

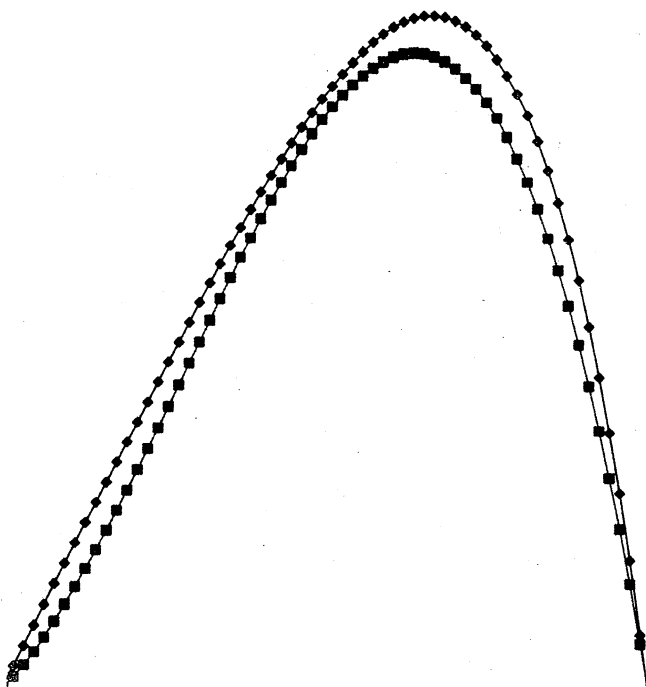
# *Function*

**A School Mathematics Journal**

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*Function* is a refereed mathematics journal produced by the Department of Mathematics and Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

*Function* deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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## EDITORIAL

This last issue of Volume 21 of *Function* includes three feature articles. The front cover figure is a contribution by one of our youngest authors. Melanie Komisarczyk is a Year 10 student who presents the conclusions of her research on approximating golfball trajectories with special cubic functions. The second contribution is from our reader and frequent contributor Bert Bolton, who derives a formula for calculating the distance to the horizon or to a mountain observed from the sea. And the third contribution is from the world of magic. A magician approached our editor and statistician Malcolm Clark, and asked him to determine how difficult it is to construct a magic square randomly. You will find here a thorough analysis of this question.

Mathematical knowledge has made possible the great technological change we have experienced in the last decades, and this change, in turn, has had a great impact on the nature of mathematics. Michael Deakin illustrates with a very simple example how some techniques which were meaningful at the time calculations were performed by hand, have a different appeal now when we perform them with calculators and computers.

For those readers interested in designing their own fractals, there is yet another type of fractal presented in the *Computers and Computing* column.

Our *Problem Corner* editor has included more new problems as well as answers to some previous ones. We also publish the outstanding performance of the Australian team at the International Mathematical Olympiad held in Mar del Plata, and the problems they had to solve to earn two gold, three silver and a bronze medal.

Happy reading!

\* \* \* \* \*

## FLYING THROUGH THE AIR: GOLFBALLS, ARROWS AND CUBIC EQUATIONS

Melanie Komisarczyk, Year 10, Wheelers Hill Secondary College

In an ideal case, where no air resistance is involved, an arrow shot from a bow, a golfball driven off the tee, or a shell fired from a gun would all follow parabolic paths. If the initial direction of motion makes an angle  $\alpha$  with the horizontal, then the path is given by the equation

$$y = x \tan \alpha - \frac{g}{2V^2} x^2 \sec^2 \alpha, \quad (1)$$

where  $V$  is the initial speed and  $g$  is a constant, the acceleration due to gravity.

However, by choosing special units of length and time<sup>1</sup>, we may arrange that  $V = 1$  and  $g = 1$  in these special units. If we do this and also rearrange equation (1), we get

$$y = Ax(R - x), \quad (2)$$

where  $A = \frac{1}{2} \sec^2 \alpha$  and  $R = \sin 2\alpha$ .

But in real life, air resistance can be important. One suggestion to take account of this is to replace equation (2) by a cubic equation:

$$y = Ax(R - x)(1 + Bx), \quad (3)$$

where  $R$  is the range. (But we may now have different formulas for  $A$  and  $R$ ; not necessarily those given above.)

Equation (3) comes from an article from *Function Vol 21, Part 1* (see the final footnote of the article).

I looked at two applications of equation (3). The first was about the path travelled by a golfball and the second concerned a relation between the travel time and the maximum height reached.

One trajectory that has been studied is that followed by a golfball in flight. In the special units, this trajectory is given by the rather complicated equation:

$$y = \frac{x}{4} + 0.05(e^{5x} - 1 - 5x) - 0.01(e^{10x} - 1 - 10x). \quad (4)$$

---

<sup>1</sup>See *Function Vol 21, Part 1*, p. 12.

About 120 years ago, a man named P G Tait invented equation (4). Tait was a keen amateur golfer and also a physicist and mathematician. His equation is different from the parabolic path because he had to consider the effects of air resistance and of aerodynamic lift caused by the backspin of the golfball. Parabolic equations don't include these factors, but with golfballs they are important.

I looked at this cubic approximation to Tait's trajectory.

$$y = 0.395x(0.315 - x)(1 + 17x). \quad (5)$$

(This approximation came from a recent Monash University Applied Mathematics Report.)

The first thing I did was to check how similar the cubic was to the more complicated function. The values given by equation (4) I called  $y_1$  and the cubic approximations (5) I called  $y_2$ .

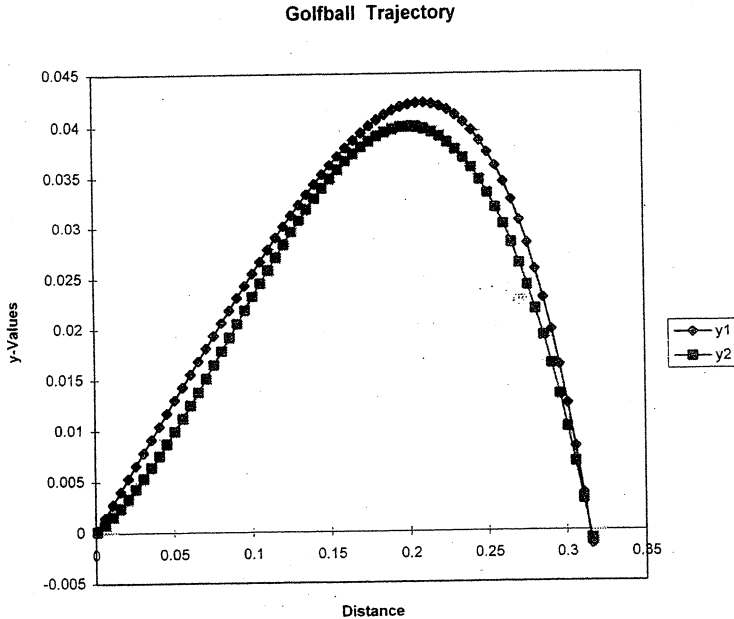
I calculated  $y_1$  and  $y_2$  by using the computer spreadsheet EXCEL. The results were graphed also using EXCEL. This gave the graph in Figure 1.

The graph shows that  $y_1$  and  $y_2$  are similar but not quite the same. Over most of the path the discrepancy is less than 10%.  $y_1$  has a steeper decline than  $y_2$  and  $y_2$  has a bigger concavity than  $y_1$ . Also the two curves ( $y_1$  and  $y_2$ ), when they are starting their ascent, start off together and then seem to separate from each other. When they near the top they join together for a moment before separating again, to join again only when they hit their destination point.

The next thing I worked on was the total time of flight and the maximum height reached by an arrow,  $T$  being the flight-time and  $H$  the height. This equation (in the special units) is

$$T^2 \approx 8H. \quad (6)$$

Equation (6) was first recorded by the mathematician J E Littlewood. He worked on gunnery tables for the British Army in World War I. I will call it Littlewood's Formula.



