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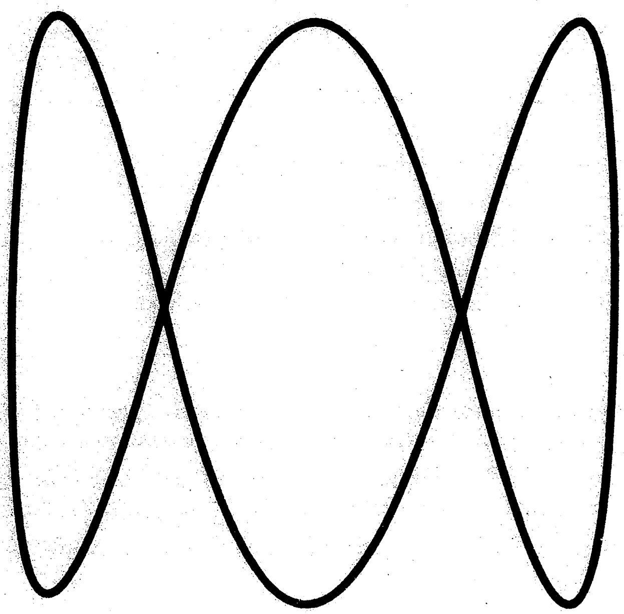

Department of Mathematics - Monash University

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Function is a refereed mathematics journal produced by the Department of Mathematics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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*$10 for *bona fide* secondary or tertiary students.

EDITORIAL

Australian readers will recognise the figure on the front cover as the ABC logo. This is actually a mathematical figure which was thoroughly investigated by the French mathematician Jules A Lissajous. We invite you to find out more about it in the *Front Cover* article.

Readers keep sending us interesting articles. This time we include an everyday example about customer arrival patterns and their implications in running a shop; Ron Adlem shows us how various areas of mathematics contribute to the modelling of this problem. We also include an article by Ian McDowell and Ken Evans about hailstone sequences. If you like analysing number patterns, this article is for you!

Evariste Galois is amongst the greatest mathematicians, and is regarded as the founder of group theory. But he unfortunately died too young in a duel. You will find in this issue a review of another book about his life which brings an interesting new version of the circumstances of his death.

It is widely believed that the Hebrew Bible uses the value of π approximated to the integer 3. However, this belief has been queried; the authors of the *History of Mathematics* column, Michael Deakin and Hans Lausch, explain the rationale for the alternative view that the Hebrews had a very accurate value for π.

Have you ever used file compression programs to make some room on your disks? You will find in the *Computers and Computing* column an introduction to how these programs manage to store a file using sometimes as little as ten percent of the original storage space.

We received another letter from Kim Dean with a puzzling conclusion made by Dr Fwls. We invite our readers to dispute his result.

We hope you will feel challenged by the problems included in this issue. Send us your solutions if you'd like to see them published in *Function*.

Happy reading!

* * * * *

THE FRONT COVER

The Figure of the ABC

Bert Bolton, University of Melbourne

In 1996, the Australian Broadcasting Corporation (ABC) ran a series of advertisements on its TV channel in which a person traced in the air with the end of a finger the outline of a closed three-lobed curve.[1] (By "closed" is meant that the curve ends at its initial point without any retracing of the path.) This curve appears on our front cover for this issue. It is known as a *Lissajous figure* after the French mathematician Jules Antoine Lissajous (1822-1880), who first investigated them.

Suppose we have two coordinates represented by x and y, and suppose that the values of these are in their turn functions of a variable t, which we may take as representing time. Then for any value of t, we may compute values of x and y, and these various values give the co-ordinates of a point on a curve.[2]

If the values of x and y repeat after an interval of time T (called the *period*), then as a result of a theorem by an earlier French mathematician, Jean Baptiste Fourier (1768-1830), we may express x and y in terms of trigonometric functions. The very simplest case is given by

$$x = \cos \omega t \quad \text{and} \quad y = \cos(\omega t + \varphi), \tag{1}$$

where $\omega = \frac{2\pi}{T}$ and φ is a constant called the *phase difference*.

If we put $\varphi = 0$ in equations (1), we find $y = x$ and $-1 \leq x \leq 1$, so the curve in this case is a straight line segment; if we put $\varphi = \frac{\pi}{2}$, we find the circle $x^2 + y^2 = 1$; if φ lies between these values, we obtain an ellipse. These are all special cases of Lissajous figures. Three cases are shown in Figure 1; these are $\varphi = 0, \varphi = \frac{\pi}{2}$ and $\varphi = \frac{2\pi}{9}$. The arrows show how a point would move along the curve as it traces out the successive positions given by equations (1) as t goes from 0 to T.

We may generalise equations (1) to produce more complicated curves than those discussed so far. For example, if we put

$$x = \cos \omega t \quad \text{and} \quad y = \cos(2\omega t + \varphi), \tag{2}$$

[1] An earlier series of ads also made use of the figure, but in a different way.

[2] The curve is said to be *parametrically* specified and the equations giving x and y in terms of t are called *parametric* equations.

then we find a two-lobed figure. The case $\varphi = \frac{\pi}{2}$ is shown as Figure 2.

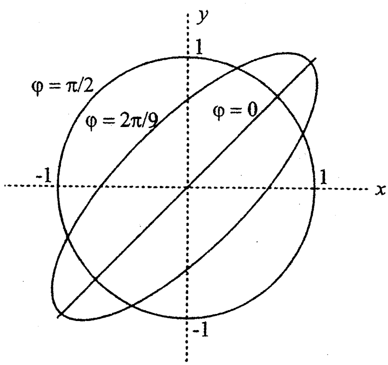

Figure 1

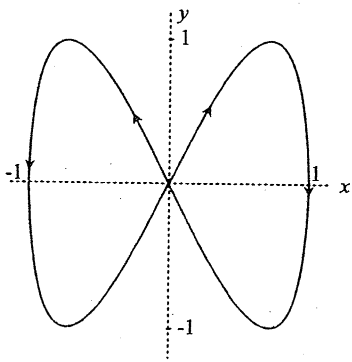

Figure 2

In order to produce the ABC logo, put

$$x = \cos \omega t \quad \text{and} \quad y = \cos(3\omega t + \varphi) \tag{3}$$

and once again set $\varphi = \frac{\pi}{2}$. This is the curve depicted on the cover and in Figure 3.

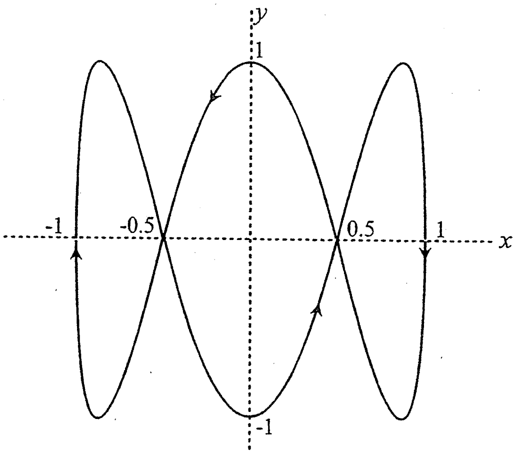

Figure 3

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AN EXAMPLE OF A MATHEMATICAL MODEL

Ron Adlem, Monash University

Applied mathematics is concerned with the process of mathematical modelling. This involves making observations and then formulating assumptions as carefully as possible. We use the language of mathematics to describe phenomena which we observe, and the rules of mathematics to draw conclusions. In this way our model can be used to "explain" the phenomena and also to make predictions. The model is a good one if it accurately describes the observations and if the predictions that are made compare favourably with further observations. Some models are what we call *deterministic*, whereas others are *probabilistic*. From deterministic models we can make accurate predictions but in probabilistic models we make a prediction with a certain probability.

