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Function is a refereed mathematics journal produced by the Department of Mathematics and Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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EDITORIAL

We welcome new and old readers alike with our 22nd volume of *Function.* We hope you find in it many interesting and enjoyable items.

Did you recognise the number on the front cover? Did it mean anything to you? Find inside this issue what the brilliant Indian mathematician Ramanujan saw in it.

We have already published in *Function* several articles about sundials. This time our regular contributor Bert Bolton provides the instructions for the construction of a simple sundial, which could give you a hands-on experience on how people lived when they did not have a clock to go by.

Michael Deakin presents here a lively account of the heated discussions amongst mathematicians about a problem involving a dog running backward and forward between a boy and a girl who are walking in the same direction and at different speeds.

Malcolm Clark's article is another evidence of the usefulness of mathematics in so many aspects of our lives. He analyses the problem of fair divisions of any goods, whether it be cakes or inheritances. Read about how mathematics can be used to solve such difficult problems.

The *History of Mathematics* column presents the first of two parts of the story of how logarithms came to play an important role in the number system. In the *Computers and Computing* section you will find a computer program to experiment with the beautiful patterns representing orbits of points in the plane.

We thank the many readers who send solutions to our problems. You will find a few new problems in the *Problem Corner* for your entertainment.

Happy reading!

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THE FRONT COVER

Ramanujan's Number

When the Indian mathematical genius Srinivasa Ramanujan (1887-1920) lay seriously ill in a London hospital, he was visited by the Cambridge mathematician G H Hardy, with whom he had worked for several years.

In an attempt to make conversation, Hardy commented that the number of the taxi he had arrived in was 1729, and he added that it seemed a rather dull number. Ramanujan replied: "No, Hardy! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways."

It is easy to check that 1729 can be expressed as the sum of the cubes of two natural numbers in two different ways: $10^3 + 9^3$ and $12^3 + 1^3$. It takes a little more effort to prove that 1729 is the *smallest* such number. But what is so remarkable is that Ramanujan was so intimately acquainted with numbers that he instantly recognised the significance of the number 1729 as soon as Hardy mentioned it.

Although Hardy's formal education was deficient, his natural talent in mathematics was of the highest order.

To find more about these and many other interesting numbers we recommend you *The Book of Numbers* by J H Conway and R K Guy, 1996, Springer Verlag.

* * * * *

Archimedes was not free from the prevailing notion that geometry was degraded by being employed to produce anything useful. It was with difficulty that he was induced to stoop from speculation to practice. He was half ashamed of those inventions which were the wonder of hostile nations, and always spoke of them slightingly as mere amusements, as trifles in which a mathematician might be suffered to relax his mind after intense application to the higher parts of his science.

Macaulay in *Lord Bacon*

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A SIMPLE SUNDIAL

Bert Bolton, University of Melbourne

In one of Robert Louis Stevenson's best historical novels, *Kidnapped*, which tells a good story based on one of the most tangled and dangerous parts of Scotland's history, two characters in 1751 are on the run from the English soldiers and have to travel by night and sleep by day. The two characters are the young man David Balfour and the soldier Alan Breck. During the day they took turns for one to watch. David tells the story in Chapter 22: "Alan took the first watch and it seemed to me I had scarcely closed my eyes before I was shaken up to take the second. We had no clock to go by; and Alan stuck a sprig of heather in the ground to serve instead, so that as soon as the shadow of the bush should fall so far to the east, I might know to rouse him". Alan had made a simple sundial and because in those days the Scots spent a lot of time in the open air far from clocks, the use of such a simple sundial was probably widely known.

The principle of the sundial is to have a thin pointer which in the sun casts a shadow on a graduated scale of hours and minutes. The pointer is often called the gnomon; this word has the first letter silent and is from the Greek for 'the interpreter'. When the sundial is at a site where the latitude is λ and its pointer is inclined to the horizontal by this angle λ, then the shadow of the pointer moves round a cylindrical scale at 15° per hour. This is the principle used for most garden sundials. But the simplest sundial, and the one that was being used in Stevenson's novel *Kidnapped*, has its pointer vertical and the shadow is watched over a horizontal surface. It is easy to make; I made one in Melbourne with a piece of dowel rod about 20 mm long and diameter about 5 mm; it was glued on to a cardboard base and it projected through a piece of white paper on which the positions of the shadow were noted at times taken from a watch reading eastern standard time (EST).

You might like to check the measurements in Table 1.

The angles for corresponding times will be approximately the same for all seasons, but the length of the shadow is longest at the winter solstice, 22 June, when the sun is lowest in the sky. From the table notice that there is no symmetry of the readings about the position of noon; thus, at three hours before noon the position is 45° west and at three hours after noon the position is 47° east. This is because the longitude of the sundial in

Melbourne was about 145° east and the centre of a symmetrical pattern will occur when the shadow will have its shortest length at local noon, which is at 12.20 EST[1].

Time	Position in degrees
	West
9	45
9.30	39
10	32
10.30	24
11	16.5
11.30	8.5
Noon	0
12.30	9
1	18
1.30	26
2	33
2.30	41
3	47
4	57
	East

Table 1

There is no reason to expect that Stevenson wanted his characters in the novel to measure the position of the shadow of the piece of heather, but they were open-air persons and would have known roughly where the shadow should move to.

* * * * *

[1]In Australia, Eastern Standard Time is defined by the noon of the longitude 150° east, which is near the middle of the longitudes for towns in the eastern states of Australia and it runs very close to the town of Eden on the south eastern coast of New South Wales. The longitude in Melbourne is about 145° east, so there is a 5° difference between Melbourne and the EST longitude of 150° east. This 5° is equivalent to a 20 minutes difference in local time, and because the earth rotates from west to east, local noon at Melbourne occurs 20 minutes after noon EST.

A BIRD, TWO TRAINS, THE BOY, THE GIRL, THE DOG AND THE FREEDOM OF HUMAN ACTION

Michael A B Deakin

Let us begin with an old problem which indeed many readers may have seen.

> *Two trains are initially 30 km apart and travelling toward one another on parallel tracks. Each is doing 30 km/h. Meanwhile, a bird flying at 60 km/h flits backward and forward between them. How far will the bird have flown when the trains meet?*

There is a very simple way to solve this problem, but I will not give it. I'd rather leave this for you to work out for yourselves, if you don't already know it. However, the answer is 30 km, and this may perhaps provide a clue.

However, as well as this simple approach there is another, less efficient one involving the use of infinite series. I want to talk about this less efficient approach. Not because I like doing things "the hard way" – that's a quite wrong view of Mathematics! Mathematics is supposed to make hard questions easy. Rather, the complexity of the less apt approach illuminates aspects of the problem that don't arise in the usual (and simpler) analysis.

It is often said that any mathematician would naturally think of the hard method rather than the simple one. There is even a piece of folklore about the problem and the great mathematician John von Neumann.[1] According to this, when the problem was put to him he answered immediately and correctly. "I see", his questioner is supposed to have said, "that you knew not to sum the series". "What do you mean", von Neumann is said to have replied, "Of course I summed the series; how else could you do it?"

Well, let us set out to sum the series. The moment we embark on this course, we notice that there is no information given about the initial position

[1] 1903-1957. Von Neumann was a mathematician of the very highest accomplishment. He was one of the founders of an entire branch of Mathematics known (somewhat misleadingly) as the *Theory of Games*. He also contributed very significantly to the development of Quantum Theory and was also one of the pioneering figures in Computer Science. He had a legendary ability at mental arithmetic. This story was also told in my History of Mathematics column in *Function Vol* 19 *Part* 4. It is probably not true.

of the bird, nor in which direction it is going. In fact, as we shall see, we don't need this information. The "simple solution" does perfectly well without it.

But suppose that we do look at the problem from the more complicated point of view. Then we *will* require this information. The situation is shown in Figure 1. Now the actual numbers are unimportant; we may alter them to suit ourselves. For simplicity, I've put the initial positions of the trains at -1 and 1. (This doesn't really change anything; it simply amounts to using a different unit of length.) We can also generalise the original problem somewhat and I will do this. Let the speed of the trains be V and that of the bird be ν.

I've put the bird initially at a point x_0 to the right of the mid-way point where the trains are destined to meet. Furthermore, I've assumed that it flies to the right in the first instance. (These *are* special assumptions, but none of them has any effect on the answer. You may convince yourself of this by means of a number of arguments; if necessary, by considering all the alternatives, although there are simpler ways, one of which will become apparent as this article proceeds.)