**Figure 1**

$y1$  is the higher curve (P G Tait's flight-path trajectory).  
 $y2$  is the lower curve and is the cubic approximation to Tait's trajectory

The next task was to test the cubic approximation. I did this with the EXCEL spreadsheet, as shown below.

In the spreadsheet I calculated  $T$  by using the following equation:

$$T = \frac{2}{9B} \sqrt{2A} \left[ (1 + 2BR)^{3/2} - (1 - BR)^{3/2} \right], \quad (7)$$

where  $A$ ,  $B$  and  $R$  are derived from equation 3.

Next I calculated  $H$  by using the equation

$$H = \frac{A}{27B^2} \left[ 2(1 + BR + B^2R^2)^{3/2} - 2 - 3BR + 3B^2R^2 + 2B^3R^3 \right]. \quad (8)$$

(These equations also came from the Monash University Applied Mathematics Report mentioned above.)

If  $BR > 1$ , equation (7) will not work because it would involve the square root of a negative number in this case. In fact, in the golfball trajectory it doesn't work, as the golfball has an aerodynamic lift caused by the backspin. Equation (5) tells us that for the golfball  $BR = 5.355$ .

Table 1 shows the output from the spreadsheet calculations. The second and third columns list the values of  $T$  and  $H$  for different values of  $BR$ . After this the values of  $T^2$ ,  $8H$  and the ratio  $\frac{T^2}{8H}$  are shown.

The spreadsheet shows that the ratio is always close to 1. It also shows that when  $BR$  gets bigger the ratio gets smaller.

We can also check if the spreadsheet is right by looking at the last number in the ratio column. If we use equations (7) and (8), we find the ratio equals  $\frac{\sqrt{3}}{2}$ , or about 0.866025, as shown in the table.

Value of BR	Value of T	Value of H	Value of T <sup>2</sup>	Value of 8*H	Ratio
0	0	0	0	0	1
0.05	0.07157312	0.00064072	0.005122711	0.005125762	0.99940485
0.1	0.14479007	0.002626486	0.020964163	0.021011891	0.997728524
0.15	0.21953836	0.006054215	0.04819709	0.048433724	0.995114279
0.2	0.29571538	0.011022634	0.087447588	0.088181074	0.991682046
0.25	0.37322612	0.017632048	0.13929774	0.141056385	0.987532322
0.3	0.4519813	0.025984136	0.204287094	0.207873085	0.982749132
0.35	0.5318958	0.036181766	0.282913139	0.289454131	0.977402319
0.4	0.61288729	0.048328843	0.375633083	0.386630741	0.971549311
0.45	0.69487489	0.06253016	0.482851106	0.500241281	0.965236426
0.5	0.77777778	0.078891289	0.604938272	0.631130309	0.958499794
0.55	0.86151374	0.097518468	0.742205927	0.780147744	0.951365857
0.6	0.94599735	0.118518519	0.894910978	0.948148148	0.943851422
0.65	1.03113765	0.141998764	1.063244854	1.135990113	0.935963123
0.7	1.11683501	0.168066966	1.247320443	1.344535728	0.927696019
0.75	1.20297628	0.196831266	1.447151924	1.574650126	0.919030774
0.8	1.28942691	0.228400136	1.662621761	1.827201091	0.909928179
0.85	1.37601659	0.26288234	1.893421645	2.103058721	0.900318011
0.9	1.46250844	0.300386892	2.138930948	2.403095134	0.890073355
0.95	1.5485141	0.341023028	2.39789593	2.728184223	0.878934754
1	1.63299316	0.384900179	2.666666667	3.079201436	0.866025404

**Table 1**  
Calculations for  $T, H$  for Littlewood's Formula

It should perhaps be said that the first value of the ratio in the spreadsheet was not calculated by the computer. Because the value of  $BR$  was 0, we find

$T = 0$  and also  $H = 0$ , so that the ratio  $\frac{T^2}{8H} = \frac{0}{0}$ , and if you put that formula in the spreadsheet the computer would not calculate it. However, it is known that the value 1 can be used in this case. Littlewood's Formula applies even for quite large values of  $BR$  (up to about 0.85).

The purpose of this investigation was to look at some consequences of the cubic law, as given by equation (3). It was found that it gave a quite good agreement in the case of the golfball, even though this is quite extreme with  $BR = 5.355$ . It also allows exploration of Littlewood's Formula

$$T^2 = 8H.$$

Even with air resistance and aerodynamic lift (caused by the backspin of a golfball), the cubic approximation is quite good. Also, even for quite big values of  $BR$  the approximation still holds. Because cubics are fairly simple functions, this allows fairly easy analysis of trajectories.

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*Melanie wrote this article during her short stay at the Department of Mathematics and Statistics of Monash University, where she did research as part of the work experience of her Year 10 program.*

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There are many things you can do with problems besides solving them. First you must define them, pose them. But then of course you can also *refine* them, *depose* them, or *expose* them, even *dissolve* them! A given problem may send you looking for analogies, and some of these may lead you astray, suggesting new and different problems, related or not to the original. Ends and means can get reversed. You had a goal, but the means you found didn't lead to it, so you found a new goal they did lead to. It's called play. Creative mathematicians play a lot; around any problem really interesting they develop a whole cluster of analogies, of playthings.

– David Hawkins in *From Cardinals to Chaos*,  
Cambridge University Press, 1988

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## HOW FAR IS THE HORIZON?

Bert Bolton, University of Melbourne

Most of us have stood on the seashore and asked "How far can we see?" and often we have been given the answer "To the horizon". This logically irritating answer then has provoked the question "But how far is the horizon?" Using Pythagoras's theorem, the distance can be calculated and the method allows further questions to be answered when we see the tops of mountains or masts above the horizon. It is assumed that the atmosphere is clear and that there are no restrictions on vision due to mist or rain.

In Figure 1 is shown a sketch of the earth with centre  $C$  and radius  $R$ . The earth is not a true sphere but the difference is small; the radius at the equator is slightly larger than the radius at the pole and this is ignored for this calculation. The observer  $O$  has eyes at a height  $h$  units above the surface of the sea. Consider a line tangential to the surface of the sea at the point  $H$ ; the line  $\overleftrightarrow{OH}$  is the line of sight of the observer to the horizon. The triangle  $CHO$  has a right angle at  $H$ . By Pythagoras's theorem,

$$OH^2 + HC^2 = OC^2$$

and using  $x$  for  $OH$

$$x^2 + R^2 = (R + h)^2.$$

Expanding the bracket gives

$$x^2 = 2Rh + h^2 = 2Rh \left( 1 + \frac{h}{2R} \right). \quad (1)$$

Consider the values of  $R$  and  $h$  and then the value of the correction term  $\frac{h}{2R}$  in the bracket in (1). The unit of distance used is the metre. The radius length of the earth is 6375 kilometres, which is the usual average value. When the observer is standing on the sand by the sea, the height of the eyes will usually be less than 2 metres. The correction term is then

$$\frac{h}{2R} \approx 1.6 \times 10^{-7}$$

where the usual equal sign is replaced by the sign for 'approximately equal'. The effect of this correction term can be ignored. There will be times when the observer can be several kilometres above the sea; we can, for example, choose  $h = 5000$ . Then the correction term is

$$\frac{h}{2R} \approx 4 \times 10^{-4}$$

and can still be ignored. Equation 1 then reduces to

$$x = \sqrt{2hR}. \quad (2)$$

If  $OH = x$  km,  $OP = h$  m,  $CH = 6375$  km, then this equation yields

$$x = 3.57h^{1/2}. \quad (3)$$

Equation 2 also applies to the calculations that arise when from ships at sea, a mountain seems to rise out of the water. If its height is known, equation 3 can be used to give its distance. As an example, when the height of the mountain is 1 kilometre = 1000 metres, then equation 3 gives 112 kilometres, which is the distance from the mountain to the horizon; to this must be added the distance of the observer to the horizon also calculated by equation 3.