Let us suppose that a supermarket manager is interested in finding a pattern among customer arrivals. We will assume that the manager chooses a particular interval of time on a given day of the week, for example from 6.00 pm to 7.00 pm Friday evenings. Having collected data for a few weeks, such as the times that customers arrive at a certain checkout, he looks for any particular patterns. Let us assume he notices that if he subdivides the one-hour interval of observation into smaller subintervals, the number of customer arrivals in one interval seems to have no relation to the number in any other interval. However, he does notice that if the subintervals are very short, e.g. half a minute, then generally either zero or only one customer arrives, i.e. rarely are there more arrivals than one. Furthermore, he notices that, again considering small subintervals of time, the longer the interval, the more likely it is that a customer arrives.

After the manager has looked at the data and made these observations, a mathematician might refine them into the following, more mathematical, assumptions:

Assumption 1: The number of customer arrivals is independent from one interval to another.

Assumption 2: The probability of more than one customer arrival in a small interval is negligible.

Assumption 3: The probability of a customer arriving during any small time interval is proportional to the length of the interval.

Before we translate these assumptions into mathematical language, let us define a piece of notation. Let $P_n(t)$ represent the probability of having n customer arrivals in an interval of length t minutes. (For example, $P_4(2)$ represents 4 arrivals in 2 minutes.)

From our knowledge of probability, we can now state:

$$P_n(t) \geq 0 \quad \text{for all permissible values of } n \text{ and } t$$

$$P_0(t) + P_1(t) + P_2(t) + \ldots = 1 \tag{1}$$

If δt denotes a small interval of time, then from Assumption 2, we can deduce

$$P_n(\delta t) = 0 \text{ if } n \geq 2$$

Hence from equation (1) we have

$$\begin{aligned} P_0(\delta t) + P_1(\delta t) &= 1 \text{ or} \\ P_0(\delta t) &= 1 - P_1(\delta t) \end{aligned} \tag{2}$$

From Assumption 3 we can deduce that for a small interval of time δt, the probability of having a customer arrival is:

$$P_1(\delta t) = k\ \delta t \tag{3}$$

where k is a positive constant. Hence, substitution into equation (2) gives

$$P_0(\delta t) = 1 - k\ \delta t. \tag{4}$$

From probability theory we have that for two independent events A and B, the probability of both happening at the same time is calculated as the product of the probabilities for each event, that is,

$$Pr(A \cap B) = Pr(A) \times Pr(B).$$

Therefore from Assumption 1 we have

$$P_0(t + \delta t) = P_0(t) \times P_0(\delta t). \tag{5}$$

Substitution for $P_0(\delta t)$ from equation (4) in equation (5) gives

$$\begin{aligned} P_0(t + \delta t) &= P_0(t)(1 - k\ \delta t) \\ &= P_0(t) - k\ P_0(t)\ \delta t \end{aligned}$$

Rearranging,

$$\frac{P_0(t + \delta t) - P_0(t)}{\delta t} = -kP_0(t)$$

The quotient on the left hand side of this last equation might look familiar to you. It is the change produced in the function $P_0(t)$ over the time interval δt divided by the change in time δt. As δt becomes smaller, this quotient approaches the derivative of $P_0(t)$. Note that the right hand side of the equation does not depend on δt, therefore by letting $\delta t \to 0$, we have

$$\frac{d[P_0(t)]}{dt} = -kP_0(t).$$

This is a differential equation with solution

$$P_0(t) = e^{-kt} \tag{6}$$

which you can easily check. A derivation of this solution appears in the appendix.

Expression (6) gives the probability of no customer arrivals in an interval of t minutes.

Now consider an interval of time $t + \delta t$ and divide this interval into two subintervals of lengths t and δt.

We have,

Probability that one customer arrives in time $t + \delta t$

$=$ Probability of 1 arrival in interval t and 0 arrivals in interval δt

$+$ Probability of 0 arrivals in interval t and 1 arrival in interval δt

that is,

$$P_1(t+\delta t) = P_1(t)P_0(\delta t) + P_0(t)P_1(\delta t). \tag{7}$$

This equation makes use of the probability rules:

$$Pr(A \cup B) = Pr(A) + Pr(B) \quad \text{for mutually exclusive events } A \text{ and } B,$$

$$Pr(C \cap D) = Pr(C) \times Pr(D) \quad \text{for independent events } C \text{ and } D.$$

Substituting from equations (3), (4), and (6) into equation (7) gives:

$$P_1(t+\delta t) = P_1(t)(1 - k\,\delta t) + e^{-kt}k\,\delta t.$$

Rearranging gives

$$\frac{P_1(t+\delta t) - P_1(t)}{\delta t} = -kP_1(t) + ke^{-kt}.$$

The left hand side of this equation is again a quotient that leads to a derivative when $\delta t \to 0$,

$$\frac{d[P_1(t)]}{dt} = -kP_1(t) + ke^{-kt}.$$

This is another differential equation. Its solution,

$$P_1(t) = kte^{-kt} \tag{8}$$

is derived in the appendix.

We have now deduced expressions (6) and (8) for the probabilities that 0 and 1 customer arrival occurs in an interval of length t.

Formulae for further probabilities can be derived in a similar manner. For example, it may be shown that

$$P_2(t) = \frac{e^{-kt}(kt)^2}{2}$$

$$P_3(t) = \frac{e^{-kt}(kt)^3}{6}$$

We can rewrite all of these expressions in a slightly different way. Thus

$$P_0(t) = e^{-kt} = \frac{e^{-kt}(kt)^0}{0!}$$

$$P_1(t) = kte^{-kt} = \frac{e^{-kt}(kt)^1}{1!}$$

$$P_2(t) = \frac{e^{-kt}(kt)^2}{2} = \frac{e^{-kt}(kt)^2}{2!}$$

$$P_3(t) = \frac{e^{-kt}(kt)^3}{6} = \frac{e^{-kt}(kt)^3}{3!}.$$

Intuition might lead us to conjecture that, in general, the probability of n customer arrivals in a time interval of length t would be given by

$$P_n(t) = \frac{e^{-kt}(kt)^n}{n!}.$$

In fact, this can be proved by mathematical induction. For such a proof, there are two steps involved.

Step 1. The statement is proved for $n = 1$.
Step 2. If the formula is assumed true for a particular value of n, say i, then it is proved true for the next value of n, i.e. $i + 1$.

The formula has already been proved true for the case when $n = 1$ (and $n = 0$). It is left as an exercise for the reader to verify step 2.

If we replace kt by λ, and $P_n(t)$ by $Pr(X = n)$, where X is the number of customers, then we have the well-known Poisson formula:

$$Pr(X = n) = \frac{e^{-\lambda}\lambda^n}{n!} \qquad (n = 0, 1, 2, \ldots).$$

This formula gives the probability of the number of events (customer arrivals in our case) that occur per unit time interval where λ is the average or mean.

Having obtained this probability model, the supermarket manager can now collect some data on customer arrivals and estimate the mean λ. One way of doing this would be to choose a unit time interval (e.g. a five-minute interval) and record the number of customer arrivals in each of several unit intervals. The average number of arrivals can then be used to estimate λ. With this information, the manager can calculate the probabilities that $0, 1, 2, \ldots$ arrivals will occur in a unit interval. A further set of data could be collected, and the frequencies with which $0, 1, 2, \ldots$ arrivals occur can be compared with the theoretical ones derived from using the Poisson model. Assuming the predicted and the observed frequencies are similar, the manager then has a useful tool to decide, for example, whether a further checkout should be used to reduce the queue length. Since different arrivals patterns probably exist for different times of the day, and for different days of the week, it will most likely be necessary to estimate λ for these different times.

This formula can be applied to any process which is based on assumptions corresponding to the assumptions of our supermarket manager. The Poisson process often occurs with electrons being emitted from an electron tube, accidents occurring on a freeway, telephone calls at an exchange, and many other situations involving apparently random events.

