Function

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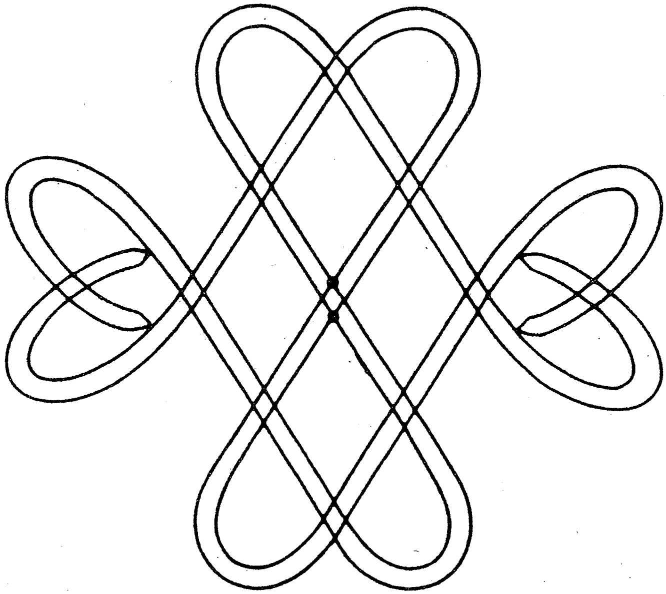

Department of Mathematics - Monash University

Function is a refereed mathematics journal produced by the Department of Mathematics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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*\$10 for *bona fide* secondary or tertiary students.

EDITORIAL

The figure on the front cover is an illustration from our *History Column* about ethnomathematics – a term which is very fashionable these days. It is a Malekula sand-drawing design using four-fold replication of a basic figure, which is to be drawn without lifting the finger from the sand. Michael Deakin uses this design to question the way its mathematical significance is sometimes presented to give dignity to some cultures normally not regarded as of advanced numeracy.

We include three feature articles in this issue of *Function*. F Mifsud and K Spiteri present a classical method – already known by the ancient Greeks – for constructing a regular pentagon, using only an ungraduated ruler and a compass. M Deakin, observing a special number pattern generated using the golden ratio which was presented in a previous article about Wythoff's game, investigates the patterns that emerge when other irrational numbers such as $\sqrt{2}$ are used. Peter Grossman gives a mathematical explanation for the "Rule of 72", which is often used to find an approximation to the doubling time of anything that is growing.

The *Computers and Computing* article gives an insight into how searching algorithms work to find a pattern in text documents, World Wide Web information pages, or CD-ROMs. It also outlines the structure of a simple algorithm which you may want to implement in your preferred programming language.

Many thanks to all readers who sent solutions to our problems. We publish all those that reach us before the due date. As usual, we include a few new problems in the *Problem Corner* for your entertainment. We also challenge you with the problems set for the participants of the *Ninth Asian Pacific Mathematics Olympiad.*

We hope you find in this issue of *Function* many interesting and enjoyable items.

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CONSTRUCTION OF A REGULAR PENTAGON

F Mifsud and K Spiteri

A method for the construction of a regular pentagon, using ruler (ungraduated) and compass, was known to the ancient Greek geometers. It is given in Euclid's *Elements* (Euclid: approx. 330–275 BC), a collection of mathematics books, some of which were used as school textbooks to the beginning of this century.

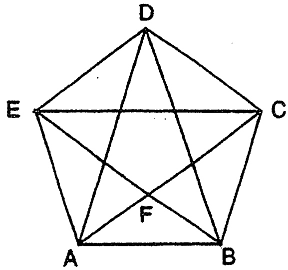

Figure 1

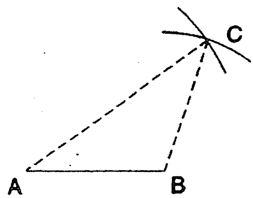

Figure 2

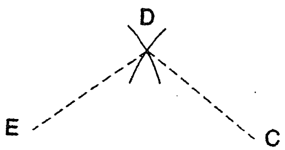

Figure 3

The construction procedure shown in this article[1] depends on first determining the length of a diagonal in terms of the length of a side. (The five diagonals are shown in Figure 1.) Once this is done, a side, say $\overline{AB}$, is drawn with any given length. Another vertex, C, is then determined by the intersection of an arc with centre at B and radius length AB, and an arc with centre A and radius length equal to the diagonal length (as in Figure

[1] A slightly different version of this construction was given in *Function, Vol* 15 *Part* 5.

2). Similarly the point E can be constructed. Finally D can be constructed as the intersection of arcs with centres E, C and radius length AB (Figure 3).

Now let $AB = 1$ unit. It remains to determine the diagonal length, $AC = x$ units, and then to construct an interval of this length. Let $AF = y$ units, where F is as shown in Figure 1. Some properties of the regular pentagon are required. By symmetry, all diagonals have the same length (as we have already assumed) and each diagonal is parallel to one side of the regular pentagon. In particular, in Figure 1,

$$\overline{EC} \parallel \overline{AB}, \quad \overline{ED} \parallel \overline{AC}, \quad \overline{EB} \parallel \overline{DC}.$$

From the second and third of the pairs of parallel intervals, it follows that $EFCD$ (Figure 1) is a parallelogram (in fact, a rhombus). From the first pair of parallel intervals, it is easy to show that the triangles EFC and BFA are similar. Hence

$$\begin{aligned} \frac{EC}{AB} &= \frac{FC}{AF} && (1)\\ &= \frac{DE}{AF} && (FC = DE \text{ from parallelogram } EFCD)\\ \text{so } \frac{x}{1} &= \frac{1}{y} && (AF = y \text{ units, } AB = DE = 1 \text{ unit})\\ \text{therefore } y &= \frac{1}{x} \end{aligned}$$

Now

$$\begin{aligned} AF + FC &= AC\\ \text{so } \frac{1}{x} + 1 &= x \quad (FC = DE = 1 \text{ unit})\\ \text{therefore } 1 + x &= x^2\\ \text{and } x^2 - x - 1 &= 0\\ \text{therefore } x &= \frac{1+\sqrt{5}}{2}\\ &\quad (x \text{ is the positive root of the quadratic equation}) \end{aligned}$$

The construction of a line interval of length $\frac{1+\sqrt{5}}{2}$ units is described in stages below and is shown in Figure 4.

First draw $\overline{AB}$ one unit long (a side of the pentagon). With centres A, B and the same radius length, draw arcs to intersect at R and S. Let $\overleftrightarrow{RS}$, the perpendicular bisector of $\overline{AB}$, intersect $\overline{AB}$ at M, so $AM = MB = \frac{1}{2}$ unit.

With centre M and radius length $AB = 1$ unit, draw an arc to intersect $\overleftrightarrow{RS}$ at N. By the theorem of Pythagoras,

$$BN = \sqrt{1^2 + (1/2)^2} \text{ units} = \frac{\sqrt{5}}{2} \text{ units.}$$

With centre M and radius length BN, draw an arc to intersect $\overrightarrow{AB}$ at P so $MP = \frac{\sqrt{5}}{2}$ units and hence $AP = (\frac{1}{2} + \frac{\sqrt{5}}{2})$ units $= \frac{1+\sqrt{5}}{2}$ units. Hence $\overline{AP}$ has the length of the diagonal of a regular pentagon with side length AB.