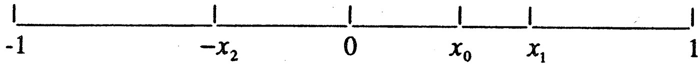

Figure 1

The bird meets the right-hand train at a point x_1, at which point it turns round and flies toward the other train, eventually meeting it at a point $-x_2$. And so on. (To avoid cluttering the diagram, I've illustrated only these two meeting-points.)

The first thing to do will be to find a formula for x_1. The bird, when it first meets the train on the right, has travelled a distance $x_1 - x_0$ at speed ν, and so has taken a time $\frac{x_1-x_0}{\nu}$. The right-hand train has travelled a distance $1 - x_1$ at speed V, and so has taken a time $\frac{1-x_1}{V}$. *And these times must be equal.* Thus

$$\frac{x_1 - x_0}{\nu} = \frac{1 - x_1}{V}.$$

Solving this equation for x_1 gives:

$$x_1 = \frac{\nu + x_0 V}{\nu + V}. \tag{1}$$

We may use the same technique to find x_2. This I leave as an exercise to the reader. The result is:

$$x_2 = \frac{(\nu - V)x_1}{\nu + V}.$$

In the same way, we may find x_3 in terms of x_2, x_4 in terms of x_3, and in general x_{n+1} in terms of x_n. In fact, we find (for $n \geq 1$):

$$x_{n+1} = \frac{(\nu - V)x_n}{\nu + V}.$$

So the successive values of x_n (other than x_0) form a geometric sequence with first term x_1 and common ratio $\frac{\nu - V}{\nu + V}$. If we form a *series* from this sequence, we find

$$\begin{aligned} S &= x_1 + x_2 + x_3 + \ldots + x_n + \ldots = \\ & x_1 + \frac{\nu - V}{\nu + V}x_1 + \frac{(\nu - V)^2}{(\nu + V)^2}x_1 + \ldots + \frac{(\nu - V)^{n-1}}{(\nu + V)^{n-1}}x_1 + \ldots . \end{aligned}$$

Using the formula for the sum of a geometric series, we find $S = \frac{(\nu+V)x_1}{2V}$. In view of Equation (1), we can write this (after a little work) as:

$$S = x_1 + x_2 + x_3 + \ldots = \frac{\nu}{2V} + \frac{x_0}{2}. \tag{2}$$

Now consider the total distance the bird flies. This will be

$$(x_1 - x_0) + (x_1 + x_2) + (x_2 + x_3) + \ldots,$$

in other words, $-x_0 + 2S$. Substitute now from Equation (2) to find:

$$\textit{Total distance} = -x_0 + 2\left(\frac{\nu}{2V} + \frac{x_0}{2}\right) = \frac{\nu}{V}, \tag{3}$$

which is independent of x_0. We may also at this point check the answer. Remember that the speed of the bird (ν) in the initial statement of the problem was *exactly twice* that of the trains (V), so the total distance flown must be twice the chosen unit of distance (here 15 km) so that we have recovered the answer I quoted earlier.

Now I *did* say that my reason for "doing this problem the hard way" was not out of some perverse fascination. Rather, I want to show its connection with other such problems, to share with you some anecdotes, partly historical and partly personal, and to show some surprising further considerations.

Attentive readers may have noticed a special feature of the original problem, one preserved even in my generalisation. But I assumed that the trains were travelling toward one another *at the same speed.* It is perfectly possible to relax this assumption and have the left-hand train travelling to the right at speed U, say, while the right-hand train travels to the left at speed V. As an exercise, can you solve this problem? Having done so, can you then find a simpler argument to say why your answer is correct?

In fact, the trains need not be coming towards one another – if they aren't, there is only a simple change of sign involved. This leads us to another famous (or should I say infamous?) problem: the boy, the girl and the dog.

This was first proposed by A K Austin of the University of Sheffield. The US journal *Mathematics Magazine*[2] includes a section of "Quickie Problems", apparently difficult questions, with wonderfully simple answers, and Austin posed one of them. *Function* ran Austin's problem but with slightly different figures (we metricated it!) and it appeared as our Problem 5.1.2 back in 1981. Here is how it went in our version:

> *A boy, a girl and a dog go for a walk down the road, setting out together. The boy walks at a brisk 8 kph, while the girl strolls at a leisurely 5 kph. The dog frisks backward and forward between them at 16 kph. After one hour, where is the dog, and in what direction is it facing?*

Austin's solution was that the dog might be anywhere between the two and facing in either direction. He justified this answer by "letting all three reverse their motion until they came together at the starting point at the starting time". The famous *Scientific American* columnist Martin Gardner ran this problem[3] and accepted Austin's answer. At that stage he could not have been aware that it had attracted severe criticism in a subsequent issue of *Mathematics Magazine*.[4]

Four different criticisms were published. Their details differed, but in essence all made the same point: whereas the telescoping motion (when the film is run in reverse, so to speak) is well-defined, the expanding motion is not. How would the dog get started?

[2]Published by the Mathematical Association of America, and somewhat similar to *Function*. However, where *Function* is directed to students in Years 11-12, *Mathematics Magazine* addresses Years 13+. The problem appears on p 56 of *Volume* 44 (1971).

[3]Posed in July and answered in August, 1971.

[4]Pp 238-239 of *Volume* 44 (1971). These did not see print till after Gardner's second discussion appeared.

As far as I know, Austin never defended his (really very elegant) solution against these attacks. However, Gardner did. His December 1971 column consisted largely of extensive quotes from a letter from the famous philosopher of science, Wesley Salmon. The detail of Salmon's analysis I will spare you; it was highly technical stuff whose general tenor may be gauged from its title: *The Paradoxes of Zeno as "Supertasks"*.[5]

My own justification of Austin's solution was much simpler. It appeared in *Function Vol* 5 *Part* 4, and proceeded by allowing the dog to cheat ever so slightly. For a brief interval of time, the dog adopts a speed somewhere (anywhere) between 5 and 8 kph. After this very short time has elapsed, it stops cheating and speeds up to 16 kph, "breaking" towards either the boy or the girl and thereafter following its programme to the letter. It makes no matter how short the "cheating" period may be; less than a microsecond if you like, or less than a nanosecond, whatever. Just as long as it exists.

I sent my solution to *Mathematics Magazine* who didn't like it and didn't publish it, though *Function* did.[6] However, this was not the end of the matter. *Function* has a Dutch counterpart, *Pythagoras*, and they have an exchange agreement with us. They ran the story under the title (in English translation) "Where is Zoef?" – Zoef being a favourite name for a dog in Dutch.

I did publish my version in a more "up-market" journal also, more "up-market" indeed than *Mathematics Magazine*. The underlying point is that there *is* an important philosophical principle at stake; much more important than Salmon realised, certainly much closer to human significance than "the paradoxes of Zeno as supertasks", and also connected with an interesting if little-known chapter in the history of scientific thought.

Early last century, the laws of (Newtonian) Physics had been explored and developed to such a point that it seemed to be possible in principle to deduce the total history of the entire universe from a knowledge of its precise

[5]For a discussion of the paradoxes of Zeno, see my History of Mathematics column in *Function Vol* 14 *Part* 3. Salmon in 1971 had just finished editing a book on the paradoxes of Zeno, and some of the pieces in it make great play with the "supertask" concept. A "supertask" is defined as one requiring the fulfilment of infinitely many smaller tasks. My own solution of the problem (see the next paragraph) *avoids* this notion. The dog reverses its direction only finitely many times, and this is true no matter how short the period of "cheating" might be. This is very much the approach that mathematicians use in their discussion of limits.

[6]Possibly because I was the editor at the time!

state at any one given moment. In 1814, the mathematician Laplace[7] put it thus:

> *An intelligence which, at one given instant, knew all the forces by which the natural world is moved and the position of each of its component parts, if as well it had the capacity to submit all these data to Mathematical analysis, would encompass in the same formula the movements of the largest bodies in the universe and those of the lightest atom; nothing would be uncertain for it*[8]*, and the future, as also the past, would be present to its eyes.*

In other words, he saw the entire universe as like a piece of clockwork whose every movement is absolutely and forever predetermined. But if this is so, then there is clearly no scope for any freedom at all; in particular, our belief that we humans are free to choose what or what not to do is illusory.