In creating this mathematical model, notice the contribution made by various aspects of mathematics – the rules of probability, direct proportion, the definition of a derivative, differential equations, and mathematical induction.

Appendix

1. Solution to $\frac{d[P_0(t)]}{dt} = -kP_0(t)$.

Inverting this equation leads to

$$\frac{dt}{d[P_0(t)]} = -\frac{1}{kP_0(t)}.$$

Transferring the constant k and integrating with respect to $P_0(t)$ gives

$$\int k dt = -\int \frac{d[P_0(t)]}{P_0(t)},$$

i.e. $-kt = \ln P_0(t) + K$ where K is a constant of integration.

Now, when $t = 0, P_0(0) = 1$. (The probability that no customers arrive in zero time is equal to 1.) Thus

$$K = 0 \text{ and } P_0(t) = e^{-kt}.$$

2. Solution to $\frac{d[P_1(t)]}{dt} = -kP_1(t) + ke^{-kt}$.

This can be solved by multiplying both sides by e^{kt} (this is called an integrating factor).

$$e^{kt}\frac{d[P_1(t)]}{dt} + ke^{kt}P_1(t) = k$$

which can be rewritten as:

$$\frac{d[e^{kt}P_1(t)]}{dt} = k.$$

This can be verified by using the product rule for differentiation. Integrating both sides of this equation with respect to t gives

$$e^{kt}P_1(t) = kt + L$$

where L is a constant of integration.

When $t = 0, P_1(0) = 0$. (The probability that one customer arrives in no time is zero.) Thus, $L = 0$ and $P_1(t) = kte^{-kt}$.

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HAILSTONE NUMBERS

Ian McDowell and Ken Evans

A sequence of natural numbers may be generated as follows:

Choose any natural number (say 11) as the first term of the sequence. To obtain the next term, two cases are considered.

(a) If the preceding term is divisible by 2 (i.e. if it is even), then divide it by 2 to obtain the next term.

(b) If the preceding term is not divisible by 2 (i.e. if it is odd), then multiply it by 3 and add 1 to obtain the next term.

The choice of 1st term as 11, which is not divisible by 2, means the 2nd term is $3 \times 11 + 1 = 34$.

The 3rd term is obtained by applying the rules (a), (b) to the 2nd term, the 4th term is obtained by applying the rules (a), (b) to the 3rd term, and so on.

Since 34 is divisible by 2, the 3rd term is $34/2 = 17$. Since 17 is not divisible by 2, the 4th term is $3 \times 17 + 1 = 52$. Proceeding in this way gives the terms

(1) 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, ...

Clearly the following terms remain in the 4, 2, 1 loop.

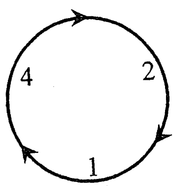

Figure 1

The question arises: what happens if a different first term is chosen? First notice that if any of the terms of sequence (1)

$$34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1$$

is chosen as first term, the same 4, 2, 1 loop is reached. Sequences with other first terms, in increasing order, have terms

(2) $3, 10, 5, 16, 8, 4, 2, 1, \ldots$

(3) $6, 3, 10, 5, 16, 8, 4, 2, 1, \ldots$

(4) $7, 22, 11, \ldots$ (then as in sequence (1))

(5) $9, 28, 14, 7, 22, 11, \ldots$ (then as in sequence (1))

In all these cases, the 4, 2, 1 loop is entered quickly. First terms not yet tested are, in increasing order,

$$12, 14, 15, 18, 19, 21, 23, 24, 25, 27, \ldots$$

The even terms may be omitted as they are halved until a smaller odd number is obtained. You may check that those sequences with first term 15, 19, 21, 23, 25 all enter the 4, 2, 1 loop quickly, but the sequence with first term 27 does not reach the 1 of the 4, 2, 1 loop until the 112th term.

A conjecture may be made at this stage, *viz.* that each sequence of the type described above (however large the first term) finally enters the 4, 2, 1 loop. In 1970, Professor H S M Coxeter (from Canada) offered a prize of \$50 to anyone who proved the conjecture or a prize of \$100 to anyone who disproved the conjecture. In more recent times the prize has been increased by others, but no-one has yet proved or disproved the conjecture. A computer has been used to test each of the first 10^{12} natural numbers as first term, but in each case the 4, 2, 1 loop is reached. This may convince you that the conjecture is true, but does not prove it!

The rule for obtaining the terms of sequence (1) may be written more formally:

Let t_n be the nth term (so that t_{n+1} is the following term). Then

$$t_1 = 11 \qquad \text{(the 1st term is chosen arbitrarily),}$$

and, for each natural number n,

$$t_{n+1} = \begin{cases} t_n/2 & \text{if } t_n \text{ is divisible by 2} \\ 3t_n + 1 & \text{if } t_n \text{ is not divisible by 2} \end{cases}$$

This formula, which enables us to transform t_n to t_{n+1}, is an example of a recurrence or iterative relationship, and the process of obtaining a term from the preceding one is called an *iteration*.

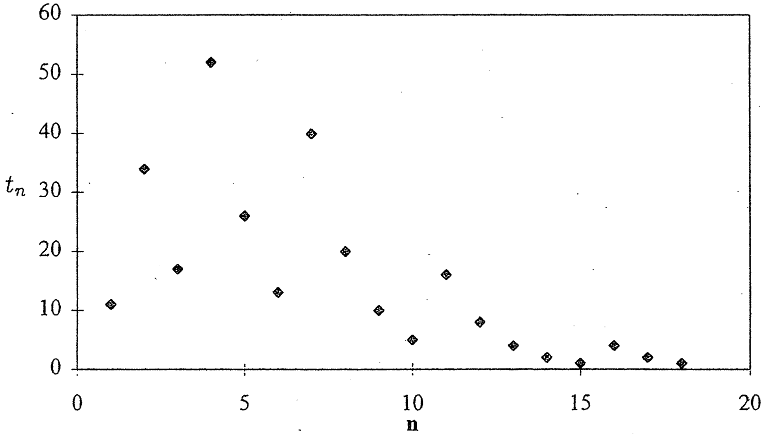

Figure 2

The graph of sequence (1) is shown in Figure 2. Successive terms go "up" and "down" somewhat like a hailstone in a cloud (which eventually grows sufficiently to fall to the ground). This accounts for the terms in the sequences being called *hailstone numbers* and the sequences themselves, *hailstone sequences.*

Sometimes if a problem is generalised, the study of the generalisation may throw light on the original problem. A generalisation of the rule for sequences (1)–(5) is

$$t_{n+1} = \begin{cases} t_n/d, & \text{if } t_n \text{ is divisible by } d \\ mt_n + i, & \text{if } t_n \text{ is not divisible by } d \end{cases}$$

t_1 is an arbitrarily chosen natural number.

Here the multiplier, m, and the divisor, d, are natural numbers, while the increment, i, is a positive or negative integer (though for the program discussed below i is chosen so that $t_n > 0$ for all n). In sequences (1)–(5), $m = 3$, $d = 2$ and $i = 1$. Varying m, d, i, as well as t_1, introduces a myriad of new sequences to investigate. These sequences are also called *hailstone sequences*, though sometimes they are called *Collatz sequences* after L Collatz, a German mathematician who studied related sequences in his research work in the 1930s.

Three examples of hailstone sequences are examined below.

(6)

$$t_1 = 7$$
$$t_{n+1} = \begin{cases} t_n/2 & \text{if } t_n \text{ is divisible by 2} \\ 3t_n + 5 & \text{if } t_n \text{ is not divisible by 2.} \end{cases}$$

This differs from sequence (1) in that the increment, i, is changed from 1 to 5. Successive terms are

$$7, 26, 13, 44, 22, 11, 38, 19, 62, 31, 98, 49, 152, 76, 38, \ldots$$

Note that a loop occurs whenever a previous term is hit again: $t_{15} = 38$ is the same as t_7.