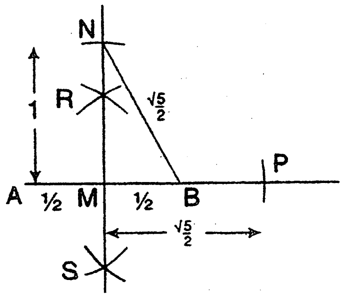

Figure 4

Note: From the ratio of equation (1), *viz* $\frac{FC}{AF} = \frac{EC}{AB}$, we obtain

$$\frac{FC}{AF} = \frac{x}{1}$$

When a point F divides a segment $\overline{AC}$ in this way, i.e. the ratio of the length of the larger part to that of the smaller part is equal to the ratio of the length of the whole to the larger part, we say that F divides $\overline{AC}$ in a *golden ratio* which is $\frac{\sqrt{5}+1}{2}$. The diagonals of a regular pentagon which intersect in the interior of the pentagon divide each other in golden ratio.

Reference: H S M Coxeter, *Introduction to Geometry*, . John Wiley & Sons, 1961, Chapter 11.

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SPECIAL IRRATIONAL NUMBERS

Michael A B Deakin

This article is a follow-up to my earlier article[1] "Number Patterns and Wythoff's Game". For the reader's convenience, some of that earlier article will be summarised here. Let $\alpha = \frac{1+\sqrt{5}}{2}$, and multiply α successively by 1, 2, 3, 4, ... This gives rise to the numbers:

$$1.618\ldots,\ 3.236\ldots,\ 4.854\ldots,\ 6.472\ldots,\ 8.090\ldots,$$

$$9.708\ldots,\ 11.326\ldots,\ 12.944\ldots,\ \text{etc.}$$

However, in each case, we take only the integral part: 1, 3, 4, 6, 8, 9, 11, 12, ... We now use *these* numbers to build up a table.

n	1	2	3	4	5	6	7	8	9	10	11	12	...
a_n	1	3	4	6	8	9	11	12	14	16	17	19	...
b_n	2	5	7	10	13	15	18	20	23	26	28	31	...
c_n	1	2	3	4	5	6	7	8	9	10	11	12	...

The numbers a_n are the integral parts just formed. This is expressed as $a_n = [n\alpha]$, where the square brackets are the symbol for "integral part of". The numbers b_n are just those integers that are skipped over in the line above. However, they may also be generated in another way. If we put $\beta = \alpha + 1$, and as can be verified $\frac{1}{\beta} + \frac{1}{\alpha} = 1$, we find $[n\beta] = b_n$. Finally, we have $c_n = b_n - a_n$. But we also have $c_n = n$.

This is a quite remarkable property of the number α, and indeed α is the *only* number for which this property holds.

If, however, we use another, but again special, value of α, namely $\sqrt{2}$, something rather similar happens, except that this time we get $c_n = 2n$. The previous article ended with a problem: to find values of α for which $c_n = 3n$, $c_n = 4n$, and so on. No-one sent us a solution, but I had a go at it myself and found something quite remarkable.

[1]See *Function, Vol* 19 *Part* 4.

Suppose we distinguish between the various values of α and write $\alpha_1 = \dfrac{1+\sqrt{5}}{2}$, $\alpha_2 = \sqrt{2}$, etc. In general, we will have a number α_m (say) and we also define another number β_m by $\dfrac{1}{\beta_m} + \dfrac{1}{\alpha_m} = 1$. Then we want

$$[n\beta_m] - [n\alpha_m] = mn. \tag{1}$$

The answer to the problem that was set is not all that difficult to find, although the proof that it is correct is somewhat technical, as is the proof that *no other number will do*. The answer is

$$\alpha_m = \left(1 - \frac{m}{2}\right) + \sqrt{\frac{m^2}{4} + 1}. \tag{2}$$

The justification is given in the Appendix.

But the thought that occurred to me is that, in one way of looking at the matter, α_1 is the *very simplest* irrational number there is.[2] This seemed to me to square with the fact that α_1 satisfies the very simplest case of equation (1). So I asked the question: is there a sense in which we can regard α_2, i.e. $\sqrt{(2)}$, as the second simplest irrational number and α_3 as the third simplest, and so on?

Well, I found that there was, although the answer was not quite as straightforward as I would have liked. To see how this worked, let us backtrack for a minute.

The reason that α_1 may be regarded as the very simplest irrational number is that we have:

$$\alpha_1 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}}} \tag{3}$$

where the "..." indicates that the process indicated continues *ad infinitum*.

The right-hand side of equation (3) is an object known as a "continued fraction"[3].

[2]For more on this, see *Function*, *Vol* 16 *Part* 5, pp. 133-139.

[3]For more on continued fractions, see *Function*, *Vol* 4 *Part* 4, *Vol* 11 *Part* 2. The particular continued fraction (3) appears in Problem 4.2.1.

A continued fraction is any expression of the form

$$a+\cfrac{b}{c+\cfrac{d}{e+\cfrac{f}{g+\cfrac{h}{i+\dots}}}}$$

where the letters represent integers. It may terminate or it may (as in the cases here being considered) go on forever. In the case where the numerators (i.e. $b, d, f, h, \ldots$) are all equal to 1, the expression is said to be a "simple continued fraction".[4]

I decided to look at "very simple continued fractions". In the very simplest case (equation (3) above), *all the numbers involved* are equal to 1. So this was what I set out to generalise. Here is how it goes.

Let $f(m)$ be given by the following expression, where once again the "..." indicates that the process indicated continues *ad infinitum.*

$$m+\cfrac{1}{m+\cfrac{1}{m+\cfrac{1}{m+\cfrac{1}{m+\dots}}}} \tag{4}$$

We then have $f(m) = m+\frac{1}{f(m)}$, fairly clearly. This is a quadratic equation in $f(m)$, namely $[f(m)]^2 - mf(m) = 1$, which we solve for $f(m)$. Its positive solution (obviously the one we want) is given by

$$f(m) = \frac{m}{2} + \sqrt{\frac{m^2}{4} + 1}. \tag{5}$$

Subtracting (5) from (2), we find

$$\alpha_m = f(m) + 1 - m = 1 + \cfrac{1}{m+\cfrac{1}{m+\cfrac{1}{m+\cfrac{1}{m+\dots}}}} \tag{6}$$

very like the form (4). So there is indeed a close relation between the numbers α_m and the very simple continued fractions $f(m)$.

[4]A continued fraction terminates if one of its numerators is zero.

Something similar happens if we look at β_m instead of α_m. The result is

$$\beta_m = f(m) + 1. \tag{7}$$

So neither β_m nor α_m is exactly the mth simplest irrational number as I have defined it, although both are very closely connected to it. For example, when $n = 2$, we find $f(2) = \sqrt{2}+1$ as the second simplest irrational number, while $\alpha_2 = \sqrt{2}$ and $\beta_2 = \sqrt{2}+2$.

Finally, we may compare equations (6) and (7) and so find

$$\beta_m = m + \alpha_m. \tag{8}$$

This equation is interesting in its own right and will come in useful in the technical discussion that follows.

Appendix

We here show that the form given by equation (2) indeed satisfies condition (1) and that no other value will do.