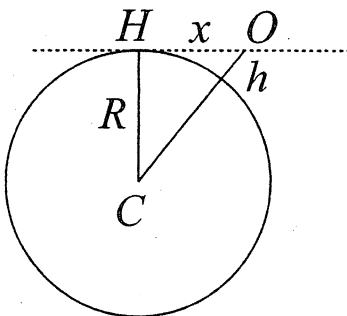


Figure 1

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Mathematics, the science of the ideal, becomes the means of investigating, understanding and making known the world of the real. The complex is expressed in terms of the simple. From one point of view mathematics may be defined as the science of successive substitutions of simpler concepts for more complex.

— William F White, in *A Scrap Book of Elementary Mathematics*

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## MAGIC SQUARES AT RANDOM?

Malcolm Clark, Monash University

One of the pleasures of life as a mathematician is the variety of questions/puzzles posed to you by the public. Recently, I had a phone call from Charles Gauci, a member of the Australian Society of Magicians, and as it turned out, a retired magician himself. His question to me was to do with “magic squares”. Apparently, a fellow member of that Society had managed to construct, by hand, a  $16 \times 16$  magic square. In other words, he had found a way of arranging the integers from 1 to 256 in a  $16 \times 16$  square in such a way that the numbers in every row, every column, and the two diagonals added up to the same total, namely 2,056.

The fact that such a large magic square had been constructed by hand, rather than by computer, was most impressive, to me and to Mr Gauci. In fact, the Society of Magicians intended to make this the leading story in their next Newsletter, with a challenge to their readers to attempt to produce this and any other  $16 \times 16$  magic squares “by computer”. But before going to print, Mr Gauci wanted some idea of how difficult such a task would be. If the task was really trivial, then the intended leading article would have to be modified, or even dropped altogether. As the Newsletter was almost ready to go to press, Mr Gauci needed an answer within 24 hours.

I had a brief conversation with Mr Gauci, in which I had to admit that I didn't really know how difficult it was to construct a  $16 \times 16$  (or similar-sized) magic square by means of a computer program. He envisaged a complete enumeration of all possible arrangements of the numbers 1 to 256 into a  $16 \times 16$  array. This method is very inefficient and not feasible in practice, since the number of possible arrangements is so enormous.

As the conversation continued, he came up with the following idea (for assessing the difficulty of the problem). “Suppose”, he said, “you put the numbers 1 to 256 into the  $16 \times 16$  array *at random*. What is the probability of one row or column adding up correctly (to 2,056)?” This question *can* be answered to a good approximation, using elementary methods of statistics but in a rather unconventional setting.

To be definite, let us focus attention on the *first* row of the random  $16 \times 16$  square. The 16 numbers in this row are a random sample from the population of integers 1 to 256. Let  $\bar{X}$  denote the average (or sample mean) of these 16

numbers in this random sample, and let  $T$  denote the corresponding sum of those numbers.

We now appeal to a very famous theorem, the Central Limit Theorem. This states that if the sample size  $n$  is large, then the sample mean is approximately normally distributed, with mean  $\mu$  and standard deviation  $\sigma/\sqrt{n}$ , where  $\mu$  and  $\sigma$  are the mean and standard deviation of the population. Furthermore, the larger the sample size, the better the approximation. So by this theorem, our  $\bar{X}$  (the average of the numbers in the first row) should be approximately normally distributed. The row total  $T$  is just 16 times  $\bar{X}$ , and so  $T$  is also approximately normally distributed. So in essence, all we have to do is work out  $\mu$  and  $\sigma$ ; probability statements regarding  $T$  can be determined by finding the area under the appropriate bell-shaped normal curve.

By symmetry, the population mean  $\mu$  is the half-way point (or median), 128.5. The population standard deviation  $\sigma$  can be computed by applying the usual formula<sup>1</sup>

$$\sigma = \sqrt{\frac{\sum(Y_i - \bar{Y})^2}{N}}$$

to the data  $\{1, 2, \dots, 256\}$ , giving  $\sigma = 73.90$ . Hence  $\bar{X}$  is approximately normally distributed, with mean  $\mu = 128.5$  and standard deviation  $\sigma_{\bar{X}} = 18.475$ .

The row total  $T$  is just a scaled-up version of  $\bar{X}$ , so it too must be approximately normal, but with mean and standard deviation 16 times that of  $\bar{X}$ , i.e.  $\mu_T = 2,056$  and  $\sigma_T = 295.6$ .

The probability that this first row of the square adds up correctly is equal to the probability that this random sum  $T$  equals 2,056. We can use the preceding normal distribution to approximate this probability. But there is a slight complication.

The random sum  $T$  can take only integer values, whereas all normal distributions are continuous, so we need to apply a "continuity correction". With suitable scaling, probabilities regarding  $T$  can be represented by the areas of the bars in the relative frequency histogram shown in Figure 1. We are interested in the area of just one of these bars, the one centered on 2,056. It is clear from this diagram that this area is approximately the same as the

<sup>1</sup>Strictly speaking,  $\sigma$  should be reduced slightly, to allow for the fact that our population is finite.

area under the approximating normal curve, from 2055.5 to 2056.5. And area means probability!

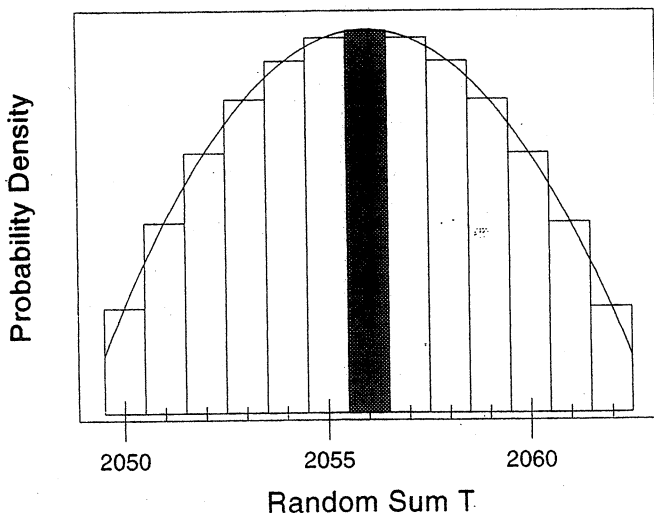


Figure 1

Hence we can write

$$P(T = 2056) \simeq P(2055.5 \leq Y \leq 2056.5) \quad (1)$$

where  $Y$  is a normal random variable, with the same mean and standard deviation as  $T$ . As all students of statistics know, we can compute this latter probability by converting it to an equivalent statement regarding the standard normal variable  $Z$  (the  $Z$ -score), and then look up the tables of standard normal probabilities. Alternatively, normal probabilities can be computed directly by computer (e.g. Excel or Minitab) or calculator. The required normal probability turns out to be  $p = 0.0013496$ .

So that's the answer, at least approximately! Or is it? We have computed the probability that one *particular* row, the first row, adds up correctly. This is not the question posed by Mr Gauci: he wanted the probability of either *any* of the 16 rows adding up correctly, or *any* of the 16 columns. Although he didn't mention it specifically, he presumably would be interested in the possibility of either one of the two diagonals adding up correctly too.

Furthermore, it could happen that *two or more* rows or columns add up by chance. We should not exclude such a possibility from our calculations. If it turns out that this happens fairly frequently, then it possibly indicates that it is fairly easy to construct a  $16 \times 16$  magic square by hand.

So the real question is, what is the probability that *at least one* of the rows, columns or diagonals adds up correctly? How can we answer this?

The 16 rows, 16 columns and 2 diagonals make up 34 subsets of 16 numbers from the random permutation of the numbers  $\{1, 2, \dots, 256\}$ . By symmetry, each of these 34 subsets must have the same probability,  $p$ , of adding up correctly. By our calculations for one of these subsets (the first row),  $p \simeq 0.0013496$ . Intuition suggests that these subsets are likely to be approximately independent. For example, if the first row happens to add up correctly, then the probability that the second row also adds up correctly is still approximately 0.0013496.