(7)

$$t_1 = 1$$
$$t_{n+1} = \begin{cases} t_n/2 & \text{if } t_n \text{ is divisible by 2} \\ 3t_n + 5 & \text{if } t_n \text{ is not divisible by 2.} \end{cases}$$

This differs from (6) only in the choice of t_1. Successive terms are 1, 8, 4, 2, 1, ... This sequence has a loop beginning at t_1, but a different loop from (6). Further investigation of this family of sequences is left to the reader.

(8)

$$t_1 = 7$$
$$t_{n+1} = \begin{cases} t_n/3 & \text{if } t_n \text{ is divisible by 3} \\ 6t_n + 1 & \text{if } t_n \text{ is not divisible by 3.} \end{cases}$$

Successive terms are 7, 43, 259, ...

Here the terms continue to increase, i.e. $t_{n+1} > t_n$ for all n. The reason is that, if the 1st term is not divisible by 3, no term will be divisible by 3, because $6t_n + 1$ always leaves a remainder of 1 on division by 3. You may be able to find a condition for the general sequence to diverge in this way.

Some characteristics of the sequences which may be investigated are:

(a) The maximum and minimum terms of the sequence. For example, the maximum and minimum terms of (1) are $t_4 = 52$ and $t_{15} = 1$ respectively.

(b) The maximum and minimum terms in the loop (if a loop occurs). For example, the maximum and minimum terms of the loop in (6) are 152 and 19 respectively. The loop maximum is the same as the sequence maximum but the sequence minimum is 7.

(c) The "length" of a sequence. This is the number of iterations (steps) from t_1 to the first occurrence of the first term that does recur. For example, in (1), 4 is the first term to recur and there are 15 iterations from $t_1 = 11$ to the first occurrence of 4. The length of (1) is 15. Sequence (8) may be regarded as having infinite length.

(d) The "length" of a loop. This is the number of iterations between successive "hits": a "hit" is any occurrence of the first term to recur. The length of the loop in sequences (1)–(5) is 3 (the number of iterations from a term 4 to the next 4). The length of the loop in (6) is 8, while in (7) the loop length is 4.

A computer program has been written by Ian McDowell to investigate some of these characteristics of hailstone sequences. The program operates essentially as follows:

(a) enter values of the initial argument, s (the first term), the multiplier, m, the increment, i, and the divisor, d.

(b) perform iterations in accordance with the sequence rule and display the progress value.

(c) test each new term against all previous terms until the sequence hits a previous term.

(d) record the maximum term reached and the value of the hit.

(e) continue the calculation of terms until another hit is made.

(f) record the maximum and minimum terms in the loop.

(g) stop and display results.

(h) increment any parameter (m, i, d, s) and commence a new run, or exit.

This program is available to *Function* readers.

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BOOK REVIEW

Evariste Galois, 1811-1832
by Laura Toti Rigatelli. Translated by John Denton
Birkhäuser, 1996 (*Vita Mathematica* Vol 11)
(168 pages, soft cover, ISBN 3-7643-5410-0)

Reviewed by Hans Lausch, Monash University

Solving polynomial (or algebraic) equations has been one of the oldest mathematical problems. Since antiquity, linear equations ($ax + b = 0$) and quadratic equations ($ax^2 + bx + c = 0$) have been successfully handled, although the idea of negative, often called "false", numbers – not to mention complex numbers – created difficulties as late as the eighteenth century. We all learned the solution formula for quadratic equations: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

By the seventeenth century, solution formulae for cubic equations ($ax^3 + bx^2 + cx + d = 0$) and quartic (or biquadratic) equations ($ax^4 + bx^3 + cx^2 + dx + e = 0$) were well known. Such equations can be solved "by algebra" as can quadratic equations, i.e. their solution formulae are made up solely from the coefficients $a, b, c, \ldots$ of the powers of x in the equation, the "basic" operations addition, subtraction, multiplication and division, and from roots (square, cubic, ...). Although the best mathematicians, including the great Leonhard Euler (1707–1783), attempted to find a solution formula of that type for quintic equations: $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$, no further progress occurred. Indeed, in 1799, the Italian mathematician Paolo Ruffini published a demonstration of his claim: no such solution formula exists. In 1824 the Norwegian mathematician Niels Henrik Abel (1802–1829) gave a flawless proof of Ruffini's theorem.

However, there are equations of any degree that can be solved "by algebra". Take, for example, the equation $x^n - a = 0$ ($n = 1, 2, \ldots$). By the definition of roots, it has as its solutions the n-th roots of a. As was discovered later, Abel subsequently planned to investigate the problem of which equations can and which can't be solved "by algebra", but died without realising his plan. In 1830, the nineteen-year-old Evariste Galois provided a comprehensive solution to this problem. Even more importantly, Galois' entirely new way of algebraic thinking turned out to be the beginning of modern algebra. For several generations, Galois has been regarded as the founder of group theory, now undisputably the best developed algebraic discipline. Its original conceptual apparatus lies at the foundations of present-day algebra. (See the reviewer's article "Evariste Galois. Revolutionary genius – failed revolutionary", *Function, Vol* 3 *Part* 2.) Yet the name Galois touches many a mathematician with sadness and some with ire. A political hotspur, Galois was fatally wounded in a duel.

Relying on hitherto neglected documents – a police prefect's memoirs, a government agent's report, and an article from a Lyons newspaper – Toti Rigatelli provides a new version of the circumstances leading to Galois' death. Say no more, except that it looks as if past historians of mathematics as well as romantically affected writers have fallen victim to a thorough cover-up.

Thankfully, the book lacks romantic nonsense. The socio-political background of Galois' life is well presented, while Galois himself is not subjected to unnecessary "psychological analysis". The biographical part, 114 pages, dominates the book, and an appreciation of Galois' work, 24 pages, offers passages – valid then as now – that Jules Tannery deliberately omitted from his edition *Manuscrits de Évariste Galois* (Paris 1908), since, in his view, its author must have been either drunk or feverish when he wrote the passages, such as:

> *... algorithms have become so complicated that progress was impeded by this means, without the elegance with which modern geometricians have endowed their research and by means of which the mind quickly grasps all at once a large number of operations. ... My opinion here should not be confused with the conceit shown by some in avoiding all calculations, translating what is expressed very briefly by algebra into very long sentences, thus adding inappropriate linguistic complication to long operations. These people are a hundred years out of date.*

We also learn of Galois' most influential teacher, Louis-Paul-Emile Richard (1785-1839). In January 1837, Richard was awarded a medal for his service to state education. Not a creative thinker, he had considerable teaching gifts, which enabled him to maintain his pupils' mathematical alertness (Toti Rigatelli, p 35):

> *They were captivated ... by the refined mathematical language used by their teacher. Richard always kept abreast with developments in research, by reading the latest scientific articles and papers.*

A comprehensive bibliography lists Galois' works and Galois editions, other Galois biographies, biographical studies, Galois novels, plays and films (there have been one French and two Italian movies made about Galois), historical works relating to Galois' life, studies on Classical Galois Theory, and more. Originally, Toti Rigatelli's Galois biography was published in Italian under the title *Matematica sulle barricate* ("Mathematics on the Barricades"). The present translation fits very well into the superb biographical series *Vita Mathematica*, edited by Emil A Fellmann. A highly readable book that can be strongly recommended to both teachers and students!

* * * * *

Monash women leading the way

The 33rd ANZIAM (Australian and New Zealand Industrial and Applied Mathematics) Conference was held in Lorne (Victoria) on 2-6 February 1997. Each year since 1969, the T M Cherry Prize has been awarded for the best student presentation. The award commemorates Prof Sir Thomas Cherry, Fellow of the Royal Society (London) and the inaugural president of the Australian Academy of Science, one of Australia's greatest ever mathematicians.

This year's winner was Sharen Cummins of Monash University. She is the third Monash winner. It may be of interest to note that *all* our winners have been young women; the others were Anne Becker (1986, also an honourable mention in 1989) and Belinda Barnes (1994).