First up, note that

$$\beta_m = \frac{\alpha_m}{\alpha_m - 1}. \tag{A1}$$

Equation (1) is rather hard, so we will first simplify it. So we consider equation (1), but simplify it by leaving out its most difficult feature – the square brackets. This gives us

$$\beta_m - \alpha_m = m$$

and from (A1)

$$\frac{\alpha_m}{\alpha_m - 1} - \alpha_m = m, \tag{A2}$$

an equation with two solutions, one positive and the other negative. The positive root is given by

$$\alpha_m = (1 - \frac{m}{2}) + \sqrt{\frac{m^2}{4} + 1} \tag{A3}$$

and we may now use equation (A1) to show that with this value of α_m, equation (8) holds. We next need to show that this value also satisfies equation (1). This is not difficult; in fact, it follows immediately from equation (8). We have:

$$[n\beta_m] - [n\alpha_m] = [nm + n\alpha_m] - [n\alpha_m] = nm + [n\alpha_m] - [n\alpha_m] = nm.$$

(The second equality holds because nm is a positive integer.)

So the value of α_m given by equation (A3) is a solution to equation (1). All that remains now to prove is that it is the *only* one.

Suppose we tried to replace α_m, as given by equation (A3), by some other value, $\alpha_m + \varepsilon$, let us call it. Then β_m will have to be replaced by some other value, $\beta_m + \eta$, say, where

$$\frac{1}{\alpha_m + \varepsilon} + \frac{1}{\beta_m + \eta} = 1.$$

We now want

$$[n(\beta_m + \eta)] - [n(\alpha_m + \varepsilon)] = nm$$

and, by the same process as was used above, we may reduce this equation to

$$[n(\alpha_m + \eta)] - [n(\alpha_m + \varepsilon)] = 0.$$

This last equation must hold for all values of n. Now unless $\eta = \varepsilon$, there will always be some (perhaps very large) value of n for which this equation will fail (because the difference between $\alpha_m + \varepsilon$ and $\alpha_m + \eta$, even though it may be very small, can be made to amount to more than 1 if it is multiplied by a sufficiently large number – indeed, by choosing *very* large values of n we may make the left-hand side as large as we please).

Thus we require $\varepsilon = \eta$. The only reasonable solution of this equation is $\varepsilon = 0$. (This follows from the derivation of equation (A3) or it may be discovered directly by means of some rather tedious algebra; the details are omitted.)

This then completes the proof.

Note

My editorial colleague Peter Grossman has drawn to my attention an interesting article on the subject matter of my earlier article. It is Chapter 12 of Ross Honsberger's book *Ingenuity in Mathematics* (published by the Mathematical Association of America as Volume 23 of their *New Mathematical Library*). Honsberger is a prolific writer of popular mathematics, some of whose work has appeared in *Function*. A theorem similar to that proved in the earlier article is there proved and given the name "Beatty's Theorem", after a Professor Sam Beatty, one of Honsberger's teachers and the first to state the theorem (although the first published proof was by Ostrowski and Aitken[5]).

His article goes on to provide much more detail and a list of further reading; it is recommended.

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[5] Aitken was the principal subject of the *History of Mathematics* column in *Function, Vol* 19 *Part* 4.

THE "RULE OF 72"

Peter Grossman

In an article by Jessica Mathews on population growth, printed in the *Washington Post* and reproduced in *The Age* (13/6/96), there appeared a reference to a formula known as the "Rule of 72". The author began the article by quoting some figures for the annual population growth rates of various countries. She then continued:

> "To find some concrete meaning in these abstractions, apply the invaluable Rule of 72 (to find the doubling time of anything that is growing, divide its rate of increase into 72)."

Where does this rule come from? Is it correct?

In order to answer these questions, we need to examine how the size of something increases as a function of time if it is growing at a constant rate. One example of this type of growth which you are probably familiar with is compound interest, but the same principles apply to anything with a constant growth rate.

Let us suppose, then, that we have something with an initial size of P_0. For compound interest, P_0 is the *principal* (the amount initially invested), while for population growth, P_0 is the initial size of the population. Suppose, furthermore, that the growth rate is r percent per annum. (For compound interest, r is of course the *interest rate.*) After one year, the original amount will have increased by a factor of $1+\frac{r}{100}$, so the size at the end of the one-year period, P_1, will be $P_0(1+\frac{r}{100})$. After two years, the size will have increased again by a factor of $1+\frac{r}{100}$, giving a size of $P_2 = P_0(1+\frac{r}{100})^2$. In general, after n years, the size will be given by the familiar compound interest formula:

$$P_n = P_0(1+\frac{r}{100})^n.$$

The Rule of 72 states that the time in which the size doubles, or in other words, the value of n when P_n equals $2P_0$, is $\frac{72}{r}$. Let's see what happens if we substitute $P_n = 2P_0$ in the compound interest formula and solve for n:

$$\begin{aligned} 2P_0 &= P_0(1+\frac{r}{100})^n \\ \therefore 2 &= (1+\frac{r}{100})^n \\ \therefore \log 2 &= n\log(1+\frac{r}{100}) \\ \therefore n &= \frac{\log 2}{\log(1+\frac{r}{100})} \end{aligned}$$

(The logarithms could be to any base.)

This is certainly not the Rule of 72! However, the possibility remains that the Rule of 72 might be a good approximation to the correct formula. We can test this claim by tabulating values of n given by the exact formula and the Rule of 72, for a range of values of r. This is done in Table 1, which was generated using a spreadsheet package.

r	**Exact**	**Rule of 72**	**% error**
2	35.00	36.00	2.8
4	17.67	18.00	1.9
6	11.90	12.00	0.9
8	9.01	9.00	-0.1
10	7.27	7.20	-1.0
12	6.12	6.00	-1.9
14	5.29	5.14	-2.8
16	4.67	4.50	-3.6
18	4.19	4.00	-4.5
20	3.80	3.60	-5.3

Table 1

By examining the table, we see that the Rule of 72 provides an overestimate of the true value for smaller values of r and an underestimate for larger values of r. (The crossover point is the value of r for which the Rule of 72 is exact, and the figures in the table suggest that this occurs for r just less than 8. As an exercise, you might like to calculate a better numerical approximation for this critical value of r.) The rule is reasonably accurate over most of this range of values, although we see it is starting to lose accuracy for the larger values of r in the last few lines of the table. For growth rates up to about 14 percent, the Rule of 72 appears to be accurate to within about 3 percent, which is good enough to be useful for many practical purposes. For higher growth rates, we need to exercise caution, as the Rule of 72 becomes increasingly inaccurate as r increases. When $r = 100$, for example, the exact answer is $n = 1$ (obviously), but the Rule of 72 gives $n = 0.72$, which is in error by a considerable amount.

Incidentally, you might wonder why the number 72 was chosen, rather than some other number. We can see that the Rule of 72 gives good approximations for growth rates that are typical of interest rates (and this is the context in which the rule was originally formulated). A "Rule of 70" would be more accurate than the Rule of 72 for lower growth rates, but less accurate for higher rates, as you may check; indeed, some older accounting texts, perhaps written at a time when interest rates tended to be lower, quote just such a rule. Clearly, any number around the value 70-72 will give a reasonable rule, and 72 was probably chosen because it has many divisors (thus allowing the calculation to be performed easily in one's head for a number of different rates).

If you are familiar with the use of a spreadsheet, you may wish to replicate the figures in Table 1 and extend them to a greater range of values of r. You could also use the software to draw graphs of the exact and approximate values as a function of r on the same pair of axes. You should be able to see how the two curves match closely for smaller values of r, but diverge for larger values.