Let  $N$  denote the number out of  $r$  subsets which add up correctly. Then with this assumption of independence, the required probability becomes

$$P_1 = P(N \geq 1) = 1 - P(N = 0) = 1 - (1 - p)^r = 0.04488 \quad (2)$$

on substituting  $p = 0.0013496$  and  $r = 34$ . (Statistics students will recognise this as an application of the binomial distribution.)

Before getting back to Mr Gauci, I repeated the calculations but this time for random  $3 \times 3$  magic squares (see *Function Vol 4, Part 1*, pp 25-26), noting that there were now only 8 rows, columns or diagonals. As a check, I did a small simulation using Minitab. In this case, it was possible to compute the exact probability,  $p$ , of a specified row adding up to the correct row total (15 for a  $3 \times 3$  square), simply by enumerating all 84 possible samples of 3 numbers from 1 to 9. In eight of these samples, the 3 numbers add up to 15. Hence the exact probability is  $p = 8/84 = 0.0952$ , whereas the normal approximation gave 0.273. That approximation holds only when the sample size  $n$  is large; you could hardly call 3 large! Substituting  $r = 8$  and  $p = 0.0952$  in (2) gave an estimated probability of 0.551. In comparison, out of the 100 random  $3 \times 3$  squares generated by Minitab, 51 had at least one row, column or diagonal adding up correctly. This is quite good agreement, suggesting that the assumption of independence behind equation (2) is reasonable.

This story shows the importance of answering the right question, even when the customer is not sure himself what is the right question. It also

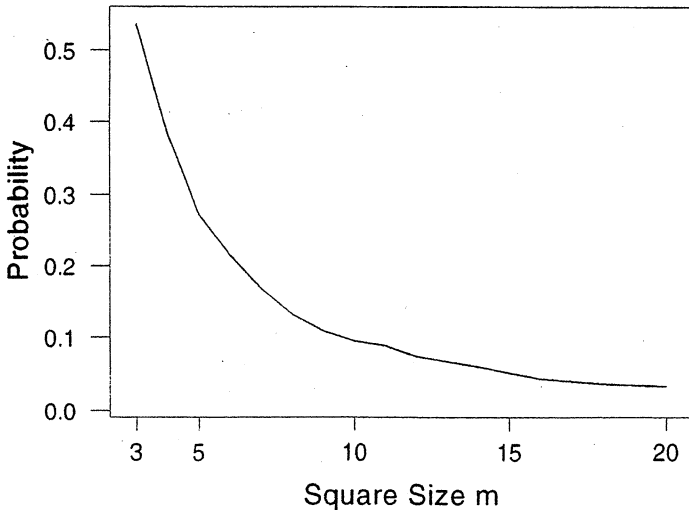
shows how a complicated problem is broken down into a number of steps. The steps are tackled one at a time, using approximations if necessary. Finally, all the pieces (in this case, random sampling, Central Limit Theorem, computing  $\mu$  and  $\sigma$ , continuity correction, binomial-type probabilities) are put together.

**Epilogue**

Some weeks later, I wrote a computer program to estimate the required probability by simulation, for squares of arbitrary size  $m \times m$ . For various values of  $m$  between 3 and 20, I generated 10,000  $m \times m$  squares at random, and simply counted the number of times that at least one row, column or diagonal added up to the correct value for a magic square. This procedure involved no assumptions, such as independence or a normal approximation. These simulated probabilities decrease as  $m$  increases, although the fastest decrease is for small values of  $m$  (see Figure 2). In fact, these probabilities can be very accurately predicted by the equation

$$P_1 = 3.03m^{-1.5}.$$

Finally, after writing the first draft of this article, I discovered an algorithm for constructing a  $16 \times 16$  magic square – in *Function*<sup>2</sup> in 1994!



**Figure 2**

<sup>2</sup>*Function* Vol 18, Part 2, pp 52-58.

## HISTORY OF MATHEMATICS

### Right before our Very Eyes

Michael A B Deakin

In my own lifetime, I have seen one of the most profound changes in the nature of mathematics in its entire history. It isn't the only such major shift ever to occur, but it is just as profound as any of the others.

I refer, of course, to the impact of the computer and the electronic calculator. Their influence has been much wider than the simple story I give here will indicate, and of course the computer now does much more than merely calculate. However, I will show one way that these devices actually alter the way in which we *think* about mathematics. It isn't merely that they remove the drudgery from much of what once was part of mathematics syllabuses, although that is very much involved. But we now look differently at the underlying mathematics itself – precisely because we have tools to remove the tedious aspects of the work.

Start with a simple example. My trusty CASIO fx-100B tells me that

$$\sqrt{2} = 1.414213562. \quad (1)$$

This is, as we know, not strictly accurate, but the machine can only display ten digits and this is the best possible ten-digit approximation there is to the true value of  $\sqrt{2}$  (an irrational number).

Now suppose we wanted to find the reciprocal of the approximation in equation (1); that is to say, we wanted an approximation to  $\frac{1}{\sqrt{2}}$  (the value of  $\sin 45^\circ$  or  $\cos 45^\circ$ ). And suppose we lived, as once I did, before efficient modern computing machinery was available.

To *divide* 1 by 1.414213562 with the aid *only* of pencil and paper is possible and in one sense not particularly difficult. It demands no great brainpower! But it is tedious, boring, daunting, offputting and *extremely* prone to error.

However, nobody ever did things that way. The reason is that the calculation can be performed much more simply. We work as follows:

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \dots \quad (2)$$



and now we simply halve the right-hand side of equation (1). With the numbers in front of us, we simply read off successive digits, and so find

$$\frac{1}{\sqrt{2}} = 0.707106781 \quad (3)$$

(to the same degree of accuracy we had before).

Given the digits of equation (1), we can quite readily produce those of equation (3) *in our heads* (and in fact I just did, but for convenience's and laziness' sake I checked them on the calculator).

This process, aimed at simplifying what at first sight seem to be horrendous division calculations, goes by the name of *rationalising the denominator*. We avoid having in the bottom line of our fraction a number that has (necessarily) infinitely many digits, and simplify the division by turning it into a straightforward rational number. The example I have given is the most spectacular such simplification there is, but for that reason it is a particularly good one to begin with.

The same principle may be used with more complicated divisions. Here is another example. The Golden Ratio,  $\varphi$  let us call it, is the number

$$\varphi = \frac{1 + \sqrt{5}}{2} \simeq 1.618033989, \quad (4)$$

and once again it would be extremely tedious, without a calculator or computer, to find its reciprocal by sheer bull-at-a-gate methods.

We have, after a trivial rearrangement,

$$\frac{1}{\varphi} = \frac{2}{\sqrt{5} + 1} \quad (5)$$

and we need to simplify (rationalise) that awkward denominator.

The trick is to use the "difference of two squares" formula. We notice that

$$(\sqrt{5} + 1)(\sqrt{5} - 1) = (\sqrt{5})^2 - 1^2 = 5 - 1 = 4. \quad (6)$$

Now 4 is a *much* easier number to divide by than is  $\sqrt{5} + 1$ , so on the right-hand side of equation (5), we multiply top and bottom by  $\sqrt{5} - 1$ . This gives

$$\frac{1}{\varphi} = \frac{2}{\sqrt{5} + 1} \times \frac{\sqrt{5} - 1}{\sqrt{5} - 1} = \frac{2(\sqrt{5} - 1)}{4} = \frac{\sqrt{5} - 1}{2}. \quad (7)$$

(This example is very much to the point. The Golden Ratio  $\varphi$  is in fact *defined* as the number which satisfies the equation

$$\frac{1}{\varphi} = \varphi - 1. \quad (8)$$

We see from equations (4) and (7) that this is so. Notice also another point: equation (8) is a quadratic equation and so in fact possesses *two* solutions. They are  $\frac{1 \pm \sqrt{5}}{2}$ , and we choose the positive root. The negative root is  $-\frac{1}{\varphi}$ .)