* * * * *

Mathematics stands forth as that which unites, mediates between Man and Nature, inner and outer world, thought and perception, as no other subject does.

– Froebel

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LETTER TO THE EDITOR

I recently had a further note from my Welsh correspondent Dai Fwls ap Rhyll. Readers may know that Dr Fwls delights in finding difficulties with the standard results taught in mathematics. He sends me from time to time the results of some new enquiry or other – never explaining himself fully but always leaving me wondering as to the adequacy of standard mathematics.

His latest letter focuses on logarithms. He began by reminding me that the usual story has it that the value of

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$$

is $\ln 2$, the natural logarithm of 2. My calculator tells me that $\ln 2 = 0.69314\dots$. To test the theory, I added 3000 terms of the series on my computer and found a value of about 0.693, so it seemed that the theory might be correct.

However, Dr Fwls disagrees. He gets 0 as the correct value. Here is his argument. He begins with

$$L=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$$

and then multiplies by 2 to get

$$\begin{aligned} 2L &= 2-\frac{2}{2}+\frac{2}{3}-\frac{2}{4}+\frac{2}{5}-\frac{2}{6}+\frac{2}{7}-\frac{2}{8}+\frac{2}{9}-\frac{2}{10}+\dots \\ &= 2-1+\frac{2}{3}-\frac{1}{2}+\frac{2}{5}-\frac{1}{3}+\frac{2}{7}-\frac{1}{4}+\frac{2}{9}-\frac{1}{5}+\dots . \end{aligned}$$

The next step is to group together those terms with the same denominator. This gives

$$2L=(2-1)-\frac{1}{2}+\left(\frac{2}{3}-\frac{1}{3}\right)-\frac{1}{4}+\left(\frac{2}{5}-\frac{1}{5}\right)-\frac{1}{6}+\dots .$$

But a little simple arithmetic shows us that the right-hand side of this last equation is just L.

So we have $2L = L$, and unless we are prepared to say that $1 = 2$, we must have $L = 0$. So the "official" value of the series sum is quite incorrect and we have been misled for centuries!

Kim Dean
Union College
Windsor

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HISTORY OF MATHEMATICS

The Bible and π

Michael A B Deakin and Hans Lausch

It is widely, almost universally, believed that the Hebrew Bible gives the value of π as the crude approximation 3. This is much less accurate than the values adopted by other ancient civilisations, such as the Babylonian, the Egyptian and the Chinese. An early article in *Function*[1] gave the biblical source as:

> "And he made a molten sea, ten cubits from the one brim to the other; it was round all about, and his [i.e. its] height was five cubits; and a line of thirty cubits did compass it round about."[2]

This is from **I Kings 7:23**, but it is repeated in a later verse (**II Chronicles 4:2**):

> "Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about."

A few observations are in order before we proceed. First, there are some apparent differences of wording between the two passages. These arise because the *English translation* here employed was produced by a committee; they are not to be found in the *Hebrew original.* (We shall see later on that there are subtle differences in the Hebrew versions, but the main one is not, nor could it be, reflected in the English.)

Second, the obvious interpretation is that the biblical author used a value of $\pi = 3$. The underlined word "line" is quite often rendered as "circumference",[3] and so the ratio of the circumference (30 cubits[4], about 15m) to the diameter (10 cubits "from brim to brim", about 5m) is clearly 3.

[1]See *Function Vol* 4, *Part* 1, pp 6-11.

[2]The translation used is the Authorised (King James) Version. The "molten sea" was a tank, cast in metal, which was molten during the actual casting. However, once it was cast, the tank held water (not molten metal).

[3]It means more precisely "tape measure" or else the measurement produced by means of a tape measure.

[4]Quite remarkably, but also puzzlingly, the Septuagint (an early Greek translation of the Hebrew Bible) "corrects" the circumferential measure to 33 cubits!

It is thus widely held that the Hebrews of this era[5] used the value $\pi = 3$, and thus had a rather inaccurate mathematical tradition.[6] This belief has, however, been queried and we will use this column to explain the rationale for an alternative view. We base our discussion on two sources, one fairly easy to come by, the other anything but. The harder-to-find article has much more detail.[7]

The former is a brief discussion in the journal *The Mathematics Teacher*[8] by Alfred Posamentier and Noam Gordan of the City University of New York. The latter is an extended discussion by Shlomo Belaga and published some years ago by the Canadian Society for the History and Philosophy of Mathematics. This published version is very hard to come by and has also been updated. A copy of the later text has reached us via email.[9]

Belaga gives some background. The molten sea was a large, bronze reservoir set on the backs of twelve bronze oxen and placed in the court of Solomon's temple. With the dimensions as given, its capacity would have been about 45000 litres. It was one of the greatest engineering works ever undertaken by the Hebrew nation and its size compares with that of some of the very largest church bells cast in modern times. It is certainly very dubious that an engineering work on this scale could have been carried out by a people who genuinely believed that $\pi = 3$. (Although very possibly they didn't use a specific value, but rather relied on scale drawings, mechanical instruments and the like.)

However, there is a suggestion that they *did* have a value for π, a very accurate value, and one that is encoded in the original Hebrew of the very passages quoted above. Posamentier and Gordan state that this proposal was first put forward about 200 years ago by Rabbi Elijah of Vilna[10], one of the great modern Jewish biblical authorities. However, Belaga tends to attribute it to a Rabbi Max Munk.[11]

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[5]Solomon lived around 950 BCE.

[6]There are even cases of fundamentalist Christians claiming that the value 3 must be correct as it has been divinely revealed! House Bill No. 246 (1897) of the Indiana State Legislature has been so interpreted by some (although the full truth is considerably more complicated). It narrowly failed to pass.

[7]There are other articles, but even harder to find, or else in Hebrew.

[8]*Volume* 77(1), Jan 1984, pp. 47, 52.

[9]Our thanks to Mark McKinzie, and to Michael Closs for the earlier account.

[10]Elijah ben Solomon Zalman, the Gaon of Vilna (1720-1797). Vilna is now known as Vilnius and is the capital of modern Lithuania.

[11]Belaga was aware of the attribution to the Gaon of Vilna, but could not find any relevant passage in his writings.

The key to the suggestion is the Hebrew word for **line**, occurring in the text of both the passages given above. In the original Hebrew, the two passages are almost identical, the principal difference being this word.[12] The second passage was copied from the first by a scribe known as Ezra[13] some 400 years after the first was written.

Look at Figure 1. The left-hand illustration gives the Hebrew word as it appears in **I Kings 7:23**; on the right is the form in **II Chronicles 4:2**. Remembering that *Hebrew is read from right to left*, we see that the later version omits the final letter. The earlier version is spelt as **Qof, Vav, He**, that is to say **QVH**.[14] In the second version, the final **He** (**H**) is omitted. According to Belaga, this accords with a tradition under which the earlier, *written*, **QVH** would actually be *read* as **QV**. Ezra wrote the word, however, as it was meant to be read.

קוה   קו

Figure 1

Two different renderings of the Hebrew word for **line** (circumference). To the left, the literary form as used in **I Kings 7:23**, that to the right is the "reading form" rendered explicit in **II Chronicles 4:2**.

We thus have *two different versions of the word for circumference*, and this leads on to the interesting part of the story. In Hebrew[15] there is a technique for writing numbers as words; it goes by the name "gematria". According to this, the letter **Qof** has the numerical equivalent 100, the letter **Vav** the value 6 and the letter **He** the value 5. Thus the word we have

[12] There is just one other, very minor, difference in the wording. As far as we know, no significance has ever been attached to this.

[13] Tradition ascribes the first to the prophet Jeremiah (around 600 BCE).

[14] Vowels are not written in Hebrew. The letter **Q** is roughly equivalent to the English **K**.