The population growth rates in Jessica Mathews's article ranged from 3.2 percent to 4.9 percent. These rates fall well within the range of values for which the Rule of 72 provides a good estimate. Given that population growth rates are estimates anyway (and that they are unlikely to remain constant for very long), the use of the rule in the article is perfectly reasonable. Nevertheless, the author's assertion that the Rule of 72 applies to "anything that is growing", without any further qualification, was just a little careless.

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Newton was the greatest genius that ever existed, and the most fortunate, for we cannot find more than once a system of the world to establish.

– Joseph Louis Lagrange

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HISTORY OF MATHEMATICS

Ethnomathematics

Michael A B Deakin

A few months ago I came across an interesting story that I would like to share with my readers, but it also brought back into my mind other matters that have concerned me for some time and I want also to use this column to raise some of these more general issues.

1. A Somali Poem

I recently learned via the Internet of a Somali poem that has a mathematical flavour to it and which in particular can be related to the notion of a limit, so central to calculus. To reach the relevant site, first go to < http://www.dejanews.com/forms/dnq.html >. A dialogue box will then appear and under "SEARCH OPTIONS" click on "All" for the **Keywords Matched** and "Old" for the **Usenet Database**. Then, in the **Search For** box, type **somali fox calculus**. This will give you eight news items on the matter. The first part of this article is based on these postings, most especially the fourth, seventh and eighth.

They are concerned with a poem called **Qayb Libaax**, written in Somali, and associated with the Dervish movement.[1] The story line of the poem runs like this.

> The family of wild animals killed a camel and set about dividing the meat for their consumption. The lion (king of the beasts in Somali tradition as well as in our own) ordered the hyena to make a fair division. The hyena apportioned the flesh as follows: "One half for the king [lion] and the other half for the rest of us". This division displeased the lion, who punished the hyena, injuring its eye. The lion then asked the fox to take on the task – the fox being associated with cunning and opportunism in Somali tradition (again as also in our own). The fox produced a modified version of the hyena's apportionment: "One half of the camel meat for the

[1]The Dervishes are a branch of the Sufic (or mystic) stream of Islamic religion. They arose in the 12th and 13th centuries, and Dervish communities are still to be found today, despite the disapproval of mainstream Islamic thought. One of the postings describes the poem as "one of the last poems of the Dervish movement". I'm not quite sure what this means, but it may refer to the 13th century.

king [lion]; from the remainder, again one half for the king and so on". The lion then asked the fox "When did you learn this fairness and justice?", to which the fox replied "When I saw the injured eye of the hyena".

The mathematical point, of course, is that the lion gets everything. This can be discovered by summing a geometric series:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots=1 \tag{1}$$

and this is the route taken by some of those posting the various discussions.

What leaves me a little uneasy, however, is the somewhat exaggerated claims that some commentators seem to make for this fable. Among other things it is presented as an independent discovery of the paradoxes of Zeno.[2] This seems to me to claim far too much. In the first place, these latter constitute an elaborate and subtle argument as to the nature of Space and of Time. Are these to be thought of as continuous, and hence infinitely divisible, or else as composed of atom-like "places" and "instants"? There are four possibilities (each of Space and Time may or may not be infinitely divisible) and the four paradoxes are designed to show the impossibility of all of them. The conclusion we are invited to draw is that Space and Time are illusions.[3]

Thus the primary purpose of the Zeno analysis is metaphysical rather than mathematical; the mathematics is incidental, although important to the argument. Nonetheless, it is there, and at one point it comes very close to the point of the Somali poem.

The particular paradox in question is the first, known as the *Dichotomy*.[4] This takes Space to be infinitely divisible, in order to arrive at a contradiction. It may be presented in its starkest form by considering a journey. Before we can reach our destination, we must first reach the halfway point, and before we can reach *this*, we must reach the quarter-way point, etc. How could we ever get started? Now of course if we add up the half, the quarter and so on, then we get equation (1), and the sum of all these fractions (as an infinite series) is 1, the entire journey.

[2]The paradoxes of Zeno were discussed in this column of *Function*, *Vol* 14 *Part* 3 (June 1990).

[3]A similar point may be made in respect of a Buddhist version of one of these paradoxes; again see my earlier article.

[4]Meaning "division in two".

The important mathematical point, however, is that while we may *assign* the sum 1 to the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$, in a very important sense its sum is *not* 1. I remember puzzling over exactly this point as a child, and not ever managing to resolve it. For *no matter how many terms of the series we take, we always fall short of* 1.

The actual mathematical resolution of the question of the sum is quite subtle. If we take (say) n terms of the series, we may show that the sum is $1 - 2^{-n}$. (I leave the proof to the reader; the actual arithmetic is quite simple.) We *never do sum infinitely many terms*. Such a task would be impossible. Rather, we note that we can, *in only finitely many steps*, get not to 1 itself, but *arbitrarily close to it*. So then we say that we *assign* the value 1 to the infinite case. No value short of 1 will serve (and of course no value greater than 1 would make sense.) But this is a completely new sort of sum, and we've only had such sums for something less than 300 years. Certainly Zeno never considered this subtle logic; nor does the Somali poem.

Rather we, from a more informed standpoint, find this *implicit* in Zeno as also in the fable of the shared camel. If we are to do this with the latter, it is perhaps more useful not to consider the geometric series at all, but instead to proceed from a much more elementary consideration. *There is no provision for any of the other animals to get any of the meat*; this is the simple reason why the lion gets the lot. (At one point in the poem, the other animals complain to the fox on this very account.) If we want to put a "mathematical spin" on this insight, we may do so. After a finite number of meals, the lion has eaten $1 - 2^{-n}$ of the camel and 2^{-n} remains. No other animal has yet eaten, nor may any do so now, for half of what remains is the lion's. This applies whatever the value of n. At no point may any animal other than the lion touch the carcass.

But notice that this mathematical analysis is *my* interpretation of the situation; there is no vouch for it in the text of the poem. If we are to understand what *that* is saying, then we need other background. Now I know next to nothing of Somali history and culture, so what I am about to say is offered only very tentatively.

But it seems to me that the poem is making not a mathematical point, nor a metaphysical one (as Zeno was when he queried the nature of Space and Time); rather the point of the poem is *moral*. It concerns "fairness and justice". I see it as a (somewhat rueful) recognition that "might" can take precedence over "right" in this imperfect real world we inhabit. (The other animals complain, surely with justice, about the unfairness of the fox's ruling.) Even if this interpretation is not strictly correct, I would hold that *some such point* is the main thrust of the poem; its purpose is not primarily mathematical at all. (In the original Somali, the "mathematics" occupies only some $7\frac{1}{2}\%$ of the total. The rest is concerned with the situation itself, the actual happenings, the dialogue on "fairness and justice" and then with applications of the story to other situations.)

I give this story in detail in part because it is interesting and the poem certainly not widely known, partly because it gives me the chance to explain once again some very central mathematics, and also because it is a convenient springboard to the discussion of a more general question: the validity of "ethnomathematics".

2. Ethnomathematics

Much, and to be quite frank most, of today's mathematics is a clear product of Western culture as that culture has developed since (say) the days of Newton and Leibniz (around about 1700). Of course the work of these two great mathematicians and their contemporaries and successors elaborated an already rich tradition: Euclid, Archimedes and the other mathematicians of ancient Greece. This is the clear pattern of the main lines of mathematical endeavour.

However, the tapestry of mathematics is richer than this simple description allows. There was early work in Babylon and in Egypt, in China and in India. Other cultures (Hebrew, Japanese, Javanese, Korean, Mayan[5], Persian and Tibetan) also reached high levels of numeracy, but without having major influence on the mathematics of today.