Once we have equation (7), of course, it's easy to compute  $\frac{1}{\varphi}$ , especially if we notice that it's merely  $\varphi - 1$ . That's a calculation we can *all* do in our heads!

So now let's generalise. If we have *any* number of the form

$$\frac{a + b\sqrt{N}}{c + d\sqrt{N}},$$

where the  $a, b, c, d$  and  $N$  are all nice, rational numbers, we can *always* rationalise the denominator by multiplying above and below by  $c - d\sqrt{N}$ . This gives

$$\frac{a + b\sqrt{N}}{c + d\sqrt{N}} \times \frac{c - d\sqrt{N}}{c - d\sqrt{N}}.$$

Resisting the temptation to cancel the terms on the right, which would merely restore the *status quo ante*, we multiply out to find

$$\frac{(ac - bdN) + (bc - ad)\sqrt{N}}{c^2 - Nd^2},$$

and this will always have a rational denominator.

This has important theoretical consequences. They are most easily seen in the case  $N = -1$ . In that case,  $a + b\sqrt{N}$  and  $c + d\sqrt{N}$  are both *complex numbers*. The calculation we have just performed shows that, in more usual notation,

$$\frac{a + ib}{c + id} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}. \quad (9)$$

The denominator,  $c^2 + d^2$ , is now an ordinary, common or garden, *real number* and we talk of "*realising* the denominator".

Now equation (9) is a formula. It tells us the standard form for the quotient of two complex numbers. It's the sort of thing many well-intentioned

students feel they need to memorise. Honestly, don't bother! I first saw this formula 45 years ago and I have never even tried to commit it to memory. Nor did I race to look it up just now as I wrote this part of the article. Knowing the *process* of realising the denominator, I worked it out as I went along.

But the full import of equation (9) lies not in its details, but in its theoretical standing. Its rather messy-looking right-hand side has the form  $A + iB$ . In other words, it too, like the  $a + ib$  and the  $c + id$  from which it arose, is a straightforward complex number.

This shows that complex numbers can be divided and that the result is just another complex number. This is actually a quite remarkable finding. If we try, for example, to divide integers, the result need not be an integer. We cannot divide vectors, and matrices cannot be "divided" without our introducing lots of qualifications. "Division Algebras" are rare species!

So what began as a computational convenience has now been shown to have quite important theoretical consequences.

But now we no longer need the computational convenience. Silicon "brains" don't care about getting bored. Routine mindless drill is OK by them and they're very largely immune to careless error.

Let's now get back to that first example. If I get my trusty little CASIO onto the job, I press  $\boxed{2} \boxed{\sqrt{\quad}}$  and so see the number 1.414213562 displayed. To get its reciprocal needs two more keystrokes  $\boxed{\text{INV}} \boxed{1/x}$  and presto! 0.707106781 comes up. Instantly. Were I to use the "rationalise the denominator" approach, I would go, once the 1.414213562 appeared:  $\boxed{\div} \boxed{2} \boxed{=}$ . And, of course, get the same answer, but less efficiently: 3 keystrokes instead of 2.

So nowadays rationalising the denominator doesn't have the same urgent appeal it once possessed. To show how the computer makes different demands, let me give you a final example. I'll start with a very simple equation

$$x - 1 = 0. \quad (10)$$

Very obviously, the solution is  $x = 1$ .

But now suppose that equation (10) isn't quite the correct equation for the purpose at hand. More accurate is, let us say,

$$\varepsilon x^2 + x - 1 = 0, \quad (11)$$

where  $\varepsilon$  is very small. (This isn't, incidentally, a piece of pure mathematical quibbling; whole areas of economic modelling use equations *very like* equation (11).)

Now equation (11) is a quadratic equation and it will have two solutions given by the standard formula, which in this case goes

$$x = \frac{-1 \pm \sqrt{1 + 4\varepsilon}}{2\varepsilon}. \quad (12)$$

This equation's roots depend on the value of  $\varepsilon$ , which we have supposed to be small. Surely, we would expect, and rightly as it turns out, one of these roots would be near  $x = 1$ . In fact, theory tells us that *this* root is approximately  $1 - \varepsilon$ , as you can check by substituting back into equation (11). (In the economic context, this is the root of importance.)

But now comes the rub. If  $\varepsilon$  is small enough, then the computer, which always stores numbers as *finite* strings of digits, will see  $1 + 4\varepsilon$  as no different from the number 1. So the computer will "think" that the roots are  $\frac{1}{2\varepsilon}(-1 \pm \sqrt{1})$ : in other words, 0 and  $-\frac{1}{\varepsilon}$ . The  $-\frac{1}{\varepsilon}$  is the unimportant root, which the computer has got reasonably well. The other root is *very* badly wrong!

We do much better by taking equation (12) and doing what would once have been unthinkable. We rationalise the *numerator* and put the square root down below. This gives

$$x = \frac{2}{1 \pm \sqrt{1 + 4\varepsilon}}. \quad (13)$$

(I leave it to readers to check for themselves that this is the correct answer.)

And now suppose the computer fails to "see" that little  $\varepsilon$ ? Well, no great matter. The uninteresting root wanders off to infinity, where we may safely leave it. The other is simply given as 1, which is much, much better than 0 as an estimate of the true value.

This example has many more riches, which well repay explanation. I leave that to my readers.

But I hope you will agree that, between equations (12) and (13), the computer has turned mathematics upside down!

# COMPUTERS AND COMPUTING

## Crocheting the Modern Way

Cristina Varsavsky

I have always been amazed at the many different patterns that can be created with a crochet hook drawing the yarn into intertwined loops. I used to spend many hours watching my grandmother as she was magically creating in front of my eyes daisies, roses, bells, . . . I even had a go at it myself, but that was many years ago when life did not seem to be as busy as it is these days.

One of this winter's Sunday afternoons I was observing a pattern produced by my grandmother from a mathematician's point of view. I discovered that it could be described as a pattern with a self-similarity property, that is, as the first few iterations in the process of the creation of a fractal. So I decided to use the computer rather than the crochet hook to reproduce it. It turned out to be pretty simple; the screen output of the program I wrote appears in Figure 1.

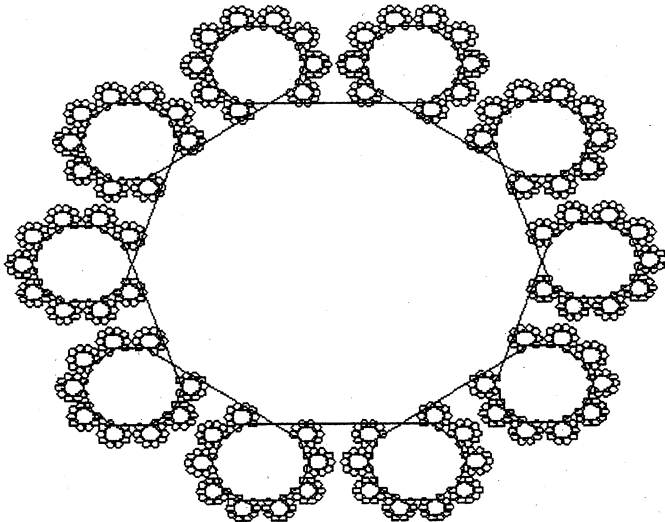


Figure 1

The basic figure is a regular polygon with ten sides – known as a *decagon*. A smaller decagon appears at each vertex, which in turn, has an even smaller decagon branching off each vertex that is not part of the bigger decagon. This process could in theory continue to create the crochet fractal, but in practice only the first few iterations can be appreciated with our eyes.