Laplace saw this as a consequence of his position, and indeed argued for it:

> *The [freedom of human action] is a figment of the mind, which ... convinces itself that it has acted of itself and without constraint.*

This extreme position has been espoused by subsequent thinkers as well, but most of us would think that some flaw must lie in the argument if it results in something so manifestly contrary to human experience. Last century and early in our own, there was much discussion of the Laplace Demon (as Laplace's "intelligence" came to be called). This died down with the coming of the Uncertainty Principle in Quantum Mechanics.[9]

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[7]Pierre-Simon, marquis de la Place (1749-1827), one of the greatest mathematicians of all time.

[8]I digress to tell an interesting, if somewhat irrelevant, story. *Function*, as a matter of policy, and I, as a matter of courtesy, try where possible to use gender-neutral language. In French on the other hand, every noun comes already equipped with a gender and the learner must memorise in each case whether the thing described is masculine or feminine. In French, *intelligence* (the actual word is the same in both languages) is feminine. So what Laplace actually said was "nothing would be uncertain for *her*". I once read a translation in which the word "her" was rendered in English as "him". This really is dominant-gender stuff! Grammatically correct in English (as well as being gender-neutral) is "it". There is no exact equivalent in French to this English word.

[9]I have seen a quote attributed to Max Planck, one of the founders of the Quantum Theory, to the effect that now that the uncertainty principle was enunciated he was happy, because it was suddenly clear that "even the electron" had free will. To my regret, I am

It makes a big difference to our understanding of ethics and human responsibility, of course, if we believe that all human action is predetermined. "The laws of Physics made me do it!" I rather doubt that many judges would accept this as a defence. So last century there was a reaction to Laplace's extreme view. Three names stand out, and it is interesting, in the light of today's understanding, what it was they said. They were Maxwell[10], Boussinesq[11] and Saint-Venant.[12]

What they said then takes on a new significance in the light of modern developments. All three, one way or another, challenged Laplace's assumption that the laws of nature are *perfect* descriptions of the world. Any small deviation, *no matter how small*, may make it impossible to use them predictively.

Just as the dog "cheats", but only infinitesimally, so perhaps nature may "cheat". The dog's "cheating" may be so subtle that we will never catch it out. The period of the "cheating" can be made sufficiently small as to evade the accuracy of any measuring device, be it every so good. They saw nature as obeying what we refer to as laws but with some room for their being ever so slightly inexact. What is in question is the matter of *stability*. A pencil may stand on its point *if it is exactly vertical*. But in real life it can *never* be *exactly* vertical. So it falls over. But we are quite unable to say *in which direction it will fall*.

Just as we can never detect the dog's "cheating", which affects its subsequent position, so we cannot detect those minute deviations from the vertical that determine the direction of fall. And so also, according to Maxwell and the others, we cannot detect the minute influences of the human will, except by their subsequent effects. Nowadays we see such situations routinely studied in the theory of chaotic systems, but that is another story.[13]

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now no longer able to trace this quote. However, later on the Nobel Prize winner J C Eccles attempted a model of the brain in which quantum effects formed the basis of freedom of human action. (See Chapter 8 of his *The Neurophysiological Basis of Mind.*) Nowadays we do not take this route.

[10] 1831-1879. A Scottish physicist, and of the very first rank. He formulated the laws of electromagnetic theory, which are now named after him.

[11] 1842-1929. A French scientist, remembered mostly for his work on fluid flow. The Boussinesq Number (to do with wave flow in open channels) preserves his name.

[12] 1979-1886. Another French scientist, who worked on the theory of elastic bodies (i.e. deformable solids). In this area of theory, Saint-Venant's Principle commemorates him.

[13] See *Function Vol 2 Part 5*.

DIVIDING THE CAKE

Malcolm Clark, Monash University

The problem of fair division, whether it be dividing up a cake, or distributing an inheritance amongst a number of heirs, is as old as the hills. Only recently have such problems been tackled by mathematicians. A recent book, *Fair Division: From cake-cutting to dispute resolution* by Steven J Brams and Alan D Taylor (Cambridge University Press, 1996), offers a new approach to the problem, by emphasising various criteria defining different notions of fairness, and providing algorithms for achieving a fair division of goods.

Dividing a cake between two people is often done by allowing one person to cut the cake and the other to choose which piece he or she wants. The cake need not necessarily be homogeneous, e.g. one part of it could have cherries on top, and another part have strawberries. This procedure, known as "I cut, you choose", gives the first person an incentive to be reasonably equitable in cutting the cake. But the resulting division of the cake could still result in one person envying the other. This could happen, for example, if both people preferred strawberries to cherries on top, and the initial cut had one piece with only strawberries on top.

This procedure is not very satisfactory when it comes to dividing up an estate: some or all of the items may be indivisible, and the heirs may value the items differently. For example, suppose that two people, Bob and Carol, are left a collection of antiques by an aunt: china figures, crystal glasses, and chairs. Bob is very keen on the crystal glasses, and quite likes the chairs, but is not very interested in the china figures. In contrast, Carol very much wants the chairs, is quite keen on the glasses, but is similarly indifferent to the china figures. What is the best way of dividing the collection between Bob and Carol?

In general, a desirable allocation of the goods or items should be:

(1) efficient: there is no allocation which is better for both Bob and Carol;

(2) equitable: Bob's valuation of his allocation is the same as Carol's valuation of her allocation;

(3) envy-free: neither person would trade his or her allocation for that of the other.

In addition, the procedure for determining the allocation should be able to deal with items which cannot be divided.

It is clear that when there are two "players", conditions (2) and (3) are equivalent. But it can be shown that this is not the case with more than two "players".

Returning to Bob and Carol, how do we allow for their different preferences, and is it possible to make an allocation which satisfies all three criteria?

Firstly, we ask Bob and Carol to specify how much they value each item relative to the other items, by allocating points out of 100. Suppose in fact that their allocated points are as shown in the following table. These points confirm, for example, that Bob values the crystal glasses very highly, assigning them 67 out of his 100 points.

Items	**Bob**	**Carol**
Crystal glasses	67	34
Chairs	27	61
China figures	6	5
	100	100

Brams and Taylor offer two algorithms for distributing a number of goods (items) between two people, namely Proportional Allocation and the Adjusted Winner procedure.

Proportional Allocation:

Suppose that Bob's valuations for goods $G_1, G_2, \ldots, G_k$ are $x_1, x_2, \ldots, x_k$ respectively, while Carol's valuations are $y_1, y_2, \ldots, y_k$.

Each of the k goods $G_1, G_2, \ldots, G_k$ are divided between Bob and Carol in proportion to their valuations. For example, Bob receives a proportion $x_1/(x_1 + y_1)$ of good G_1, $x_2/(x_2 + y_2)$ of G_2 and so on.

Applying this procedure to the collection of antiques, Bob is awarded $67/101 = 0.6634$ of the crystal glasses, $27/88$ of the chairs, and $6/11$ of the china figures. Since Bob valued all of the crystal glasses at 67 points, his allocation of 0.6634 of the crystal glasses corresponds to 44.45 of his valuation points. Similarly, his allocation of the chairs corresponds to 8.28 points, and the china figures, 3.27 points. Hence Bob's allocation gives him a total of 56.00 of his valuation points. A similar calculation shows that

Carol's allocation is also valued at 56.00 points. (Notice that this procedure is better than the naive one of splitting each item 50:50, in that each person receives more than half of their 100 points).

In this case, the allocation is equitable, and since there are only two people involved, it is also envy-free. This is always the case with Proportional Allocation, as is proved in the Appendix. However, the allocation is not necessarily efficient. The Adjusted Winner procedure described below leads to an equitable envy-free allocation which is better for *both* Bob and Carol.

Another disadvantage of this procedure is that it requires every item to be divided in varying proportions. In some cases, this may be feasible. For example, if there happened to be, say, 50 crystal glasses, Bob and Carol could agree that 33 of these would go to Bob and 17 to Carol. This would give an allocation in approximately the correct ratio. Alternatively, the collection of crystal glasses could be sold, and the proceeds sub-divided between Bob and Carol in the ratio 67:34. This is unsatisfactory, because (a) the points given by Bob and Carol do not necessarily represent monetary value, and (b) some items may not be able to be sold, e.g. personal diaries.