Possibly there are others we could add to this list.[6] And certainly special mention should be made of the Arab mathematicians[7], who not only preserved much of the ancient Greek heritage, but also added to it in many meaningful ways (and whose influence *has* been felt in the mathematics we learn today).

All these traditions are clearly mathematical in that lengthy, involved and precise arguments are advanced by means of symbolic techniques, either written or embodied in some type of hardware.

---

[5]See the cover story in *Function, Vol* 12 *Part* 4.

[6]For instance, many people might include as mathematics the wonderful navigational feats of the Polynesians.

[7]See my History of Mathematics column in *Function*, *Vol* 15 *Part* 2.

I will refer to these cultures in what follows as "cultures of high numeracy". But it then follows that there are other cultures that are *not* of high numeracy. Some people find this a disturbing conclusion. I don't, I'm afraid. Rather, some cultures had a need to develop (applied) mathematics; others even had the leisure to go on to develop pure mathematics. But these factors are not by any means universal.

Thus the mathematics involved in establishing calendars is very important once a society becomes reasonably complex. For example, of the groups I mentioned above, the Javanese and the Mayans showed their greatest mathematical prowess in this area. However, if the society has a less complex structure, then accurate calendars are less important to it and may well not be developed.

It is the same with other areas of mathematics. We define concepts as we need them. So when I read that the *Ormu*[8] word for "nine" prior to outside contact was *nen-rohi-fraja-nitje-ma*, I deduce that the *Ormu* made little use of the concept "nine". Evidently their traditional way of life had little call for it.

However, this has disturbed some people and they see such analyses as this as demeaning to groups such as the *Ormu*. They feel that (e.g.) the *Ormu* must have had a mathematical tradition and that if we think otherwise, the fault is ours for not recognising it. This is one origin of the rise of Ethnomathematics. It has become very fashionable in recent years.

What researchers in this area present for our consideration are various traditional customs and artifacts that have a strongly "mathematical" flavour. These may be games, intricate patters and designs, numeration devices, methods of keeping trade tallies, clan systems regulating who may or may not marry, and probably other such aspects of the cultures involved.

There is, for example, a game called *Mancala* (Arabic for "transferring") which may be found in one form or another in many African countries. It is deceptively simple to describe but extremely difficult to play well. Many even uneducated Africans excel at it.

Then again, many cultures have intricate geometric art. For example, the *Malekula* of Vanuatu have intricate sand-patterns which are to be drawn without lifting one's finger from the sand. We may analyse this endeavour in terms of *graph theory*[9] which deals with exactly such questions. See Figure 1

[8]A language from Irian Jaya.

[9]See *Function, Vol* 13 *Part* 1, pp. 20-27.

which shows a Malekula design produced by the four-fold replication of the element shown to the right of the full pattern.

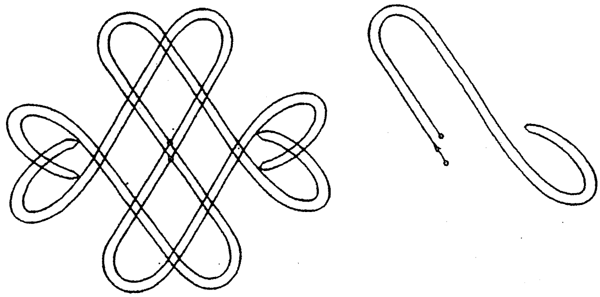

Figure 1. A Malekula sand-drawing (left), and (right) the basic unit which is replicated. It may be traced by following the direction of the arrow. (Adapted from a diagram in the article "Ethnomathematics" in the *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences.*)

What I find unconvincing about such cases is the way they are presented. *We* may analyse Malekula sand-art in terms of graph theory, but there is little evidence presented that the Malekula themselves do. There is a basic theorem of graph theory dealing exactly with the question of when a line drawing may be traced in one movement, and there is no evidence given at all that the Malekula are familiar with it. Certainly they must have found many intricate and appealing designs that may be drawn in this way, but this is not the same thing.

When the mathematician Euler was presented with a puzzle about whether the bridges of the town Königsberg could be traversed in a certain way (equivalent to drawing a pattern in one movement), he solved the matter by proving the theorem in question and then applying it to this particular case. He thus showed that the task was impossible. *That* is mathematics; to consider all the possible ways one might try to do the task and thus to eliminate all of them is not really mathematics, and Euler didn't do things that way.

The same point could be made in respect of (say) the clan structure of some societies, for example the Australian *Arunta*. An article in *Function* by

Hans Lausch[10] analysed this in terms of group theory, a branch of abstract algebra. But again notice that it was the author of the article who supplied the mathematics. *We* may find it implicit in the clan structure of the *Arunta*, but this is not the same thing as saying that the *Arunta* are engaging in group theoretical discourse.

One much studied and often cited case is that of the Inca *quipu*. A *quipu* is a form of physical representation of number made from knotted cotton cords. A *quipu* may be very intricate, with as many as 2000 separate cords and with a sophisticated hierarchy of knots and a pleasing design of different colours. Essentially each *quipu* records data as one or more numerals. Again, I don't really count this as mathematics, except in the sense that counting is mathematics. I find no evidence (or even claim) that any operation beyond simple addition was ever recorded by this means.

There are essentially two thrusts to the movement for ethnomathematics. The first is the wish to give dignity to cultures not normally regarded as cultures of advanced numeracy. I too regard this as an important and laudable aim, but I think its application misguided: all cultures have dignity and we do not enhance it by assigning to some of their aspects a significance they cannot really bear.

The second thrust is educational. With this I am entirely in sympathy. If one is teaching, say, graph theory in Vanuatu, then it makes eminent sense to use local examples and knowledge. Not only will it hold the students' attention the better, show that one *does* respect the local culture and provide good non-trivial examples, but it also makes use of the specialist knowledge the students already have.

Further Reading

There is a lot of material on Ethnomathematics. M Ascher's *ethnomathematics: A Multicultural View of Mathematical Ideas* is perhaps the fullest single account. M and R Ascher wrote the article on ethnomathematics in the *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*. In this same collection is C Zaslavsky's article on mathematics in Africa: Explicit and Implicit. Both these sources give many further references. There is also a nice "Interchapter" on mathematics around the world in V Katz's *A History of Mathematics*.

* * * * *

[10] *Function, Vol* 14 *Part* 3, pp 22-27.

COMPUTERS AND COMPUTING

The Search for a Pattern

Cristina Varsavsky

Every day, loads of information come to us via the Internet, CD-ROMs, and computer disks, which is stored in large text and graphics files. These days we spend more time than ever searching through Web browsers, electronic encyclopedias, data bases, e-mail folders, and text documents.

A widely used type of search is for a pattern within a text file – a long string of letters, blanks, numbers, and other special characters. Even the simplest word processors include a pattern matching editing feature.

Some search programs we use today, either independently or within a word processor, are still based on the very simple *brute force* pattern matching algorithm. The algorithm scans the text string, usually from left to right, checking at each position of the text whether the pattern actually matches the text. This is illustrated in Figure 1 where the pattern, FUNCTION, is slid from left to right till the match is found.

```
AFMFURISTMFUNCTIONIFUNTRSMLKJHGFIO    m   n
FUNCTION                              1   1
 FUNCTION                             3   2
  FUNCTION                            3   1
   FUNCTION                           6   3
    FUNCTION                          5   1
     FUNCTION                         6   1
      FUNCTION                        7   1
       FUNCTION                       8   1
        FUNCTION                      9   1
         FUNCTION                     10  1
          FUNCTION                    18  8
AFMFURISTMFUNCTIONIFUNTRSMLKJHGFIO
```

Figure 1

You may wish to implement the following algorithm in your preferred programming language, provided you know how to open, close, and read files.