The computer code is rather straightforward. To plot a decagon on the screen we draw 10 consecutive segments of a fixed length, say 1 unit. The angle made by two consecutive segments of the decagon is  $2\pi/10 = \pi/5$  radians. This is achieved with the following program written in QuickBasic:

```
SCREEN 9: WINDOW (-1.5, -3)-(2.5, 2.7)
pi = 3.141593
x = 0: y = 0: PSET (x, y)
FOR n = 0 TO 9
    angle = n * pi / 5
    x = x + COS(angle)
    y = y + SIN(angle)
    LINE -(x, y)
NEXT n
```

The dimensions of the window have been selected using the 4:3 screen ratio, so that the decagon fits nicely in the centre of the screen. Note that the angle increases by  $\pi/5$  radians at each vertex.

To add the smaller decagons at each vertex, we proceed as with crocheting using only one piece of yarn – mathematically speaking, drawing a closed curve figure. The computer program follows:

```
SCREEN 9: WINDOW (-1.9, -2.8)-(2.9, .8)
pi = 3.141593
x = 0: y = 0: PSET (x, y)
FOR n = 0 TO 89
    vertex = n: angle = n * pi / 5
    IF vertex MOD 9 <> 0 THEN factor = .2 ELSE factor = 1
    x = x + factor * COS(angle)
    y = y + factor * SIN(angle)
    LINE -(x, y)
NEXT n
```

The figure is a closed curve made of 90 segments; the vertices of the bigger decagon are formed with the intersections of the longer line segments, so for each vertex we need to draw only 8 segments to produce the smaller decagon. The length of the segment is controlled by the IF statement: the multiples of 9 will correspond to the sides of the larger decagon, the non-multiples are for the smaller decagons. Note that because of the direction in which the computer “needlework” is done, we need to change the definition of the window.

Following the same process we can add more iterations towards the crocheted fractal:

```

SCREEN 9: WINDOW (-1.9, -2.8)-(2.9, .8)
pi = 3.141593: factor = .2: iterations = 4
x = 0: y = 0: PSET (x, y)
FOR n = 0 TO 10 * 9 ^ (iterations - 1) - 1
  vertex = n: level = 0: angle = n * pi / 5
  WHATLENGTH:
  IF vertex MOD 9 <> 0 OR level >= iterations - 1
    THEN GOTO DRAWING
  vertex = vertex / 9: level = level + 1
  GOTO WHATLENGTH
DRAWING:
  length = factor ^ (iterations - 1 - level)
  x = x + length * COS(angle)
  y = y + length * SIN(angle)
  LINE -(x, y)
NEXT n

```

The variable *iterations* is set to 4, but you can change it. The IF statement is to check to what level of iteration the next segment belongs: if *n* has only one factor 9, then it is the first level; if it has two factors 9, the second; if it has 3, the third, and so on.

By varying the number of sides of the starting polygon and the scaling factor defining the length of the line segments, I discovered that my computer can be as creative as my grandmother was: all sorts of new shapes appeared on the screen. Perhaps you'd like to give it a try too.

## LETTER TO THE EDITOR

I have finally found time to read the April issue of *Function* and was interested in the letter of Kim Dean. Dr Fwls is obviously a good lateral thinker, but he has made a mistake that a decent year 9 student wouldn't make. I agree that when adding a series of numbers you can add the terms in any order, but eventually all the terms must be included.

If you include only the first  $4n$  terms of the series in his work you can more easily see Dr Fwls's error.

$$\begin{aligned} L &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{4n} \\ 2L &= 2 - \frac{2}{2} + \frac{2}{3} - \frac{2}{4} + \dots - \frac{2}{4n} \\ &= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \dots - \frac{1}{2n}; \end{aligned}$$

if  $n = 2$  then

$$\begin{aligned} 2L &= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{2}{5} + \frac{2}{7} \\ &\approx L + \frac{2}{2n+1} + \frac{2}{4n-1}; \end{aligned}$$

if  $n = 3$  then

$$\begin{aligned} 2L &= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{2}{7} + \frac{2}{9} + \frac{2}{11} \\ &\approx L + \frac{2}{2n+1} + \frac{2}{2n+3} + \frac{2}{4n-1} \end{aligned}$$

In general,

$$\begin{aligned} 2L &= L + \frac{2}{2n+1} + \frac{2}{2n+3} + \dots + \frac{2}{4n-1} \\ L &= \frac{2}{2n+1} + \frac{2}{2n+3} + \dots + \frac{2}{4n-1} \end{aligned}$$

The terms on the right are all positive, and as there are  $n$  terms then

$$\begin{aligned} n \cdot \frac{2}{4n-1} &< \frac{2}{2n+1} + \frac{2}{2n+3} + \dots + \frac{2}{4n-1} < n \cdot \frac{2}{2n+1} \\ \frac{2}{4 - \frac{1}{n}} &< \frac{2}{2n+1} + \frac{2}{2n+3} + \dots + \frac{2}{4n-1} < \frac{2}{2 + \frac{1}{n}}. \end{aligned}$$



As  $n$  approaches infinity

$$\frac{1}{2} < \frac{2}{2n+1} + \frac{2}{2n+3} + \dots + \frac{2}{4n-1} < 1$$

Therefore  $L$  lies between the values .5 and 1, and is not 0, as Dr Fwls would have us think.

Peter Bullock  
Norwood Secondary College  
Ringwood

\* \* \* \* \*

Mathematics may be likened to a large rock whose interior composition we wish to examine. The older mathematicians appear as persevering stone cutters slowly attempting to demolish the rock from the outside with hammer and chisel. The later mathematicians resemble expert miners who seek vulnerable veins, drill into these strategic places, and then blast the rock apart with well placed internal charges.

Howard W Eves in *Mathematical Circles*  
1969, Prindle, Weber and Schmidt

\* \* \* \* \*

"I think you're begging the question," said Haydock, "and I can see looming ahead one of those terrible exercises in probability where six men have white hats and six men have black hats and you have to work it out by mathematics how likely it is that the hats will get mixed up and in what proportion. If you start thinking about things like that, you would go round the bend. Let me assure you of that!"

Agatha Christie in *The Mirror Crack'd*  
1962, Bantam Books

\* \* \* \* \*

## PROBLEM CORNER

### SOLUTIONS

#### PROBLEM 21.3.1

Does the pattern observed in the following sums continue?

$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

#### SOLUTION

We need to establish first just what the pattern is. For each natural number  $n$ , row  $n$  has  $n + 1$  numbers on the left and  $n$  on the right, a total of  $2n + 1$  numbers. The total number of numbers in the first  $n$  rows is therefore:

$$3 + 5 + 7 + \dots + (2n + 1)$$

This expression is an arithmetic series<sup>1</sup> with sum  $n^2 + 2n$ . Therefore the numbers in the first  $n$  rows can be listed as  $1, 2, 3, \dots, n^2 + 2n$ . Hence, row  $n$  ends with  $n^2 + 2n$ . It follows that row  $n - 1$  ends with  $(n - 1)^2 + 2(n - 1)$ , so row  $n$  begins with  $(n - 1)^2 + 2(n - 1) + 1$ , which equals  $n^2$ . Thus, the left-hand side of row  $n$  is:

$$n^2 + (n^2 + 1) + (n^2 + 2) + \dots + (n^2 + n) \quad (1)$$

and the right-hand side of row  $n$  is:

$$(n^2 + n + 1) + (n^2 + n + 2) + (n^2 + n + 3) + \dots + (n^2 + n + n) \quad (2)$$

Expressions (1) and (2) are both arithmetic series, with sums  $(n + 1)(2n^2 + n)/2$  and  $n[2(n^2 + n + 1) + (n - 1)]/2$  respectively. A little routine algebra reveals that both expressions equal  $n(n + 1)(2n + 1)/2$ . Therefore (1) and (2) are equal, as required.

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<sup>1</sup>For the benefit of readers who have not studied series, an *arithmetic series* is an expression of the form  $a + (a + d) + (a + 2d) + \dots + [a + (n - 1)d]$ , where  $a$  is the first term and  $d$  is the common difference. The series evaluates to  $n[2a + (n - 1)d]/2$ , as may be proved by induction on  $n$ .