Adjusted Winner Procedure:

This procedure is preferable to Proportional Allocation, because it is efficient as well as equitable, and requires only one item to be divided. It goes like this.

1. Let X be the sum of Bob's points for all goods that he values more than Carol, and similarly let Y be the sum of Carol's points for all goods that she values more than Bob. We suppose that $X \geq Y$; if this is not the case, we interchange the roles of Bob and Carol. (In the example below, $X = 73, Y = 61$.)

2. Assign the goods so that Bob initially gets all the goods where $x_i \geq y_i$, and Carol gets the rest.

3. List the goods in an order $G_1, G_2, G_3, \ldots,$ so that:

   (a) Bob values goods $G_1, G_2, \ldots, G_r$ at least as much as Carol.

   (b) Carol values goods $G_{r+1}, \ldots, G_k$ more than Bob does.

   (c) $1 \leq x_1/y_1 \leq x_2/y_2 \leq \ldots \leq x_r/y_r$.

   Since $X \geq Y$, Bob enjoys an advantage over Carol, and is helped additionally by being assigned initially all goods that both people value equally.

4. Transfer from Bob to Carol as much of G_1 as is necessary to ensure that the allocation is equitable, i.e. the point totals are the same. If equal totals are not achieved, even with all of G_1 transferred, we next transfer G_2, G_3, etc. (*in that order*) from Bob to Carol. (It is the order given by 3 (c), starting with the smallest ratio, which ensures efficiency.)

To apply this algorithm to the collection of antiques, we list the items in the order specified in step 3, and highlight in bold the larger of the two valuations for each item.

Item	**Bob** (x_i)	**Carol** (y_i)	x_i/y_i
G_1 : China figures	**6**	5	1.20
G_2 : Crystal glasses	**67**	34	1.97
G_3 : Chairs	27	**61**	

Here $r = 2$, $X = 73$, $Y = 61$, and Bob is initially allocated all of G_1 and G_2. This is clearly not equitable, since Bob's total points are 73 and Carol's 61. Applying step 4 of the algorithm, we transfer *all* of G_1 from Bob to Carol. Bob's total points are reduced to 67 (since he valued G_1 at 6 points), while Carol's total becomes $61 + 5 = 66$ points (since she valued G_1 at 5 points).

The allocation is still not (quite) equitable, and we must transfer a proportion α of G_2 from Bob to Carol, so that the point totals are equal. Hence α must satisfy the equation:

$$67(1 - \alpha) = 5 + 34\alpha + 61,$$

yielding $\alpha = 1/101 \simeq 0.01$.

Hence Carol ends up with all the chairs and china figures, and about 1% of the crystal glasses. Bob is allocated 99% of the crystal glasses. Both Bob and Carol end up with a total of 66.3 of their 100 points, an 18% improvement over the Proportional Allocation algorithm, and a 33% improvement over a simple 50:50 split on each item.

The order in which the transfers are made (given by 3(c)) is important. You may verify that if the first transfer had been part of G_2 rather than G_1, both Bob and Carol would have fared worse, achieving only 65.0 of their points.

The proof that the Adjusted Winner procedure leads to an efficient allocation depends on showing that there can be no better allocation for both

Bob and Carol. This is nontrivial and requires an extended mathematical argument (see Brams and Taylor, pp. 85-88, for details). There are two things that make the argument work: (1) we start with an efficient distribution (giving each person all the goods he or she values most) and (2) we adjust for equitability in a prescribed order, starting with the smallest ratio x_i/y_i.

Equitability is built in to the algorithm by construction, and since only two people are involved, this implies that the allocation is envy-free as well.

The Adjusted Winner procedure usually requires *one* item (in this case, the crystal glasses) to be divided between the two people. This is preferable to Proportional Allocation or the naive 50:50 rule, where *every* item has to be divided. If this last item is something such as a picture which cannot be divided, then it could be sold, and the proceeds divided between Bob and Carol in the correct ratio. Alternatively, such an item could be shared on a time basis: Bob has it for a certain percentage of the year, and Carol has it for the remainder.

There is much fascinating material in Brams and Taylor's book. For example, they apply their Adjusted Winner algorithm to the negotiations between the U.S. and Panama over the use of the Panama Canal. They also consider what happens when the two "players" do not tell the truth, i.e. the valuation points announced by Bob and Carol are not their true valuations. They conclude that Proportional Allocation encourages both Bob and Carol to be truthful, but the Adjusted Winner algorithm is susceptible to manipulation by announcing false valuation points.

Nevertheless, the Adjusted Winner approach produces an allocation to two people which is efficient, equitable, envy-free, and requires only one item to be sub-divided. It's a different story when the items have to be divided amongst three or more people. It turns out that no such algorithm is possible. To find out what to do in such cases, read the book!

Appendix

Proof that Proportional Allocation is equitable and envy-free

We may assume that the valuation points $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$ are proportions rather than percentages. Hence

$$0 \leq x_i \leq 1, \quad 0 \leq y_i \leq 1 \tag{1}$$

$$x_1 + x_2 + \ldots + x_k = 1 \tag{2}$$

$$y_1 + y_2 + \ldots + y_k = 1 \tag{3}$$

Let P_B and P_C denote Bob's and Carol's total points under Proportional Allocation. These are weighted sums of their valuation points for each item multiplied by their fractional allocations. Hence

$$P_B = \frac{x_1^2}{x_1 + y_1} + \frac{x_2^2}{x_2 + y_2} + \ldots + \frac{x_k^2}{x_k + y_k}$$
$$P_C = \frac{y_1^2}{x_1 + y_1} + \frac{y_2^2}{x_2 + y_2} + \ldots + \frac{y_k^2}{x_k + y_k}$$

To see that $P_B = P_C$, notice that

$$\begin{aligned} P_B - P_C &= \frac{x_1^2 - y_1^2}{x_1 + y_1} + \frac{x_2^2 - y_2^2}{x_2 + y_2} + \ldots + \frac{x_k^2 - y_k^2}{x_k + y_k} \\ &= \frac{(x_1 - y_1)(x_1 + y_1)}{x_1 + y_1} + \frac{(x_2 - y_2)(x_2 + y_2)}{x_2 + y_2} + \ldots + \frac{(x_k - y_k)(x_k + y_k)}{x_k + y_k} \\ &= (x_1 - y_1) + (x_2 - y_2) + \ldots + (x_k - y_k) \\ &= (x_1 + x_2 + \ldots + x_k) - (y_1 + y_2 + \ldots + y_k) = 0 \end{aligned}$$

applying (2) and (3). Hence $P_B = P_C$.

To show that Proportional Allocation is envy-free, we must show that

$$P_B \geq \frac{1}{2} \text{ and } P_C \geq \frac{1}{2},$$

i.e. both Bob and Carol attain at least one-half their total points.

Since $P_B = P_C$, it suffices to show that

$$P_B + P_C = \frac{x_1^2 + y_1^2}{x_1 + y_1} + \frac{x_2^2 + y_2^2}{x_2 + y_2} + \ldots + \frac{x_k^2 + y_k^2}{x_k + y_k} \geq 1.$$

It is sufficient to show that, for each i,

$$\frac{x_i^2 + y_i^2}{x_i + y_i} \geq (x_i + y_i)/2 \tag{4}$$

(This is because if we sum inequality (4) over $i = 1, 2, \ldots, k$, the left-side becomes $P_B + P_C$, while the right side equals 1, by (2) and (3).)

We prove (4) by contradiction, i.e. by showing that if (4) is not true, this leads to a contradiction.

Suppose (4) is not true, i.e.,

$$\frac{x_i^2 + y_i^2}{x_i + y_i} < (x_i + y_i)/2$$

This implies that

$$2x_i^2 + 2y_i^2 < x_i^2 + 2x_i y_i + y_i^2,$$

or, after re-arrangement,

$$x_i^2 - 2x_i y_i + y_i^2 < 0.$$

This is equivalent to the statement

$$(x_i - y_i)^2 < 0,$$

which is a contradiction. Hence (4) must be true, and both P_B and P_C must be greater than or equal to $\frac{1}{2}$.

* * * * *

There is a delta for every epsilon (Calypso)

There's a delta for every epsilon,
It's a fact that you can always count upon.
There's a delta for every epsilon
    And now and again,
    There's also an N.

