^2, PC^2 in that order will be in arithmetic progression if and only if $PA^2 + PC^2 = 2PB^2$. Hence the assertion follow if and only if $e - PB\cos\theta - PB\sin\theta = 0$, that is if and only if $BC = PE + BE$. Now $BC = PE + BE$ holds if and only if $PE = EC$ and this equality holds if and only if $\angle PCB = 45°$. Hence PA^2, PB^2, PC^2 are in arithmetic progression if and only if P lies on the diagonal AC.

Also solved by Carlos Victor (Rio de Janeiro, Brazil) and Julius Guest (East Bentleigh, Victoria).

PROBLEM 22.5.3 (Juan-Bosco Romero Marquez, Valladolid, Spain)

Find all integer solutions of the equation

$$(x^2 + y^2)^2 = (x + y)^3 \text{ with } x \geq 0 \text{ and } y \geq 0.$$

SOLUTION (Carlos Victor, Rio de Janeiro, Brazil)

Clearly $(0,0)$ is a solution. Now let $x + y \neq 0$ so that $\left(\dfrac{x^2 + y^2}{x + y}\right)^2 = x + y$.

Since $x + y$ is a positive integer, $\dfrac{x^2 + y^2}{x + y}$ is also a positive integer, say $\dfrac{x^2 + y^2}{x + y} = k$. Then $x + y = k^2$ and $x^2 + y^2 = k^3$, eliminating y gives

$$2x^2 - 2k^2x + k^4 - k^3 = 0$$

with solution $x = \dfrac{2k^2 \pm 2k\sqrt{2k - k^2}}{4}$. We must have $2k - k^2 \geq 0$ so $k = 1, 2$. The case $k = 1$ leads to $x = 0,\ y = 1$ or $x = 1,\ y = 0$, and $k = 2$ gives (2, 2). So the only non-negative integer solutions are (0, 0), (0, 1), (1, 0) and (2, 2).

A partial solution was received from Julius Guest (East Bentleigh, Victoria).

PROBLEM 22.5.4 (Republic of Slovenia 38th Mathematics Competition for Secondary School Students, April 1994)

Prove that every number of the sequence 49, 4489, 444889, 44448889, ... is a perfect square (in every number there are n fours, $n-1$ eights and a nine).

SOLUTION (Carlos Victor, Rio de Janeiro, Brazil)

The n^{th} member of the sequence is

$k = 444\ldots488\ldots89$,with n fours, $n-1$ eights and a nine.

so that $k = 9 + 8\,(10^1 + 10^2 + \ldots + 10^{n-1}) + 4(10^n + 10^{n+1} + \ldots + 10^{2n-1})$

$$= 9 + 8\left(10\left(\frac{10^{n-1}-1}{9}\right)\right) + 4\left(10^n\left(\frac{10^n-1}{9}\right)\right)$$

$$= \frac{4\cdot 10^{2n} + 4\cdot 10^n + 1}{9}$$

$$= \left(\frac{1+2\cdot 10^n}{3}\right)^2$$

Hence k is a perfect square.

PROBLEM 22.5.5

Find a function $f: R \to R$ such that:

(i) $f'(x)$ exists, and $[f(x)]^2 + [f'(x)]^2 = 1$, for all $x \in R$;

(ii) $f(x) > 0$ for all $x > 0$, and $f(x) < 0$ for all $x < 0$.

(There is exactly one such function.)

SOLUTION (Carlos Victor, Rio de Janeiro, Brazil)

The equation $y^2 + \left(\frac{dy}{dx}\right)^2 = 1$ gives $\frac{dy}{dx} = \pm\sqrt{1-y^2}$. This differential equation has solutions $y = \pm\sin x$, and also $y = \pm 1$. The conditions (i) and (ii) can only be satisfied by defining

$$f(x) = \begin{cases} \sin x \ , & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 1 \ , & x > \frac{\pi}{2} \\ -1 \ , & x < -\frac{\pi}{2} \end{cases}$$

PROBLEM 22.5.6

A not very bright student, using a calculator, obtained the following incorrect result in an examination:

"$\ln x = 2$, so $x = 2.88539$ (to 5 d.p.)"

What erroneous but plausible reasoning led the student to obtain this answer?

SOLUTION (Julius Guest, East Bentleigh, Victoria)

The student may have reasoned if $\ln x = 2$ then $x = 2/\ln 2 = 2.88539$ (to 5 decimal places).

Another solution was received from Carlos Victor (Rio de Janeiro, Brazil).

PROBLEMS

Readers are invited to send in solutions (complete or partial). All solutions received by 1 July 1999 will be acknowledged in the August 1999 issue, and the best solutions will be published.

PROBLEM 23.2.1 (from Crux Mathematicorum with Mathematical Mayhem)

Suppose that a, b, c are the sides of a triangle with semi-perimeter s and area A. Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < \frac{s}{A}.$$

PROBLEM 23.2.2 (from Mathematical Mayhem)

The quartic $5x^4 - ax^3 + bx^2 + cx - d = 0$ has roots 2, 3, $\frac{N}{271}$ and $\frac{11111}{N}$. Determine the value of $(a + b + c + d)$.

PROBLEM 23.2.3 (from Mathematical Mayhem)

Show that for all positive integers a, b, c, d the polynomial $x^{4a} + x^{4b+1} + x^{4c+2} + x^{4d+3}$ is divisible by $1 + x + x^2 + x^3$.

PROBLEM 23.2.4 (from Crux Mathematicorum with Mathematical Mayhem)

Find the smallest integer in base eight for which the square root (also in base eight) has digits 10 immediately following the 'decimal' point. In base 10 the answer would be 199 with $\sqrt{199} = 14.10673\ldots$.

PROBLEM 23.2.5 (from Crux Mathematicorum with Mathematical Mayhem)

An autobiographical number is a natural number with ten digits or less in which the first digit of the number (reading from left to right) tells us how many zeros are in the number, the second digit tells you how many 1's, the third digit tells you how many 2's and so on. For example, 6,210,001,000 is autobiographical. Find the smallest autobiographical number and prove that it is the smallest.

Errata: A misprint appeared in the previous issue of Function in the solution to Problem 22.4.2 where a number 1 is shown instead of a lower case letter l. Also, the numbering of the problems was incorrect: they should be numbered as 23.1.1, 23.1.2, 23.1.3, 23.1.4 and 23.1.5. We apologise to our readers for these misprints.

* * * * *

We do not listen with the best regard to the verses of a man who is only a poet, nor to his problems if he is only an algebraist; but if a man is at once acquainted with the geometric foundation of things and with their festal splendour, his poetry is exact and his arithmetic musical.

– R W Emerson in *Society and Solitude, Chapter 7, Works and Days.*

Published by Department of Mathematics & Statistics, Monash University