[15] The ancient Greeks used a similar convention.

rendered in English transliteration as **QVH** also has the numerical meaning $100 + 6 + 5$, i.e. 111. The other version reads as 106.

Now form the ratio $\frac{111}{106}$ and multiply this by 3, the "surface" or "apparent" value of π, as given explicitly in the text. The result is $\frac{333}{106}$. In decimal terms, this is 3.141509 ... . This a value of π accurate to about $\frac{1}{4}$ of a percentage point!

Thus far we have followed essentially the account by Posamentier and Gordan (but embellishing it from Belaga's fuller version). What follows uses Belaga's further and more detailed exploration of the matter. However, before we can get to this, we need some background.

The number π is, as is now known, irrational. However, for many practical purposes, we use *rational approximations* to its true value (which we can never know in full). $\frac{333}{106}$ is one such rational approximation, and the more familiar $\frac{22}{7}$ is another. There is a systematic way to produce such rational approximations and this is via the use of *continued fractions*.[16]

A *simple continued fraction* is an expression of the form

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \frac{1}{\cdots}}}}$$

and its *convergents* are the fractions produced by ignoring the rest of the expression at some point or other. The *convergents* in a certain technical sense are *the best rational approximations that can be produced for some suitably restricted denominator.*[17]

Now the continued fraction expansion for π is[18]

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \frac{1}{\cdots}}}}}}$$

and its convergents are:

$$\pi_0 = 3, \quad \pi_1 = 3 + \frac{1}{7} = \frac{22}{7}, \quad \pi_2 = 3 + \frac{1}{7 + \frac{1}{15}} = \frac{333}{106},$$

$$\pi_3 = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = \frac{355}{113}, \quad \pi_4 = \ldots = \frac{103993}{33102}, \quad \text{etc.}$$

[16]See *Function, Vol* 4 *Part* 4; *Vol* 11 *Part* 2.

[17]For a precise statement of this theorem, see p 124 of *Continued Fractions* by C D Olds.

[18]This forms the cover design of Olds's book.

Each of these convergents is the best approximation that can be achieved without extra complexity. Roughly, 3 is the best integer approximation; $\frac{22}{7}$ is the best approximation that can be achieved with a denominator less than 8; $\frac{333}{106}$ is the best approximation that can be achieved with a denominator less than 107, and so on.[19]

But now look at the rabbinical value for π. It turns out to be π_2. The *surface meaning* of the text gives the value π_0, but this is deceptive; those in the know (so the story goes) see hidden in the text the much more accurate value π_2.

The question is what to make of all this. Either the rabbinical tradition is right, and the author of **I Kings** inserted surreptitiously into his text an extremely accurate value of π, or else we have a most remarkable numerical coincidence. Which is it?

The question is not one susceptible of being decided absolutely one way or the other. However, we ourselves incline to the view that there is a most remarkable coincidence at work here and that it has no significance beyond this. Of course not everyone will agree with us.

However, here are our reasons.

First, there is the question of how we are to know that a cipher exists and why we are to choose this particular method of decoding. There are many instances of spurious ciphers being "decoded". Perhaps the most extended such is a use of gematria and other techniques to "prove" that Queen Victoria was the true author of Tennyson's poem *In Memoriam*.[20] (This latter was produced as a satirical attack on the theory that Francis Bacon was the true author of Shakespeare's plays and indeed played a considerable part in discrediting it.)

Next up, the relative error in π_2 is, as we have seen, about $\frac{1}{4}$ of a percentage point. That of π_3 is much, much smaller, being *less than one hundred-thousandth of one percentage point*! But now look at the denominators. We have in π_2 a denominator of 106, while in π_3 the denominator is only marginally larger, 113. So if we want a good approximation to π, we might as well not bother with π_2, but go immediately to π_3.

Now there *is* a most remarkable numerical coincidence in the value of π_3. Remember it works out to be $\frac{355}{113}$. If we start at the bottom left of this

[19]For a nice visual approach to the "best approximation" theorems underlying this, see pp 77-79 of Olds's book.

[20]This appears in the book *Essays in Satire* by R A Knox.

fraction and follow the digits round in the pattern of the letter **S**, we find the simple and memorable pattern: 113355.[21]

This pattern in π_3 clearly has no great significance, although it makes for a highly accurate and easily memorable approximation to π, one that deserves to be more widely known. We incline to the view that we should see the appearance of π_2 in the same light.

If more need be said, then it could perhaps be found in other places in the Hebrew scriptures. The word here represented as "line" is to be found in its **QVH** form at **Jeremiah 31:39** and at **Zechariah 1:16**, and in its **QV** form at many other places. In all these cases, there is no reference to circular measure, the "line" or "tape measure" is stretched out **straight**. The type of analysis Belaga applies to **I Kings 7:23** could also be applied, for example, to **Zechariah 1:16**, but it is a little hard to see how the ratio $\frac{111}{106}$ could be given any significance in this different context.[22]

However, there will be those who see the biblical pattern as significant, and certainly they have a case, even a strong case. For ourselves, however, we're not convinced.

* * * * *

In questions of science the authority of a thousand is not worth the humble reasoning of a single individual.

– Galileo

* * * * *

[21]This pattern was pointed out to us by Dr Russell Smith of CSIRO. It is of course *pure* numerical coincidence, depending on our use of base ten in the representation of numbers.

[22]English translations of the texts read: "And the measuring line shall yet go forth over against it upon the hill Gareb, and shall compass about to Goath." and "Therefore thus saith the LORD; I am returned to Jerusalem with mercies; my house shall be built in it, saith the LORD of hosts, and a line shall be stretched forth upon Jerusalem". These are given in the Authorised Version. In the case of the former, the Revised Version is more explicit: "And the measuring line shall yet go out straight onward unto the hill Gareb, and shall turn about unto Goah."

COMPUTERS AND COMPUTING

Saving Space

Cristina Varsavsky

Have you ever experienced the frustration of the computer telling you that you have run out of disk space? How many times have you felt hopeless because that text file was so big that it did not fit on the floppy disk? Experienced computer users know that file compression is usually a good solution to the space problem, that they can save a significant amount of space by using one of the many shareware programs for file compression.

How do these programs work? How is it possible to store a file using sometimes as little as only 10% of the original space without losing information? This is only possible because in general, most computer files have a great deal of redundancy. Take for example your text files; they are stored as character after character, often with long strings of consecutive spaces or formatting characters. Each of these characters is usually stored using seven or eight bits, so your text files become long sequences of 0's and 1's, which are read in chunks of seven or eight.

A common file compression technique consists of breaking away from this fixed length storage procedure and using fewer bits for the characters which appear more often. This is of course language dependent; we know for example that in an English text the occurrence of the letter A is likely to be much higher than that of the letter Q, so any code for compression of English text files should use fewer bits to store the letter A than the letter Q. But file compression programs are not language dependent but rather file dependent; they usually create a code for each file, where the shortest codewords are for the most frequently occurring letters in that particular file.

A code of this type must be carefully designed to avoid ambiguity in the decoding process. For example, if the file consisted only of the word FUNCTION, and we used the following code (note that the shortest codeword is for N!):

N	0	F	1	U	01	C	10	T	11	I	100	O	101

the compressed file would consist of the following string of 0's and 1's:

101010111001010

using only 15 bits instead of 56. This is a significant saving, but useless as the decoding is not unique; the encoded string could be decoded, for example, as CCCTCUNC or also as OUNTION. The problem with the above code is that some of the codewords are part of others; and since the length of the codewords is not fixed, the decoder runs into trouble right at the start: when it reads the first 1 it doesn't know whether the 1 corresponds to F or it is just the first bit of C, T, I or O.

In 1952, D Huffman designed a very useful algorithm for coding an alphabet of any size so that the least frequently occurring characters correspond to the longest codewords. Let us show how this works with a very simple example. Imagine that a 200-letter text file consists only of vowels which occur with the following frequencies: A, 50 occurrences; E, 49 occurrences; I, 33 occurrences; O, 45 occurrences; U, 23 occurrences. We display this information as shown in Figure 1, using 5 nodes labelled with the number of occurrences of the vowels and sorted by that number.