But one condition I must give:
The epsilon must be positive.
A lonely life all the others live,
    In no theorem
    A delta for them.
How sad, how cruel, how tragic,
How pitiful, and other adjec-

Tives that I might mention.
The matter merits our attention.
If an epsilon is a hero,
Just because it is greater than zero,
It must be mighty discouragin'
To lie to the left of the origin.

This rank discrimination is not for us,
We must fight for an enlightened calculus,
Where epsilons all, both minus and plus,
    Have deltas
    To call their own.

– Tom Lehrer
Amer. Math. Monthly **81**, June-July 1974

* * * * *

HISTORY OF MATHEMATICS

Michael A B Deakin

The Story of Logarithms – Part 1

Let us begin with something very basic: the operations of simple arithmetic. The most elementary of these is addition, and when we first encounter this, it is an operation on the natural (or counting) numbers[1]: $1, 2, 3, 4, 5, 6, \ldots$. Very early on in our school careers, we learn to *add* two natural numbers, e.g. $2 + 3 = 5$. The mechanics of this operation need not occupy us here; rather I want to point to two abstract (mathematical) aspects of the matter.

The first is that the "answer" to an addition problem (a "sum") is always possible and if we add two natural numbers, we always get another natural number as the result. Probably we never learn this property explicitly in the early years of primary school. It seems obvious. Mathematicians, when they speak technically of this property, say

The natural numbers are closed under addition.

The other thing to notice is that it does not matter what order we employ when we add the two numbers: $2+3 = 3+2$, and the analogous property holds whatever numbers we add. Mathematicians, when they speak technically of this property, say

Addition is commutative.

We can put all this together by means of a simple statement:

If a and b are natural numbers, then

$$a + b = b + a = c, \tag{1}$$

where c is a third natural number.

[1]I give the most natural of two possible versions; some people include 0 as a "natural number". This has certain mathematical advantages although *psychologically* it seems less obvious!

But now take equation (1) and examine it differently. It expresses a relationship between the three natural numbers a, b and c. So far we have been concentrating on the scenario in which a and b are given and the question is to determine c. But what if, say, a and c were given and our problem was to determine b? Or, if b and c were the givens and a the unknown?

The first thing to notice is that these two new problems are really the same basic problem; this is because the order of addition is unimportant. If we ask $2+? = 5$ or if instead we put it as $? + 2 = 5$, it makes no difference. This sort of problem is referred to under the name of *subtraction*, and of course we are all familiar with it.

But now notice that, *with the natural numbers alone*, not all subtraction sums are possible. $? + 2 = 5$ has the simple answer 3, but $? + 5 = 2$ has no answer as long as we remain within the system of natural numbers.

The natural numbers are not closed under subtraction.

In order to allow us to proceed, we need to expand our number system. The new expanded system is called the system of *integers*. These come in three "flavours": there are the *positive integers*, which in essence are the natural numbers we have just been considering, there are the *negative integers*, which relate to the positive ones in a relatively straightforward way, and finally there is a special integer called *zero*, which is in a class all of its own[2].

The negative integers relate in a one-to-one way to the positive in that every positive integer like 3 (which, strictly speaking, we should now call $+3$) has an accompanying negative integer called (in this case) -3. Where $? + 2 = 5$ has the answer $+3$, $? + 5 = 2$ has the answer -3. 0 is the special integer[3] that answers questions of the sort $? + 2 = 2$.

When we subtract two numbers we are in essence reversing an addition process. Subtraction is said to be the operation *inverse to addition.*

As well as using inverse processes, we may also imagine the repetition of the basic process. And here we need yet a further insight. It does not matter at all if in adding (say) $2 + 3 + 4$ we first add the 2 and the 3 and then add

[2]The details of this explanation would be different in detail had we included 0 as a natural number.

[3]And note that I am here glossing over an important question: Why should $a - a$ always give the same answer, irrespective of the value of a? It does, but this is a theorem that, strictly speaking, requires proof!

the 4 to the result, or if we do the $3+4$ bit first, and so on. Mathematicians, speaking technically, say that *addition of natural numbers is associative.*

This means that if we form a string of (say) 3 separate 2s: $2+2+2$, this will have an unambiguous answer. We write it as 2×3, and, as we know, the answer is 6. We early on learn to *multiply*, as this new operation is called, any two natural numbers, and indeed it is not much harder (using rules like "minus by minus equals plus", etc) to multiply any two integers.

And indeed things are much as they were for addition of the natural numbers. We even have a *commutative law of multiplication*, which says that the order in which we multiply numbers is not important. This is actually rather surprising, because 2×3 means $2+2+2$ whereas 3×2 means $3+3$.

Now suppose we look at the inverse operation for multiplication. We are asking questions like $? \times 3 = 6$, or $3 \times ? = 6$. And once again, this time because *multiplication* is commutative, these two questions are really the same question. The inverse operation for multiplication is *division*, and once more we need to extend the number system to accommodate it. Questions like $3 \times ? = 6$ can be answered within the system of integers, but questions like $3 \times ? = 7$ cannot.

We need to expand even further the basic number system to include *fractions*, or *rational numbers* as they are often called. So this last question can be answered by reference to the number $\frac{7}{3}$. The integers are now replaced by rational numbers of the special form in which the lower line is always 1. This new system is closed under addition, subtraction, multiplication and division, with just one exception:

We may not divide by 0.

This is for two reasons. Division provides the answer to questions like $3 \times ? = 7$, and the answer is provided by the process of dividing by (in this instance) 3. Division by 0 would be the attempt to answer a question like $0 \times ? = 7$, and it is obvious that no answer is possible, since whatever number we use to multiply by, the right-hand side will always be 0.

There is however one exception to the rule I have just outlined. Suppose the question were $0 \times ? = 0$. This time, it is not a case of there being no solution. Rather we have an embarrassment of solutions: *any old number will do!* This case is said to be *indeterminate.*

However, with this one exception of the embargo on division by zero, we have a system that allows all the operations of elementary arithmetic and all the natural properties we ascribe to numbers under these operations.

Now consider the effect of repeating multiplication. Suppose we have $2 \times 2 \times 2$. Again, as with addition, multiplication is associative and so an unambiguous meaning may be attached to the expression and the answer is 8. We used to write $2^3 = 8$, which is (and always was) a wretched notation. Nowadays, under pressure from the requirements of the computer, we have instead $2 \wedge 3 = 8$, and this is much better. This new operation ($\wedge$) is called *exponentiation* (or earlier *involution*).

But now notice a new complication.

Exponentiation is **not** *commutative.*

To see this we need only consider the case of $3 \wedge 2$, which is 3×3, which is 9, not 8.

It thus follows that when we seek an operation inverse to exponentiation, we will find *not one such operation but two.* We may ask the question $? \wedge 3 = 8$, or else we could ask $2 \wedge ? = 8$, and this time these are different questions.

Questions of the first type give rise to an operation that used to be called evolution, which is now misleading (it has nothing to do with Darwin's theory). Another name was *root extraction* (which sounds like some awful dental procedure!). Actually there is no need for such a new name because it turns out that when we try to apply this operation to the various new types of number we introduced (negative and rational numbers), we see that all that is involved is exponentiation itself.

Suppose we tried to make sense of the expression $8 \wedge \frac{1}{3}$. This turns out to be the answer to our first question and so this number is 2 (at least on what has been said so far).

However, we need two further extensions of the number system to accommodate all these questions of exponentiation.

In the first place, the rational numbers alone cannot provide meanings for expressions like $2 \wedge \frac{1}{2}$. Such questions need the introduction of a further type of number, called an *irrational number.* Between them the rational and the irrational number make up the system of *real numbers.*

But this system is not enough either. There are other questions like $(-1) \wedge \frac{1}{2} = ?$ that cannot be answered within the real number system. For that we need to introduce yet another extension of the number system and this is called the *system of complex numbers.*

Things do seem to be getting away from us, don't they?

However, we should not despair. This is as complicated as things get. All rather unexpected, as we have yet to consider the other type of inverse operation: questions of the type $2 \wedge ? = 8$.

This operation is termed *logarithmation*, and the answer is usually written $\log_2 8$, and pronounced "log of 8 to base 2". This is a poor notation, but so far no standard alternative has achieved acceptance. In this article I will write (e.g.) $8 \ddagger 2$, but this is sheer idiosyncrasy on my part. No one else has ever done this and probably no one else ever will.