PROBLEM 21.3.2 (from a British television game show; submitted by Prof H C Bolton, University of Melbourne)

Use each of the numbers 3, 6, 25, 50, 75, 100 exactly once, together with the four basic operations of arithmetic (+, −, ×, /) and parentheses, to obtain an expression equal to 952.

SOLUTION supplied by Prof H C Bolton

One possible solution is  $[(100 + 6) \times 3 \times 75 - 50]/25$ .

PROBLEM 21.3.3

Prove that  $x^x y^y \geq x^y y^x$  for all positive real values of  $x$  and  $y$ , with equality holding only if  $x = y$ .

SOLUTION

If  $x = y$  then clearly  $x^x y^y = x^y y^x$ . Thus we need only show that if  $x > y$  or  $x < y$  then  $x^x y^y > x^y y^x$ . We consider the two cases.

**Case 1:**  $x > y$ . Then  $x - y > 0$ . Raise both sides of the inequality  $x > y$  to the power of the positive number  $x - y$ , to obtain  $x^{x-y} > y^{x-y}$ . Now multiply both sides by  $x^y y^y$  to obtain the required result.

**Case 2:**  $x < y$ . Then  $y - x > 0$ . Raise both sides of the inequality  $x < y$  to the power of the positive number  $y - x$ , to obtain  $x^{y-x} < y^{y-x}$ . Now multiply both sides by  $x^x y^x$  to obtain the required result.

PROBLEM 21.3.4

Prove that  $\sin(\cos x) < \cos(\sin x)$  for all real values of  $x$ .

SOLUTION

Both sides of the inequality are periodic with period  $2\pi$ , so we need only look at values of  $x$  within the interval  $[-\pi, \pi]$ . Further, it is readily checked that both sides are even functions<sup>2</sup>, so it is sufficient to prove the assertion for  $x$  in  $[0, \pi]$ . The fact that  $\cos$  is a decreasing function on  $[0, \pi]$  is used in two places in the proof below.

**Case 1:**  $0 \leq x < \pi/2$ . In this case  $\cos x > 0$ , so  $\sin(\cos x) < \cos x$ . Also, since  $\sin x \leq x$  and both  $x$  and  $\sin x$  are in  $[0, \pi]$ , we must have  $\cos x \leq \cos(\sin x)$ . Therefore  $\sin(\cos x) < \cos(\sin x)$ .

<sup>2</sup>A function  $f$  is *even* if  $f(-x) = f(x)$  for all values of  $x$ .

**Case 2:**  $\pi/2 \leq x \leq \pi$ . In this case,  $\cos x \leq 0$ , so  $\sin(\cos x) \leq 0 < \cos 1$ . Since  $\sin x \leq 1$ , and both  $\sin x$  and 1 are in  $[0, \pi]$ , we have  $\cos 1 \leq \cos(\sin x)$ . Therefore  $\sin(\cos x) < \cos(\sin x)$  in this case also.

**PROBLEM 21.3.5** (from *Mathematical Digest*, University of Cape Town)

Find all real numbers  $x$  for which  $3^x + 4^x = 5^x$ .

**SOLUTION** from *Mathematical Digest*

Clearly  $x = 2$  is a solution. Dividing by  $4^x$  we have  $\left(\frac{3}{4}\right)^x + 1 = \left(\frac{5}{4}\right)^x$ . The function on the left-hand side is decreasing, that on the right-hand side is increasing. Hence there is at most one solution.

**PROBLEM 21.3.6** (from *Mathematical Spectrum*)

A polynomial function of degree  $n$  is such that  $p(x) \geq 0$  for all  $x$ . Prove that

$$p(x) + p'(x) + p''(x) + \dots + p^{(n)}(x) \geq 0$$

for all  $x$ .

**SOLUTION** from *Mathematical Spectrum*

Write

$$f(x) = p(x) + p'(x) + p''(x) + \dots + p^{(n)}(x).$$

Since  $p(x) \geq 0$  for all  $x$ ,  $n$  must be even and the leading coefficient of  $p(x)$  is positive. So  $f$  is also of even degree and has a positive leading coefficient. Thus, if  $f(x)$  takes its least value at  $x = X$ , then  $f'(X) = 0$ . Now,

$$f(x) = f'(x) + p(x).$$

Hence  $f(X) = p(X) \geq 0$ . Hence  $f(x) \geq f(X) \geq 0$  for all  $x$ .

We received solutions to Problems 21.3.1, 21.3.3, 21.3.4, 21.3.5 and 21.3.6 from Carlos Alberto da Silva Victor (Nilópolis, Brazil), and solutions to Problems 21.3.1, 21.3.3, 21.3.4 and 21.3.5 from Keith Anker.

## PROBLEMS

*Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received by 23 February 1998 will be acknowledged in the April 1998 issue, and the best solutions will be published.*

## PROBLEM 21.5.1 (K R S Sastry, Bangalore, India)

Show that the infinite arithmetic progression 2, 7, 12, 17, ... does not contain any triangular number. (The  $r$ th triangular number,  $T_r$ , is given by  $T_r = \frac{1}{2}r(r+1)$ .)

## PROBLEM 21.5.2 (K R S Sastry, Bangalore, India; modified by the editors)

Let  $ABC$  be an isosceles triangle with  $AB = AC$ , and let  $O$  be its circumcentre. Let  $D$  be a point on  $\overline{AB}$  such that  $AD = BC$ . If  $C, O$  and  $D$  are collinear, prove that  $\angle BAC = 20^\circ$ . (Hint: It is possible to construct a proof without using coordinate geometry or trigonometry. Begin by finding the degree measure of  $\angle BDC$ .)

## PROBLEM 21.5.3 (K R S Sastry, Bangalore, India)

Let  $\overline{AD}$  be an altitude of triangle  $ABC$ . If the reciprocals of the lengths  $AD, AB$  and  $AC$  are the side lengths of a right-angled triangle, find the maximum value of the degree measure of angle  $BAC$ .

## PROBLEM 21.5.4

A list of finite sequences of numbers is constructed as follows. The first sequence is just the number 1. Each sequence apart from the first "describes" the previous sequence in the following sense: the first number in the previous sequence is listed, followed by the number of times it occurs consecutively, then the next number is listed, followed by the number of times it occurs consecutively, and so on.

Thus:

First sequence: 1

Second sequence: 1, 1 (since the first sequence can be described as "one 1")

Third sequence: 2, 1 (since the second sequence is "two 1s")

Fourth sequence: 1, 2, 1, 1 (since the third sequence is "one 2, one 1")

Fifth sequence: 1, 1, 1, 2, 2, 1 (since the fourth sequence is "one 1, one 2, two 1s")

etc.

Which numbers can occur as terms in these sequences?

PROBLEM 21.5.5 (from *Mathematical Spectrum*).

Find all prime numbers  $p$  for which  $2p - 1$  and  $2p + 1$  are both prime.

Many readers will have seen examples of fallacious “proofs” that  $0 = 1$ . Most of these proofs involve an attempt to divide both sides of an equation by an expression that turns out to be zero. The problem below is a little more subtle.

PROBLEM 21.5.6

Find the mistake in the following reasoning:

“Let  $I = \int \frac{1}{x} dx$ . Integrate by parts:

$$I = \int 1 \times \frac{1}{x} dx = x \times \frac{1}{x} - \int x \left( -\frac{1}{x^2} \right) dx = 1 + \int \frac{1}{x} dx = 1 + I$$

Now subtract  $I$  from both sides to obtain  $0 = 1$ .”