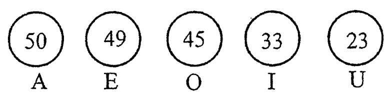

Figure 1

Now we construct a tree as follows: we merge the two nodes labelled with the two smallest numbers to form a new node; this we label with the corresponding sum and place in the appropriate position as shown in Figure 2a. Next we create a new node with the two smallest ones and proceed as before (Figure 2b); we continue this process until all nodes are part of the same tree. The final tree appears in Figure 3, where the least frequently occurring vowels are far down in the tree, and the most frequent vowels are closer to the root. This Huffman coding tree is then used to encode the vowels by traversing the tree from the root to the corresponding letter and appending a 1 for each turn to the left branch, and a 0 for the right branch. So we have the following encoding:

A 00 E 11 I 011 0 10 U 010

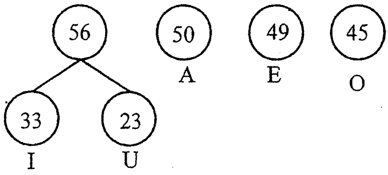

Figure 2a

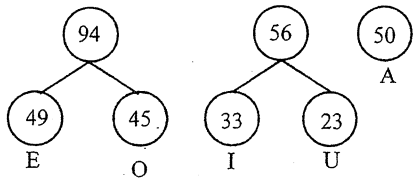

Figure 2b

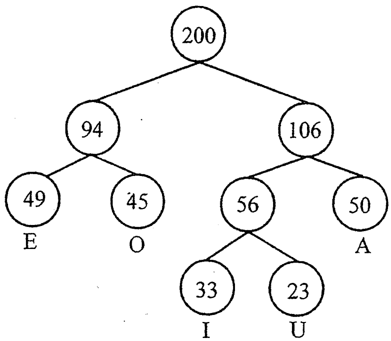

Figure 3

As you can see, none of the binary codewords is a prefix of another; to decode a string we read the bits one by one traversing the tree from the root, taking the right branch if it is a 0, and the left if it is a 1, until we find a dead end corresponding to a vowel; then we start from the root again until a dead end is reached, repeating this procedure until all bits have been read and the original string of vowels is reconstructed. For example, the string 010100111100 will decode as UOIEA.

Let us now compress the file which consists of only one paragraph:

FUNCTION IS A SCHOOL MATHEMATICS JOURNAL

First we work out the statistics. Only 16 characters appear with the following frequencies:

A	C	E	F	H	I	J	L	M	N	O	R	S	T	U	Sp
4	3	1	1	2	3	1	2	2	3	4	1	3	3	2	5

The Huffman code tree appears in Figure 4, from where we have the following code-words:

A	011	C	1110	E	00101	F	00100
H	11111	I	1101	J	00111	L	11110
M	0001	N	1100	O	010	R	00110
S	1011	T	1010	U	0000	Sp	100

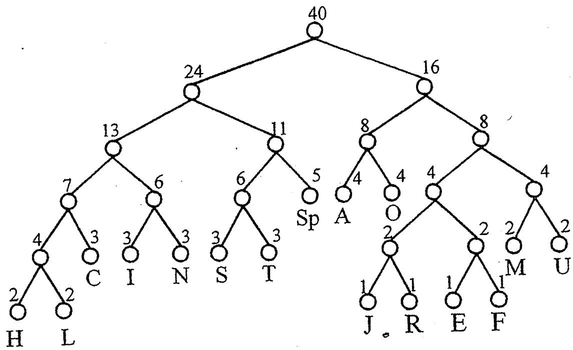

Figure 4

With this code, the file is coded with 155 bits, 44% shorter than the original 280 bits. The saving is not actually that big, because the coded file has to be stored together with the compression code; in practice it is not worthwhile compressing short files, as the compressed file may end up being bigger.

This file compression technique can be used not only with text files of a reasonable length but also with many other kinds of files. Note that in general, not every file is compressible with the same compression program; there will always be files that will become longer when compressed. Can you see why?

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 20.5.1 (from *Mathematical Mayhem*, University of Toronto)

Three distinct integers a, b and c form an arithmetic progression, and $a+b, b+c$ and $c+a$ form a geometric progression (not necessarily in that order). If $a+b+c=-1197$, find the triplet (a,b,c) which satisfies these conditions.

SOLUTION

Since a, b and c form an arithmetic progression, b is the average of a and c, so $a+c=2b$. Thus $a+b+c=3b$, so $3b=-1197$ and hence $b=-399$.

There are three cases to consider.

Case 1: $a+b$ is the middle term of the geometric progression. Then $\frac{a+b}{b+c} = \frac{c+a}{a+b}$, so $\frac{a+b}{3b-a} = \frac{2b}{a+b}$ because $a+c=2b$. Multiplying out the fractions gives $a^2+2ab+b^2=6b^2-2ab$. Therefore $a^2+4ab-5b^2=0$, so $(a-b)(a+5b)=0$. Since the integers are distinct, $a \neq b$. Hence $a=-5b=-5\times(-399)=1995$ and so $c=2b-a=2\times(-399)-1995=-2793$.

Case 2: $b+c$ is the middle term of the geometric progression. Then $\frac{b+c}{a+b} = \frac{c+a}{b+c}$, so $\frac{3b-a}{a+b} = \frac{2b}{3b-a}$. By similar reasoning to Case 1, we obtain $(a-b)(a-7b)=0$. This equation yields $a=-2793$, $b=-399$, $c=1995$, which is the solution in Case 1 with a and c reversed.

Case 3: $c+a$ is the middle term of the geometric progression. Then $\frac{c+a}{a+b} = \frac{b+c}{c+a}$, so $\frac{2b}{a+b} = \frac{3b-a}{2b}$. This leads to the equation $(a-b)^2=0$, which does not give any further solutions to the problem.

PROBLEM 20.5.2

Let α, β and γ be the radian measures of the angles of a spherical triangle (a figure on the surface of a sphere whose sides are parts of three great circles). If the sphere has radius 1, prove that the area of the triangle is $\alpha+\beta+\gamma-\pi$. (Hint: there is a simple and elegant proof which begins by extending the sides to form three great circles. Consider the areas of the pairs of "orange slices" formed by the great circles taken in pairs.)

SOLUTION

The surface area of a sphere with radius 1 is 4π, so each of the two orange slices formed by the great circles crossing at angle α has area $(\alpha/2\pi) \times 4\pi = 2\alpha$. A similar argument applies to the other orange slices, so the total area of the six orange slices is $4(\alpha+\beta+\gamma)$. These six slices cover the sphere, but three of them overlap in the spherical triangle, and the other three overlap in a copy of the spherical triangle on the opposite side of the sphere. Therefore $4(\alpha+\beta+\gamma) = 4\pi + 4A$, where A denotes the area of the spherical triangle. Solving for A gives $A = \alpha+\beta+\gamma-\pi$.

PROBLEM 20.5.3

Each point in the plane is coloured red or blue, and not all points are the same colour. Prove that it is always possible to find:

(a) an interval of unit length joining two points of the same colour;

(b) an interval of unit length joining two points of different colours.

SOLUTION

(a) Draw an equilateral triangle with unit sides. It is always possible to find two vertices of the triangle with the same colour, and the interval joining these two vertices satisfies the requirements.

(b) Pick any red point and any blue point, and join them by a zigzag path made up of unit intervals. At least one of the intervals must have end-points of different colours.

PROBLEM 20.5.4 (from *Mathematical Spectrum*)

Find all natural numbers n such that $2^n + n^2$ is a perfect square.