However, the challenge is to take all the numbers of the complex number system and to make sense of expressions like $a \ddagger b$ for (if possible) all complex numbers a and b. For the most part, this can be done, but as with division there are a few exceptions. For example, we may not use $b = 0$. (Can you see why?) Nor can we have $a = 0$. (Again, can you see why?)[4]. Nor can we have $b = 1$. (Once again, can you say why?)

The full story is very complicated and lies outside the scope of *Function*, but it needs to be said that, after these complications are fully explored, we reach the end of the story. We do not need to consider the effect of repeated exponentiation (although in a very limited sense, this can be done). There is a simple reason for this:

Exponentiation is not associative.

For example:

$$(2 \wedge 3) \wedge 4 = 4096,$$

but

$$2 \wedge (3 \wedge 4) = 2\ 417\ 851\ 639\ 229\ 258\ 349\ 412\ 352.$$

Quite a difference!

Even if the various numbers involved in the expressions are all the same, the answers are not. For example:

[4]And it could have been said earlier that there are other restrictions applying at earlier stages of the story. For example, 0^0 is indeterminate.

$$(3 \wedge 3) \wedge 3 = 19683,$$

but

$$3 \wedge (3 \wedge 3) = 7\,625\,597\,484\,987.$$

This means that we reach a "natural end" to the processes we have been considering, and the story ends with the complex numbers. However from now on, I will restrict consideration to the case of real logarithms of real numbers.

We are concerned to find numbers that answer questions of the form: $b\wedge? = a$. That is to say, we wish to ascribe meaning to expressions of the form $a \ddagger b$. To make sure that we remain within the system of *real numbers*, we restrict a to be a positive real number and the same for b. In fact, we may take b to be greater than 1. (As an exercise, see what happens if $0 < b < 1$.)

If we write $c = a \ddagger b$, we speak of c as being the logarithm of the number a to the base b. Readers of *Function* should be familiar with the laws of logarithms, and in particular the *change of base formula*. You will probably have seen it in the form:

$$\log_B a = \frac{\log_b a}{\log_b B}$$

but I rewrite it here in my idiosyncratic notation in order to emphasise that we are here concerned with basic properties of the number system and its standard operations. We have:

$$a \ddagger B = (a \ddagger b)/(B \ddagger b). \tag{2}$$

The thrust of this is that we may choose some standard base b and do all our work in that base. If for whatever reason, we need later to work in some other base B, then all we need do is divide the numbers involved by a standard factor $B \ddagger b$.

In practice, only three standard bases are used today. These are 10, 2 and e. The base 10 is used because it is the *base* (in another sense!) of our number-system. The base 2 is employed because of its relation with binary scales, the simplest possible available and the basis of much computer theory.

The third choice, e, is perhaps the most puzzling to the beginner. e is an irrational number whose approximate value is

$$2.718\,281\,828\,459\,045\,235\ldots$$

Not the most obvious choice, you would think. However, we do not land on this number out of mere caprice or contrariness, but rather because this choice is in very large measure forced on us.

In fact it is *computationally* easier to use base e than it is in many instances to use apparently simpler choices like 10 or 2.

Logarithms to base 10 were for centuries used to simplify difficult computations, and this was still very much the case when I went to school. The use of logarithms to make calculation simpler occupied much of the Year 10 syllabus when I was in my teens. Nowadays, we do not need this; our little pocket calculators have made life much simpler.

Nonetheless, logarithms remain vitally important; they are a fundamental part of the number system. In my next column, I shall explore how we came to learn of logarithms, their properties and their uses.

* * * * *

> On the day of Cromwell's death, when Newton was sixteen, a great storm raged all over England. He used to say, in his old age, that on that day he made his first purely scientific experiment. To ascertain the force of the wind, he first jumped with the wind and then against it; and, by comparing these distances with the extent of his own jump on a calm day, he was enabled to compute the force of the storm. When the wind blew thereafter, he used to say it was so many feet strong.

– Parton James, *Sir Isaac Newton*

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COMPUTERS AND COMPUTING

Attractive Orbits

Cristina Varsavsky

Fractals are very popular mathematical objects; their beauty captures mathematicians and non-mathematicians alike. In *Function* we have looked at fractals from many different perspectives, and have included computer programs for our readers to discover and experience their beauty by themselves[1]

We discussed the Mandelbrot set in *Function, Vol* 20 *Part* 2, in which we included a program to generate a colourful map of the complex plane. Some enthusiastic readers sent us their pieces of computer art stemming from that article. The Mandelbrot set is not just a pretty picture: it actually displays the intricate behaviour of complex numbers under the very simple iteration

$$z_{n+1} = z_n^2 + c \tag{1}$$

where the square is calculated using the definition of the product of complex numbers. With the starting number $z_0 = 0$, and formula (1), a sequence of complex numbers is produced: c, c^2+c, $(c^2+c)^2+c$, and so on. This sequence is called the *orbit* of c. Some orbits will escape towards infinity, others will remain bounded; the set of all *prisoners* is known as the Mandelbrot set. The coloured map of the complex plane is produced by counting, for each complex number c, the number of iterations needed for the orbit to escape a particular circle.

In this article we will take a different approach: we will focus on individual orbits and plot them on the screen. Each complex number corresponds to a point on the plane, so we can think of transformation (1) as a transformation of points of the plane. Using formula (1) and the definition of the product of complex numbers we can write this transformation in coordinate form:

$$\begin{aligned} x_{n+1} &= x_n^2 - y_n^2 + a \\ y_{n+1} &= 2x_n y_n + b \end{aligned} \tag{2}$$

where a and b are the coordinates of the point corresponding to the complex number c. The point (x_n, y_n) is transformed into the point (x_{n+1}, y_{n+1}).

[1]See for example *Function, Vol* 18 *Parts* 1, 2, 4, and 5.

In general, we can produce orbits using any pair of functions of two variables $f(x,y)$ and $g(x,y)$:

$$\begin{aligned} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{aligned} \tag{3}$$

The choice of the functions f and g is limitless and some very interesting orbits can be produced. As a general rule, the functions used should not produce *escapees*, but bounded orbits.

Very simple examples are provided by linear transformations, which we already discussed in *Function, Vol* 18 *Part* 5

$$\begin{aligned} x_{n+1} &= \cos\alpha\, x_n - \sin\alpha\, y_n \\ y_{n+1} &= \sin\alpha\, x_n + \cos\alpha\, y_n \end{aligned} \tag{4}$$

The orbit of any point will be a circle containing the point and will depend on the chosen angle α: if $\alpha = 2\pi/n$ the orbit will cycle over a fixed number of points, otherwise the orbit will (eventually) cover the whole circle.

A computer program to produce an orbit is very simple; if we use Quick-Basic as the programming language, we only need to define the window size, the iteration, and choose a starting point (x0,y0):

```
SCREEN 9 : WINDOW (-20,-15) - (20,15)
x = x0 : y = y0
    FOR n = 0 TO iterations
        PSET (x,y)
        oldx = x : oldy = y
        x = f : y = g
    NEXT n
```

Where f and g are the functions defined in terms of oldx and oldy, and the variable iterations counts the number of points to be plotted.

Inspired by the great works of the physicists and mathematicians Mira and Gumowski, I produced a few pretty pictures using the function

$$f(x,y) = ax + by + \frac{2a|x|}{1+|x|} \tag{5}$$

and modifying the corresponding lines in the computer program as follows:

```
x = a oldx + b oldy + 2 a abs(oldx) / (1+abs(oldy))
y = f(x,oldx) - oldx ,
```

choosing different values for a and b, but keeping these between -1 and 1 to ensure that the orbit does not escape towards infinity. For example, Figure 1 shows the first ten thousand points, using $a = 0.28$, $b = 0.9998$ and the starting point $x = 0$, $y = 12$.