SIMPLE SEARCH

```
1.  m := 1 ; n := 1
2.  While m ≤ M and n ≤ N          REM M = text string length
                                   REM N = pattern length
    2.1. If string(m) ≠ pattern(n) then
         2.1.1.m := m − n + 2 ; n := 1
         else
         2.1.2.m := m + 1 ; n := n + 1
3.  If n = N + 1 then
    3.1. Output "Pattern found at position " m − n
```

In the brute force approach, the program keeps a counter, m, for the position within the text string and another one, n, for the position within the pattern, counting from left to right in both cases. For each pair (m, n), it compares the corresponding character in the text (string(m)) with the character in the pattern (pattern(n)). If the characters match, then both m and n are incremented. If the two characters do not match, then n is reset to 1, which corresponds to the beginning of the pattern, while m is set to the next character of the string to be compared with the first character in the pattern. The algorithm stops when either a match is found or the right end of the text string is reached.

The algorithm is traced in Figure 1, where bold is used to indicate matching characters, and underlining for the pattern characters that cause a mismatch. A new line is started each time 2.1.1 is executed; the values of m and n at the time of the mismatch are displayed on the right.

The largest number of comparisons this algorithm performs is MN; the worst case occurs when n always reaches the value N but the last character in the pattern mismatches the text string, forcing m to be backed up. But this rarely occurs in ordinary text; in our example, n is 1 most times and the number of comparisons is 21. In fact, testing the algorithm with ordinary text indicates that the average number of comparisons needed to find a match in the ith position within text is usually around $N + i$.

It is interesting to note that more efficient algorithms have been introduced only recently. One of them was developed by D E Knuth, J H Morris, and V R Pratt, and was published in 1976. These mathematicians observed that m does not need to be decremented at all: when a mismatch occurs after a few text string characters match the pattern, we should use this information rather than backtracking over all those characters we already know. This is shown in Figure 2, where each time a mismatch is found, the pattern is shifted to the right past the last string character inspected, yielding the match with only 18 comparisons. But this will not work with a pattern that partly matches itself in some characters. This is easily seen in the example shown in Figure 3, where the first eight characters in the pattern, MATHEMATICS, match the text: if we slid the pattern to the right past the mismatch caused by H, we would miss MATHEMATICS which starts within the string of matching characters. This is because the pattern repeats its first three characters within itself.

```
AFMFURISTMFUNCTIONIFUNTRSMLKJHGFIO
FUNCTION
 FUNCTION
   FUNCTION
      FUNCTION
       FUNCTION
        FUNCTION
         FUNCTION
          FUNCTION
AFMFURISTMFUNCTIONIFUNTRSMLKJHGFIO
```

Figure 2

The Knuth-Morris-Pratt algorithm sets up a table for the pattern, which indicates how far to move the pattern depending on the position within the pattern where the mismatch occurs. In our case, the table for the pattern MATHEMATICS will always indicate a move to the right by the whole length of the pattern if the mismatch occurs before the seventh character or after the ninth, but it will indicate a slide of only 5 positions to the right if the mismatch occurs at the seventh, eight, or ninth position within the pattern.

```
ANMATHEMATHEMATICSSLKRSTULBNMOPSTM
  MATHEMATICS
          MATHEMATICS
ANMATHEMATHEMATICSSLKRSTULBNMOPSTM
```

Figure 3

Around the same time as this algorithm was being developed, R S Boyer and J S Moore came up with another clever idea for improving the efficiency of pattern matching. The innovative idea consists of scanning the pattern from right to left and moving it along the text from left to right. When the last character in the pattern mismatches the character in the text string, the pattern is moved to the right by its whole length, unless the mismatching text character appears somewhere else within the pattern.

Let us illustrate this with the search for FUNCTION in the first example. As shown in Figure 4, proceeding from right to left, we first compare the S of the text string with every pattern character; since no match occurs, we move the pattern by its full length to the right. Then we see if the I in the text string is within the pattern; we find it and we slide the pattern so that the I in the pattern matches the I in the text string. Making 8 more comparisons we find that the pattern actually matches the text at position 11.

```
AFMFURISTMFUNCTIONIFUNTRSMLKJHGFIO
FUNCTION
                FUNCTION
                    FUNCTION
AFMFURISTMFUNCTIONIFUNTRSMLKJHGFIO
```

Figure 4

However, the algorithm designed by Boyer and Moore does not make as many comparisons as we show in the example; to avoid the comparison of the text character with every pattern character, the algorithm constructs a table which associates with each letter of the alphabet found in the string, the number of places the pattern should be shifted to the right if that character causes the mismatch. This table only depends on the pattern, and it is easy to construct. If a character is not in the pattern, then the corresponding entry in the table will be the length of the pattern. If a character appears in the pattern, the table entry will correspond to the rightmost position, counted from the right end, of that character within the pattern.

On top of this improved efficiency achieved by scanning from right to left, Boyer and Moore also incorporated in their algorithm the idea developed by Knuth, Morris and Pratt, that is, the setting up of right-to-left version of their skipping table in addition to the first table, choosing in each case the longer skip.

It is easy to see that the number of comparisons this algorithm makes to find a match in the ith position does not exceed $i + N$; however, extensive testing shows that in practice the performance is much better: for reasonably long alphabets and short patterns, the average number of comparisons is around i/N.

Despite all improvements achieved in the design of algorithms for spotting a pattern within a text, the search for a simpler and faster algorithm continues. An important contribution was made by R M Karp and M O Rabin in 1980, who designed an algorithm as simple as the brute force algorithm, but with the worst case number of comparisons being only $M + N$.

You can find out more about these and other algorithms in *Data Structures, Algorithms, and Software Principles in C* by Thomas A Standish, published by Addison-Wesley in 1995.

* * * * *

To solve:

$$(x+3)(2-x) = 4$$

"Either

$$x + 3 = 4, \text{ in which case } x = 1$$

or

$$2 - x = 4, \text{ in which case } x = -2."$$

The reasoning is of course incorrect, but rather surprisingly the answers are right.

The problem is discussed in E A Maxwell's *Fallacies in Mathematics* (pp 88-89). Maxwell points out that the "method" works in very many (in a certain sense, all) cases. Consider a quadratic equation with roots p, q. Then

$$(x-p)(x-q) = 0,$$

so that

$$x^2 - px - qx + pq = 0$$

and so

$$1 - p + q = 1 + q - x - p - pq + px + x + qx - x^2 = (1 + q - x)(1 - p + x).$$

So if we now say: "either $1 + q - x = 1 - p + q$ or $1 - p + x = 1 - p + q$", we get $x = p$ in the first case and $x = q$ in the second!

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 21.1.1

Show that it is possible to find an arbitarily large finite set of points in the plane such that the points are not all collinear and the distance between any two is an integer. (Rather than trying to do this in one step, you may find it easier if you first look for sets of points for which the distance between any two points is a rational number, and then rescale.)