\* \* \* \* \*

## OLYMPIAD NEWS

**The XXXVIII International Mathematical Olympiad  
Two firsts: Australia finishes with two gold medals  
and is now among the top ten IMO countries in the world**

*Mar del Plata (Argentina) was this year's venue for the IMO. Teams, most having six members, from 82 countries had to contend with six problems during nine hours spread equally over two days in succession.*

*Here are the two papers:*

**First day, July 24, 1997<sup>3</sup>**

1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard). For any pair of positive integers  $m$  and  $n$ , consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths  $m$  and  $n$ , lie along edges of the squares. Let  $S_1$  be the total area of the black part of the triangle and  $S_2$  be the total area of the white part. Let

$$f(m, n) = |S_1 - S_2|.$$

<sup>3</sup>Each problem is worth 7 points. Time:  $4\frac{1}{2}$  hours.

- (a) Calculate  $f(m, n)$  for all positive integers  $m$  and  $n$  which are either both even or both odd.
- (b) Prove that  $f(m, n) \leq \frac{1}{2} \max\{m, n\}$  for all  $m$  and  $n$ .
- (c) Show that there is no constant  $C$  such that  $f(m, n) < C$  for all  $m$  and  $n$ .
2. Angle  $A$  is the smallest in the triangle  $ABC$ . The points  $B$  and  $C$  divide the circumcircle of the triangle into two arcs. Let  $U$  be an interior point of the arc between  $B$  and  $C$  which does not contain  $A$ . The perpendicular bisectors of  $AB$  and  $AC$  meet the line  $AU$  at  $V$  and  $W$ , respectively. The lines  $BV$  and  $CW$  meet at  $T$ . Show that

$$AU = TB + TC.$$

3. Let  $x_1, x_2, \dots, x_n$  be real numbers satisfying the conditions:

$$|x_1 + x_2 + \dots + x_n| = 1$$

and

$$|x_i| \leq \frac{n+1}{2} \quad \text{for } i = 1, 2, \dots, n.$$

Show that there exists a permutation  $y_1, y_2, \dots, y_n$  of  $x_1, x_2, \dots, x_n$  such that

$$|y_1 + 2y_2 + \dots + ny_n| \leq \frac{n+1}{2}.$$

### Second day, July 25, 1997

4. An  $n \times n$  matrix (square array) whose entries come from the set  $S = \{1, 2, \dots, 2n-1\}$ , is called a *silver* matrix if, for each  $i = 1, \dots, n$ , the  $i$ th row and the  $i$ th column together contain all elements of  $S$ . Show that
- (a) there is no silver matrix for  $n = 1997$ ;
- (b) silver matrices exist for infinitely many values of  $n$ .
5. Find all pairs  $(a, b)$  of integers  $a \geq 1, b \geq 1$  that satisfy the equation

$$a^{(b^2)} = b^a.$$

6. For each positive integer  $n$ , let  $f(n)$  denote the number of ways of representing  $n$  as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are

considered to be the same. For instance,  $f(4) = 4$ , because the number 4 can be represented in the following four ways:

$$4; 2 + 2; 2 + 1 + 1; 1 + 1 + 1 + 1.$$

Prove that, for any integer  $n \geq 3$ :

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

*Never has an Australian team performed so well at an IMO. Australia came ninth and won two gold and three silver medals as well as one bronze medal. Ahead of Australia were only the teams (listed in descending order of their unofficial scores) from China, Hungary, Iran, Russia and the United States of America (ex aequo), the Ukraine, and Bulgaria and Romania (ex aequo).*

*The other countries of the Asian Pacific Mathematics Olympiad ranked as follows: 10. Vietnam, 11. Korea, 12. Japan, 14. Taiwan, 27. Colombia, 29. Canada, 30. Hong Kong - China, 32. Mexico, 37. Argentina, 39. South Africa, 41. Singapore, 48. New Zealand, 52. Thailand, 63. Indonesia, 65. Trinidad and Tobago, 66. Chile, 69. Malaysia, 73. the Philippines.*

*The members of the Australian team received medals as follows:*

Thomas Lam, New South Wales, Gold

Stephen Farrar, New South Wales, Gold

Norman Do, Victoria, Silver

Justin Ghan, South Australia, Silver

Daniel Mathews, Victoria, Silver

Jonathan Kusilek, New South Wales, Bronze.

*Congratulations to our excellent team!*

**The 1997 Senior Contest of the  
Australian Mathematical Olympiad Committee (AMOC)**

*The AMOC Senior Contest is the first hurdle for mathematically talented Australian students who wish to qualify themselves for membership of the*



team that represents Australia in the following year's International Mathematical Olympiad. This year that four-hour competition took place on 13 August.

These are the questions:<sup>4</sup>

1. On planet Rhinochromos, 19971997 male monsters are to be married to the same number of female monsters. The same number of males and females have purple noses, the others have beige noses. The matching of males and females is performed randomly by Rhinochromos government monster psychologists. Show that the number of mixed marriages, i.e. marriages of partners with different nose colours, is even.
2. Let  $c$  be a circle,  $A$  a point on  $c$ , and  $B$  and  $C$  two different points on  $c$  such that the chord  $BC$  is parallel to the tangent to  $c$  through  $A$ . Let  $P$  be on  $BC$  and let  $AP$  (extended) intersect  $c$  again in  $Q$ . Let  $k$  be the circle which touches  $BC$  in  $P$  and passes through  $Q$ . Prove that  $k$  touches  $c$  in  $Q$ .
3. Determine all pairs of integers  $(x, y)$  that satisfy the equation

$$(x + 1)^4 - (x - 1)^4 = y^3.$$

4. Let  $c$  be a circle and let  $P$  be a point in the interior of  $c$ .
  - (a) Let  $A, B, C, D$  be points on  $c$  such that the chords  $AC$  and  $BD$  intersect at right angles at  $P$ . Show that  $AC^2 + BD^2$  is constant (i.e. the same for any such four points  $A, B, C, D$ ).
  - (b) Amongst all such points  $A, B, C, D$ , show that  $AC + BD$  takes its maximum when  $AC = BD$ .
5. Let  $f$  be a function defined for all integers by

$$\begin{aligned} f(0) &= 1, \\ f(n) &= 0, \quad \text{for } -5 \leq n \leq -1, \\ f(n) &= f(n-6) + n, \quad \text{for all integers } n. \end{aligned}$$

Prove that  $\frac{(n+1)(n+5)}{12} \leq f(n) \leq \frac{n^2+6n+12}{12}$ , for all integers  $n$ .

\* \* \* \* \*

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<sup>4</sup>Time allowed: 4 hours. NO calculators are to be used. Each question is worth seven points.

## INDEX TO VOLUME 21

Title	Author	Part	Page
Bible and $\pi$ , The	M Deakin & H Lausch	2	59
Book Review: Evariste Galois, 1811-1832	H Lausch	2	55
Bow and Arrow in the Groves of Ind	M Deakin & O Steinmayer	1	12
Construction of a Regular Pentagon	F Mifsud & K Spiteri	3	78
Cracking the Combination The Good Way!	P Grossman	1	7
Crocheting the Modern Way	C Varsavsky	5	159
Ethnomathematics	M Deakin	3	89
Example of a Mathematical Model, An	R Adlem	2	44
Figure of the ABC, The	B Bolton	2	42
Flying through the Air: Golfballs, Arrows and Cubic Equations	M Komisarczyk	5	142
Golden Rectangle and its Rectangular Spiral, The	B Bolton	1	2
Hailstone Numbers	I McDowell & K Evans	2	50
How Far is the Horizon?	B Bolton	5	147
Leaning Ladders	R King	4	117
Magic Squares at Random?	M Clark	5	149
Mathematics and the Internet	C Varsavsky	4	129
Modelling Shapes with Polynomials	C Varsavsky	1	27
Non-Euclidean Geometry, A	S Ng	4	110
Olympiad News	H Lausch	1	40
		2	75
		3	106
		5	168
Right before our Very Eyes	M Deakin	5	154
"Rule of 72", The	P Grossman	3	86
Saving Space	C Varsavsky	2	65
Search for a Pattern, The	C Varsavsky	3	96
Special Irrational Numbers	M Deakin	3	81
St Pavel Florensky and Others	M Deakin	1	21
Thinking Other People's Thoughts	M Deakin	4	122

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