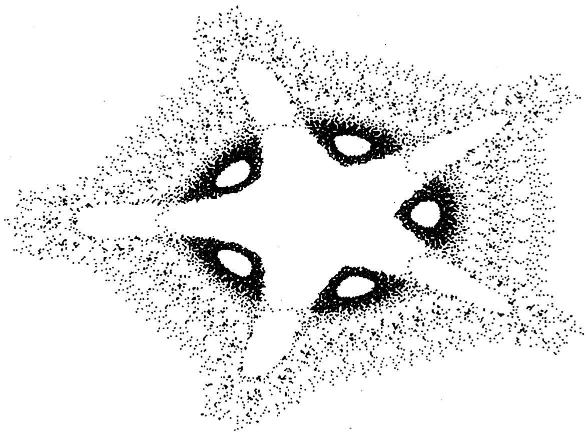

Figure 1

Using the values $a = 0.45$, $b = 0.9999$, and the starting point $x = 0$, $y = 11$ we obtain a completely different picture; Figure 2 displays 20000 points of this orbit. Figure 3 was obtained with the same starting point and the the same value for b, and $a = -0.01$.

I invite you to explore the world of these orbits by playing with the values of a and b and choosing different starting points. A world of fascinating pictures and orbit dynamics is hidden behind the iteration defined through the function (5). However, you are not constrained to this function; explore other functions such as $f(x, y) = ax + by + sin(x)$, or any other that keeps points within a certain boundary. You may need to change the definition of the window.

Over to you!

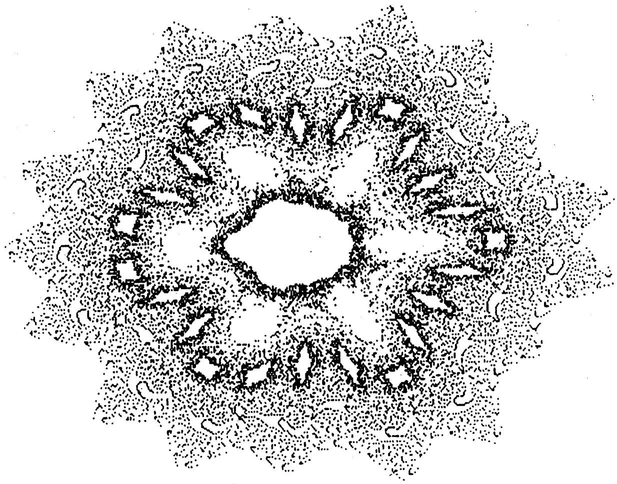

Figure 2

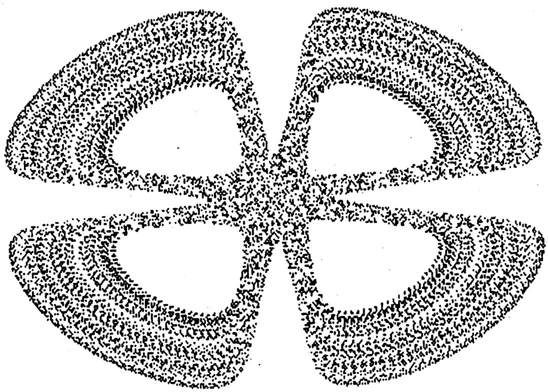

Figure 3

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 21.4.1

You have three calculating machines:

Machine A (an adder) accepts two numbers, a and b, as input, and calculates $a+b$.

Machine S (a subtractor) accepts two numbers, a and b, as input, and calculates $a-b$.

Machine Q (a "quarter-squarer") accepts one number, a, as input, and calculates $a^2/4$.

Explain how you could find the product, ab, of any two numbers a and b, using only these machines and no hand calculation.

SOLUTION by Julius Guest, East Bentleigh, VIC

Step 1: Use Machine A to find $a+b$ and denote your result by c.

Step 2: Use Machine S to find $a-b$ and denote your result by d.

Step 3: Use Machine Q to find $c^2/4$ and denote your result by e.

Step 4: Use Machine Q to find $d^2/4$ and denote your result by f.

Step 5: Finally, use Machine S to find $e-f$ which provides ab.

Proof: As $e=(a^2+2ab+b^2)/4$ and $f=(a^2-2ab+b^2)/4$, it follows that $e-f$ furnishes ab.

Solutions were also received from Keith Anker, Carlos Alberto da Silva Victor (Nilópolis, Brazil), Carson Luk (Surrey Hills, VIC), and Joáo Linneu do Amaral Prado (Sáo Paulo, Brazil).

PROBLEM 21.4.2

Find all three quadratic polynomials $p(x)=x^2+ax+b$ such that a and b are roots of the equation $p(x)=0$.

SOLUTION

Suppose firstly that a and b are distinct. Then:

$$p(x) = (x-a)(x-b) = x^2 - (a+b)x + ab.$$

Equating coefficients, we obtain:

$$-(a+b) = a \quad \text{and} \quad ab = b.$$

These two equations yield $-b = 2a$ and either $a = 1$ or $b = 0$. If $a = 1$ then $b = -2$, while if $b = 0$ then $a = 0$.

Now suppose $a = b$. Then $p(x) = (x-a)(x-c)$ for some number c. Upon expanding and then equating coefficients as before, we obtain:

$$-(a+c) = a \quad \text{and} \quad ac = a.$$

Hence $-c = 2a$ and either $a = 0$ or $c = 1$. If $a = 0$ then $b = c = 0$, while if $c = 1$ then $a = b = -\frac{1}{2}$.

The three polynomials are $p(x) = x^2 + 0x + 0$, $p(x) = x^2 + x - 2$, and $p(x) = x^2 - \frac{1}{2}x - \frac{1}{2}$.

Carlos Victor supplied an alternative solution using the following approach. From $p(a) = 0$ we obtain $b = -2a^2$, while $p(b) = 0$ yields $b^2 + ab + b = 0$. On eliminating b, we obtain an equation in a with three solutions, leading to the answer given above.

PROBLEM 21.4.3 (from *Mathematical Spectrum*)

A triangle has angles α, β and γ which are whole numbers of degrees, and $\alpha^2 + \beta^2 = \gamma^2$. Find all possibilities for α, β and γ.

SOLUTION by Carlos Alberto da Silva Victor

Since $\alpha + \beta + \gamma = 180$, we have $\alpha^2 + \beta^2 = \gamma^2 = (180 - \alpha - \beta)^2$, so:

$$\alpha = \frac{180(90-\beta)}{180-\beta} = 180 - \frac{180 \times 90}{180-\beta} = 180 - \frac{2^3 \times 3^4 \times 5^2}{180-\beta}.$$

Hence $180 - \beta$ divides $2^3 \times 3^4 \times 5^2$, and, since $\alpha > 0$, we must have $\beta < 90$. The possibilities are:

1. $180 - \beta = 100$; then $\beta = 80, \alpha = 18$ and $\gamma = 82$.
2. $180 - \beta = 162$; then $\beta = 18, \alpha = 80$ and $\gamma = 82$.

3. $180 - \beta = 108$; then $\beta = 72, \alpha = 30$ and $\gamma = 78$.
4. $180 - \beta = 150$; then $\beta = 30, \alpha = 72$ and $\gamma = 78$.
5. $180 - \beta = 135$; then $\beta = 45, \alpha = 60$ and $\gamma = 75$.
6. $180 - \beta = 120$; then $\beta = 60, \alpha = 45$ and $\gamma = 75$.

Keith Anker and Julius Guest (East Bentleigh, VIC) also sent us a solution.

PROBLEM 21.4.4 (Juan-Bosco Romero Marquez, Universidad de Valladolid, Valladolid, Spain)

Find all possible sets of six two-digit numbers $M = xy, N = yz$, $P = zu, M' = yx, N' = zy, P' = uz$ (where x, y, z and u are decimal digits, and xy, etc. denote the decimal representations of the numbers), such that M, N, P and M', N', P' are two geometric progressions with the same integer common ratio.

SOLUTION

In standard algebraic notation, $M = 10x + y$, $N = 10y + z$, $P = 10z + u, M' = 10y + x, N' = 10z + y$, $P' = 10u + z$. Denoting the common ratio by r, we obtain:

$$r = \frac{10y+z}{10x+y} = \frac{10z+y}{10y+x} \tag{1}$$

$$r = \frac{10z+u}{10y+z} = \frac{10u+z}{10z+y} \tag{2}$$

After some algebra, equations (1) and (2) yield respectively:

$$y^2 = xz \tag{3}$$

$$z^2 = yu \tag{4}$$

Equation (3) tells us that xz is a perfect square. For each possible value of x in turn from 1 to 9, we can determine the admissible values of z. Trivially, we can always take $z = x$, and in addition we find that each of the following (x, z) pairs also yields a value of xz that is a perfect square: (1,4), (1,9), (2,8), (4,1), (4,9), (8,2), (9,1) and (9,4). For each pair, we can calculate y using (3) and u using (4). After excluding the pairs that give values for u that are fractional or greater than 9, we are left with the following solutions:

$$(x, y, z, u) = (1, 2, 4, 8) \quad \text{and} \quad (x, y, z, u) = (8, 4, 2, 1)$$

and the trivial solutions for which $x = y = z = u$. Thus the only non-trivial solutions to the problem are:

$$M = 12, N = 24, P = 48, M' = 21, N' = 42, P' = 84$$

$$M = 21, N = 42, P = 84, M' = 12, N' = 24, P' = 48$$

We received solutions from Carlos Victor, Joáo Linneu do Amaral Prado, and the proposer.