SOLUTION

Many solutions are possible, but one of the simplest begins by placing one point at the origin of the x-y plane and another point at $(0,1)$. The remaining points are placed on the positive x-axis, with x-coordinates $\dfrac{2n}{n^2-1}$, $n = 2, 3, 4, \ldots, m$, where m is as large a number as we wish to choose. Then the distance from $\left(\dfrac{2n}{n^2-1}, 0\right)$ to $(0,1)$ is:

$$\sqrt{\left(\frac{2n}{n^2-1}\right)^2 + 1^2}$$

which simplifies (after some algebra, as you may check) to the rational number $\dfrac{n^2+1}{n^2-1}$. The distances between all other pairs of points are obviously rational. Therefore we can find an arbitrarily large number of points such that the distance between any two is a rational number. It remains only to scale up the coordinates of all the points by the lowest common denominator of all the rational distances between the points, and we have obtained a solution to the problem.

PROBLEM 21.1.2 (from the German mathematics magazine *Alpha*, May/June 1996)

Solve the equation $x^3 - 3y = 2$ in natural numbers.

SOLUTION

Write the equation as $x^3 = 3y + 2$. If x is divisible by 3 then so is x^3, so this is impossible. If $x = 3n + 1$ for some integer $n \geq 0$, then $x^3 = (3n+1)^3 = 27n^3 + 27n^2 + 9n + 1 = 3(9n^3 + 9n^2 + 3n) + 1$, so this is impossible also. Finally, if $x = 3n + 2$ for some integer $n \geq 0$, then

$x^3 = (3n+2)^3 = 27n^3+54n^2+36n+8 = 3(9n^3+18n^2+12n+2)+2$. Therefore x must be of the form $3n+2$, in which case $y = 9n^3+18n^2+12n+2$.

PROBLEM 21.1.3 (Claudio Arconcher, São Paulo, Brazil)

Let a and b denote real numbers. Find necessary and sufficient conditions over a and b such that:

$$ax + b\lfloor x\rfloor = ay + b\lfloor y\rfloor \quad \text{if and only if} \quad x = y$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x.

SOLUTION based on the solution by Carlos Alberto da Silva Victor, Brazil

If $x = y$ then obviously $ax + b\lfloor x\rfloor = ay + b\lfloor y\rfloor$, regardless of the values of a and b. It remains to find the conditions under which $ax+b\lfloor x\rfloor = ay+b\lfloor y\rfloor$ implies $x = y$. The implication is not valid if $a = 0$, because we could choose x and y to have the same integer part but different fractional parts. Assume $a \neq 0$. Let $x = \lfloor x\rfloor+\theta$ and $y = \lfloor y\rfloor+\varphi$, where $0 \leq \theta < 1$ and $0 \leq \varphi < 1$. Then $ax+b\lfloor x\rfloor = ay+b\lfloor y\rfloor$ is equivalent to $a(\lfloor x\rfloor+\theta)+b\lfloor x\rfloor = a(\lfloor y\rfloor+\varphi)+b\lfloor y\rfloor$, which in turn is equivalent to $(a+b)(\lfloor x\rfloor-\lfloor y\rfloor) = a(\varphi-\theta)$. If $a+b = 0$ then we cannot deduce that $x = y$, only that $\varphi = \theta$, i.e. that the fractional parts of x and y are equal. Assume $a+b \neq 0$. Then $\lfloor x\rfloor - \lfloor y\rfloor = \dfrac{a}{a+b}(\varphi-\theta)$. If the right-hand side of this equation is demonstrably less than 1 in modulus, and only then, we must have $x = y$, because $\lfloor x\rfloor$ and $\lfloor y\rfloor$ are integers. Now $-1 < \varphi-\theta < 1$, so the right-hand side *must* be less than 1 in modulus if $\left|\dfrac{a}{a+b}\right| \leq 1$, but need not be if this condition is not satisfied. The conditions on a and b that we are seeking are therefore that $a \neq 0$, $a+b \neq 0$ and $|a| \leq |a+b|$. The condition $a+b \neq 0$ can be omitted, since it is implied by the other two conditions.

Note that $|a| \leq |a+b|$ can be written more simply as $\left|1+\dfrac{b}{a}\right| \geq 1$. This in turn is equivalent to $b/a \geq 0$ or $b/a \leq -2$, so these inequalities, together with $a \neq 0$, are the required conditions.

PROBLEM 21.1.4

Let $\triangle ABC$ be a triangle with $AB \neq AC$. Let D be the point of intersection of the angle bisector of A and the perpendicular bisector of $\overline{BC}$. Prove that D is on the circumcircle of $\triangle ABC$.

SOLUTION

Let E be the point where the perpendicular bisector of $\overline{BC}$ meets the circumcircle, as shown in Figure 1. We will show that $D = E$ by showing that E is on the angle bisector of A. It is clear by symmetry that the angles BCE and CBE are equal. But $\angle BCE = \angle BAE$ (since the angles subtend the same arc), and similarly $\angle CBE = \angle CAE$. Therefore $\angle BAE = \angle CAE$, so E is on the angle bisector of A. Hence $D = E$.

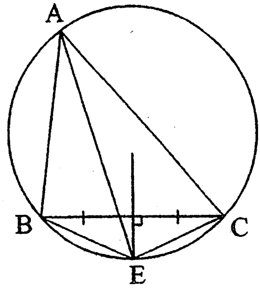

Figure 1

PROBLEM 21.1.5

Let $f(x)$ be a cubic polynomial with three distinct real roots: a, b and c. Let $v = \dfrac{a+b}{2}$. Prove that the tangent to $f(x)$ at v passes through $(c, 0)$.

SOLUTION

The problem could be solved by using calculus to find the equation of the tangent to $f(x) = k(x-a)(x-b)(x-c)$ at $x = v$, and verifying that $(c, 0)$ lies on the tangent. Another method, which gives more insight into why the result is true, runs as follows. Consider an arbitrary line, $y = px + q$, which could intersect the cubic $f(x)$ at 1, 2 or 3 points. These cases correspond respectively to $f(x) - (px+q) = 0$ having 1, 2 or 3 real roots. In particular, if the line is tangent to $f(x)$ at $x = v$, then $f(x) - (px+q) = 0$ has two real roots, one of which is a repeated root at v. Thus we can write:

$$k(x-a)(x-b)(x-c) - (px+q) = k(x-v)^2(x-r)$$

where r is the other (nonrepeated) root. Upon equating the coefficients of x^2 on both sides of this equation, we obtain $a+b+c = 2v+r$. Since $2v = a+b$, we deduce that $c = r$. Therefore the line intersects the cubic at $x = c$. But $f(c) = 0$, so the line passes through $(c, 0)$.

Carlos Alberto da Silva Victor, from Nilópolis, Brazil, submitted solutions to all of the problems.

More on some earlier problems

The proposer of Problem 20.4.3, Claudio Arconcher, has written to us to point out that we made a mistake when we paraphrased his solution in the February 1997 issue. The condition for the four points to reach the quadrilateral simultaneously is not that $ABCD$ is a cyclic quadrilateral, but rather that $ABCD$ has an inner tangent circle (a circle to which all four sides are tangent), with P at its centre. In this case, Q_1, Q_2, Q_3 and Q_4 reach the quadrilateral at the points of tangency. We apologise to readers and to Mr Arconcher for the error.

The following problem appeared in the August 1990 issue of *Function*, where it was described as "very hard". It was taken from a Hungarian mathematics competition known as the Kürschák competition. Although we invited solutions from readers at the time, we didn't receive any, and we haven't published a solution because the editors have only just now been able to solve it!