PROBLEM 21.4.5 (Claudio Arconcher, Sáo Paulo, Brazil)

Let Γ be a circle of radius r, and let $\overline{BC}$ be a chord of Γ. A point A on Γ makes one revolution around Γ. Prove that the locus of the centroid of the triangle ABC is a circle with radius $r/3$, and that this circle divides the chord $\overline{BC}$ into three equal parts.

SOLUTION

Let O be the centre of Γ, let M be the midpoint of $\overline{BC}$, and let G be the centroid of ABC. Then $AG = \frac{2}{3}AM$. Let O' be the point on $\overline{OM}$ such that $OO' = \frac{2}{3}OM$. (See Figure 1.) Then the triangles MOA and $MO'G$ are similar, and $MO' = \frac{1}{3}MO$, so $O'G = \frac{1}{3}OA = \frac{1}{3}r$. Therefore the locus of G is a circle, centred at O', with radius $r/3$. When A coincides with B, we have $BG = AG = \frac{2}{3}AM = \frac{2}{3}BM = \frac{1}{3}BC$, so the locus of G divides $\overline{BC}$ one-third of the way along its length. Similarly, when A coincides with C, we have $CG = \frac{1}{3}CB$. Therefore the locus of G divides the chord $\overline{BC}$ into three equal parts.

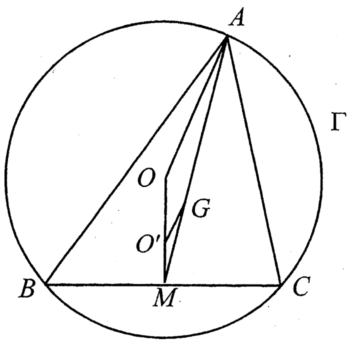

Figure 1

Solutions were also provided by Keith Anker, Carlos Victor and the proposer.

PROBLEM 21.4.6

A farmer would like to graze his animal on his neighbour's circular paddock, but the neighbour stipulates that the farmer can only use half of the paddock and the animal must be tethered on the boundary line. What is the length of the tether as a function of the radius of the paddock?

SOLUTION

Let the radius of the paddock be 1 unit, and let the length of the tether be l. Let O be the centre of the paddock, let T be the point where the animal is tethered, and let A and B be the two points on the boundary at a distance l from T. The grazed area is bounded by the arc ATB and the arc from A to B centred at T with radius l. Let α be the radian measure of the angle ATB. Then $\angle OTA = \alpha/2$, and hence $\angle OAT = \alpha/2$ also, since the triangle OAT is isosceles. Therefore $\angle AOT = \pi - \alpha$. (See Figure 2.) We note that $l = 2\cos(\alpha/2)$; this can be shown by dividing triangle OAT into two congruent right-angled triangles.

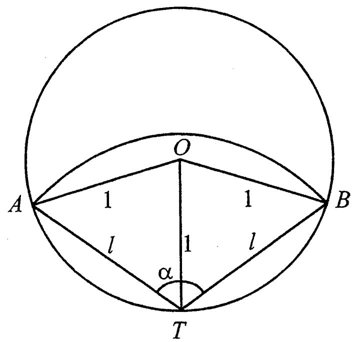

Figure 2

The grazed area equals the area of the sector, S_1, bounded by the arc ATB and the radii $\overline{OA}$ and $\overline{OB}$, plus the area of the sector, S_2, bounded by the arc from A to B centred at T and the radii $\overline{TA}$ and $\overline{TB}$, minus the area of the quadrilateral $OATB$. We evaluate each of these areas in turn.

The area of S_1 is:

$$\text{Area}(S_1) = \frac{1}{2} \times 2(\pi - \alpha) = \pi - \alpha$$

The area of S_2 is:

$$\text{Area}(S_2) = \frac{1}{2} l^2 \alpha = \frac{1}{2}\left[2\cos\left(\frac{\alpha}{2}\right)\right]^2 \alpha = 2\alpha\cos^2\left(\frac{\alpha}{2}\right) = \alpha(1 + \cos\alpha)$$

The area of the quadrilateral $OATB$ is:

$$\text{Area}(OATB) = 2 \times \text{Area}(\triangle OAT) = 2 \times \frac{1}{2}\sin(\pi - \alpha) = \sin\alpha$$

Hence the grazed area is $\pi - \alpha + \alpha(1 + \cos\alpha) - \sin\alpha$, which simplifies to $\pi + \alpha\cos\alpha - \sin\alpha$. Setting this expression equal to half of the area of the paddock, we obtain the equation $\pi + \alpha\cos\alpha - \sin\alpha = \pi/2$, which simplifies to:

$$\sin\alpha - \alpha\cos\alpha = \pi/2 \tag{5}$$

Equation (5) cannot be solved exactly for α, but an approximate numerical solution can be found using an iterative procedure such as Newton's method. The result is $\alpha = 1.905695\ldots$. We can now evaluate l, the length of the tether: $l \approx 2\cos(1.905695/2) \approx 1.1587$.

Solutions to this problem were provided by Carlos Victor and Joáo Linneu do Amaral Prado, Keith Anker, and Julius Guest.

Correction

There was an error in Problem 21.5.4 in the October 1997 issue of *Function.* The third sentence should read as follows:

Each sequence apart from the first "describes" the previous sequence in the following sense: the first number in the previous sequence is listed, preceded by the number of times it occurs consecutively, then the next number is listed, preceded by the number of times it occurs consecutively, and so on.'

A solution to the problem will appear in the next issue.

PROBLEMS

Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received by 9 April 1998 will be acknowledged in the June issue, and the best solutions will be published.

PROBLEM 22.1.1 (A Begay, Lupton, Arizona, USA)

Let $S(n)$ be the smallest positive integer such that $S(n)!$ is divisible by n, where $m!$ denotes $1 \times 2 \times 3 \times \ldots \times m$ (the factorial function).

(a) Prove that if p is prime then $S(p) = p$.

(b) Calculate $S(42)$.

(The function $S(n)$ is called the *Smarandache function*.)

PROBLEM 22.1.2 (Julius Guest, East Bentleigh, VIC)

An equilateral triangle ABC is inscribed in a circle. Let D be any point on the minor arc subtended by $\overline{AB}$. Prove that $DC = DA + AB$.

(Hint: Let E be the point on $\overline{CD}$ such that triangle ADE is equilateral. Show that the triangles AEC and ADB are congruent.)

PROBLEM 22.1.3

Let ABC be an equilateral triangle, and let P be a point inside ABC such that $AP = 3$, $BP = 4$, and $CP = 5$. Prove that the angle APB has radian measure $5\pi/6$.

PROBLEM 22.1.4 (based on a problem in *Mathematics and Informatics Quarterly*)

Let $COFFEE$ and $BREAK$ be two numbers written in decimal notation, where each letter stands for a digit. Find the numbers, given that $\sqrt{COFFEE}$ and $\sqrt[4]{BREAK}$ are integers.

PROBLEM 22.1.5 (from *Parabola*, University of New South Wales)

Let x be a real number with $x \neq \pm 1$. Simplify

$$\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \frac{8}{x^8+1} + \ldots + \frac{2^n}{x^{2^n}+1}.$$

PROBLEM 22.1.6

Prove that

$$\int_0^{\pi/2} \frac{1}{1+(\tan x)^k} dx = \frac{\pi}{4}$$

for any real value of k.

(Hint: Using the fact that $\cot x$ can be expressed in terms of $\tan x$ in two different ways, convert the integral to an integral involving $\cot x$, then convert the result to another integral involving $\tan x$.)

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