PROBLEM 14.4.7

Two lines, e and f, do not intersect the circle C. Find a construction for the line g, parallel to f, and intersecting C and e at the points A, B, E in order to make the ratio AB/BE as large as possible.

SOLUTION

Construct O, the centre of C. Construct a line through O perpendicular to f, intersecting e at P. Construct the two tangents to C passing through P. Denoting the points of tangency by A and B, the line $\overline{AB}$ is the required line. Denote by E the point where this line intersects e. Denote the midpoint of $\overline{AB}$ by D. (See Figure 2.)

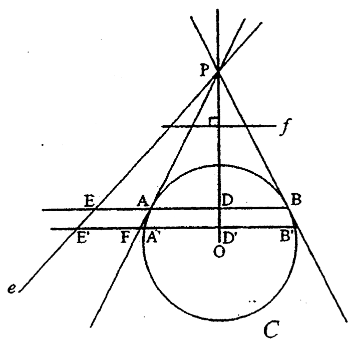

Figure 2

Observe that showing AB/BE is a maximum is equivalent to showing that EA/AD is a minimum, since $BE/AB = (EA + AB)/AB = EA/AB + 1 = EA/2AD + 1$. Consider a line parallel to $\overline{AB}$ intersecting C at A' and B', and meeting e at E'. Let D' be the midpoint of $\overline{A'B'}$, and let F be the point where $\overline{A'B'}$ intersects the tangent to C at A. From similar triangles, $EA/AD = E'F/FD'$. Now $E'F < E'A'$ and $FD' > A'D'$, so $E'F/FD' < E'A'/A'D'$. Hence EA/AD is minimal, as required. (Note that the argument carries through irrespective of which side of $\overline{AB}$ we put $\overline{A'B'}$.)

PROBLEMS

Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received by 11 August 1997 will be acknowledged in the October issue, and the best solutions will be published.

PROBLEM 21.3.1

Does the pattern observed in the following sums continue?

$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

PROBLEM 21.3.2 (from a British television game show; submitted by Prof H C Bolton, University of Melbourne)

Use each of the numbers $3, 6, 25, 50, 75, 100$ exactly once, together with the four basic operations of arithmetic $(+, -, \times, /)$ and parentheses, to obtain an expression equal to 952.

(Professor Bolton writes: "By the nature of [the problem's] origin, it had to be solved in 30 seconds, and one contestant did so ...". However, we will not impose a similar time limit on our readers!)

PROBLEM 21.3.3

Prove that $x^x y^y \geq x^y y^x$ for all positive real values of x and y, with equality holding only if $x = y$.

PROBLEM 21.3.4

Prove that $\sin(\cos x) < \cos(\sin x)$ for all real values of x.

PROBLEM 21.3.5 (from *Mathematical Digest*, University of Cape Town)

Find all real numbers x for which $3^x + 4^x = 5^x$.

PROBLEM 21.3.6 (from *Mathematical Spectrum*)

A polynomial function of degree n is such that $p(x) \geq 0$ for all x. Prove that

$$p(x) + p'(x) + p''(x) + \ldots + p^{(n)}(x) \geq 0$$

for all x.

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OLYMPIAD NEWS

The Ninth Asian Pacific Mathematics Olympiad

The Asian Pacific Mathematics Olympiad (APMO), an annual competition, was started in 1989 by Australia, Canada, Hong Kong and Singapore. Since then the number of participating Pacific Rim countries has grown to nearly twenty. Moreover, Argentina, South Africa, and Trinidad and Tobago were accepted into the APMO last year. In Australia, 22 students sat this four hour examination on 11 March:

Time allowed: 4 hours. No calculators to be used. Each question is worth 7 points.

1. Given

$$S = 1 + \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}+\frac{1}{6}} + \ldots + \frac{1}{1+\frac{1}{3}+\frac{1}{6}+\ldots+\frac{1}{1993006}}$$

where the denominators contain partial sums of the sequence of reciprocals of triangular numbers. Prove that $S > 1001$.

2. Find an integer n, with $100 \leq n \leq 1997$, such that $\dfrac{2^n+2}{n}$ is also an integer.

3. Let ABC be a triangle inscribed in a circle and let

$$l_a = \frac{m_a}{M_a},\ l_b = \frac{m_b}{M_b},\ l_c = \frac{m_c}{M_c}$$

where m_a, m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3$$

and that equality holds if and only if ABC is equilateral.

4. Triangle $A_1A_2A_3$ has a right angle at A_3. A sequence of points is now defined by the following iterative process, where n is a positive integer. From $A_n (n \geq 3)$, a perpendicular line is drawn to meet $A_{n-2}A_{n-1}$ at A_{n+1}.

   (a) Prove that if this process were continued indefinitely, then one and only one point P is interior to every triangle $A_{n-2}A_{n-1}A_n, n \geq 3$.

   (b) Let A_1 and A_3 be fixed points. By considering all possible locations of A_2 on the plane, find the locus of P.

5. Suppose that n persons $A_1, A_2, \ldots A_n (n \geq 3)$ are seated in a circle and that A_i has a_i objects such that $a_1 + a_2 + \ldots + a_n = nN$ where N is a positive integer. In order that each person has the same number of objects, each person A_i is to give to or receive from its two neighbours A_{i-1} and A_{i+1} a certain number of objects, where A_{n+1} means A_1 and A_0 means A_n. How should this distribution be performed so that the total number of objects transferred is minimal?

Australians at the XXXVIII International Mathematical Olympiad

The performance of students at the APMO as well as at the Australian Mathematical Olympiad (AMO) in February was used in selecting twelve candidates for the team, which is to represent Australia at this year's International Mathematical Olympiad (IMO). Also twelve highly gifted students with at least one more year of secondary education ahead of them were singled out for further training.

These 24 students participated in the ten day Team Selection School of the Australian Mathematical Olympiad Committee. Following a tradition, the School was held in Sydney. Participants had to undergo a day and evening programme consisting of tests and examinations, problem sessions and lectures by mathematicians. Finally, the 1997 Australian IMO Team was selected.

Mar del Plata (Argentina) is the venue of the XXXVIII IMO scheduled for July. There the Australian team will have to contend with six problems during 9 hours spread equally over two days in succession. The Australian team is:

Norman Do, year 12, Melbourne Grammar School, Victoria;

Stephen Farrar (11), James Ruse Agricultural High School, NSW;

Justin Ghan (11), Pembroke School, South Australia;

Jonathan Kusilek (12), Hurlstone Agricultural High School, NSW;

Thomas Lam (12), Sydney Grammar School, NSW;

Daniel Matthews (12), Scotch College, Victoria.

Reserve:

David Varodayan (11), Sydney Grammar School, NSW.

Good luck to them all!

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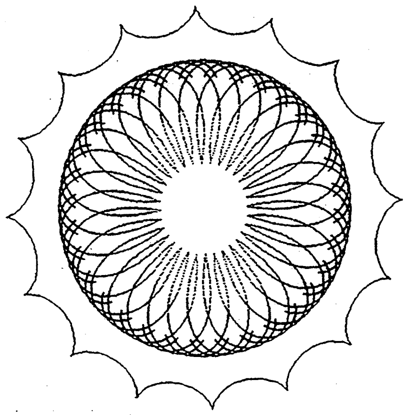

Inspired by our *Computers and Computing* column article about Bézier curves, our long time reader Julius Guest (Bentleigh, Victoria) wrote a program to produce this attractive figure.

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