SOLUTION from *Mathematical Spectrum*

Suppose $2^n + n^2 = m^2$ for some natural number m. Then $2^n = (m+n)(m-n)$, so $m+n = 2^{n-a}$ and $m-n = 2^a$ for some number a with $n-a > a$, i.e. $2a < n$. Then

$$n = \frac{1}{2}\left(2^{n-a} - 2^a\right) = 2^{a-1}\left(2^{n-2a} - 1\right).$$

If n is odd then $a = 1$ and $n = 2^{n-2} - 1$. This is impossible. If n is even then, for all $n \geq 14$,

$$\left(2^{n/2}+1\right)^2 > 2^n + n^2 > \left(2^{n/2}\right)^2$$

and $2^n + n^2$ is strictly between consecutive perfect squares. A check of the even numbers less than 14 shows that $n = 6$ is the only solution.

PROBLEM 20.5.5 **(based on a problem on the Internet)**

Find the unique natural number composed of the digits 1 to 9 (used once only) with the property that, for each n ($1 \leq n \leq 9$), the number formed by the first n digits is divisible by n.

SOLUTION

Denote the decimal representation of the number by $abcdefghi$. (Throughout this solution, strings of letters will always denote decimal representations of numbers.) Since $abcde$ is divisible by 5, $e = 5$. Since each of $ab, abcd, abcdef$ and $abcdefgh$ is divisible by an even number, b, d, f and h are the even digits 2, 4, 6 and 8 (in some order), and hence a, c, g and i must be the remaining digits 1, 3, 7 and 9.

Now, $abcd$ is divisible by 4, so cd must be divisible by 4. Since c is odd, the only possible values of d are 2 and 6. Similarly, $abcdefgh$ is divisible by 4 (because it is divisible by 8), and g is odd, so h must be 2 or 6.

Next, we use the facts that abc and $abcdef$ are divisible by 3 to deduce that $a + b + c$ and $a + b + c + d + e + f$ are divisible by 3. It follows that $d + e + f$ is divisible by 3. But $e = 5$, so $d + 5 + f$ is divisible by 3. If $d = 2$ then the only possible value of f is 8, while if $d = 6$ then $f = 4$. The number we are looking for must therefore take one of the forms $abc258ghi$ or $abc654ghi$, where $h = 6$ in the first case and $h = 2$ in the second in order to avoid repeated digits.

Since $abcdefgh$ is divisible by 8, fgh must be divisible by 8. We now know that f is 4 or 8, and in either case gh must be divisible by 8. The only possible values for gh are 32 or 72 (if $h = 2$), and 16 or 96 (if $h = 6$). The possible forms of the number are now $abc25816i$, $abc25896i$, $abc65432i$ and $abc65472i$.

Now, since $a+b+c+d+e+f+g+h+i = 45$ and $a+b+c+d+e+f$ is divisible by 3, $g+h+i$ must also be divisible by 3. With this requirement, the possible forms of the number are $abc258963$, $abc654321$, $abc654327$, $abc654723$ and $abc654729$. In each case b is the unique even digit not already used; hence the possibilities are $a4c258963$, $a8c654321$, $a8c654327$, $a8c654723$ and $a8c654729$. For each of these five cases there are two ways of choosing a and c, giving ten numbers in total. At this point, the requirement that $abcdefg$ must be divisible by 7 can be used to eliminate all of these numbers except 381654729.

It is easy to check that this number satisfies the conditions of the problem, so it is the unique solution.

PROBLEM 20.5.6 (based on a problem on the Internet)

A coin is tossed a large number of times. Before each toss, you guess the outcome of the toss.

(a) If the coin is biased so that it lands heads 75% of the time, and you guess heads 50% of the time, what percentage of guesses will be correct in the long run?

(b) If the coin lands heads 50% of the time, and you guess heads 75% of the time, what percentage of guesses will be correct in the long run?

(c) Resolve the apparent paradox in parts (a) and (b).

SOLUTION

(a) Regardless of the probability that the coin lands heads, if you guess heads 50% of the time then you will be right 50% of the time.

(b) No matter what you guess, the coin has a 50% chance of landing that way, so 50% of the guesses will be correct. It makes no difference what percentage of guesses are heads, or even whether the guesses are made randomly or according to some strategy.

(c) It might seem paradoxical to some people that the answer equals the percentage of heads guessed in the first case, but the percentage of heads tossed in the second case. The best way to clarify the problem is to look at a more general situation. Suppose the coin lands heads with probability p, and you guess heads with probability q. Then the probability of a correct guess is:

$$\begin{aligned} &Pr(\text{you correctly guess heads}) + Pr(\text{you correctly guess tails}) \\ &= Pr(\text{coin lands heads})Pr(\text{you guess heads}) \\ &\quad + Pr(\text{coin lands tails})Pr(\text{you guess tails}) \\ &= pq + (1-p)(1-q). \end{aligned}$$

If either $p = 0.5$ or $q = 0.5$ then this expression equals 0.5. Other values of p and q yield probabilities that do not equal either p or q. Notice also that the formula is symmetric in p and q; this is to be expected, as we are seeking the probability that two events (tossing a coin and guessing the outcome) give the same result. The formula does not distinguish between tossing the coin and making the guess – both are random events with two outcomes.

Solution to the problem in the special issue 'Celebrating 100 issues of *Function*'

PROBLEM

There are 25 L-shaped tiles made from 4 squares joined together, and a 10×10 board (100 squares) with the same sized squares.

Either

(i) show how to fit the tiles on the board;

or

(ii) prove it is not possible to fit the tiles on the board.

SOLUTION

Colour the first and then every alternate row of the 10×10 board black, and the rest white. Each L-shaped piece must cover three squares of one colour and one square of the other colour, no matter how it is placed. Let x be the number of L-shapes which cover three black squares. Then $25 - x$ (the rest) cover one black square each. Since there are 50 black squares, $3x + 25 - x = 50$ which gives $x = 12.5$. But x must be an integer, so it is not possible to tile the 10×10 board.

We didn't receive any solutions to this problem from readers, even though we offered a prize! We don't believe the problem was too difficult, so maybe the prize – a free subscription to *Function* – just wasn't tempting enough.

A related but more difficult problem is given below (Problem 21.2.1).

PROBLEMS

Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received by 2 June 1997 will be acknowledged in the August issue, and the best solutions will be published.

PROBLEM 21.2.1

A 5×5 square is tiled with six L-shaped pieces, leaving one square not covered. Where can that square be? (Hint: The ideas used in the solution to the problem from the special issue given above may help you to get started with this problem also.)

The following problem appeared in the October 1992 issue of *Function* (as Problem 16.5.4), but it was misprinted there. A solution has not been published. We would like to give readers a chance to try the problem in its correct form.

PROBLEM 21.2.2 (28th Spanish Mathematical Olympiad – First Round, Question 8)

Let ABC be any triangle. Two squares $BAEP$ and $CADR$ are constructed, externally to ABC. Let M and N be the midpoints of $\overline{BC}$ and $\overline{ED}$, respectively. Show that $\overline{AM}$ and $\overline{ED}$ are perpendicular and $\overline{AN}$ and $\overline{BC}$ are perpendicular.

PROBLEM 21.2.3

Three circles in the plane intersect to form seven bounded regions. In each region there is a token that is white on one side and black on the other. At any stage, you can either:

(a) flip all four tokens inside one of the circles,

or

(b) flip all tokens showing black inside one of the circles, making all the tokens in that circle white.

Starting with all tokens white, and using only (a) and (b) above, is it possible to get all the tokens white except for the one in the region common to all the circles?

PROBLEM 21.2.4

A friend challenges you to the following game. You and your friend take turns to say any one of the numbers 1, 3 and 4, and a running total is kept. (For example: you begin by saying 3; your friend replies by saying 4, bringing the total to 7; you say 3 again, making the total 10; your friend says 1, making the total 11; and so on.) The player who says the number that brings the total to 100 is the winner. (The total is not permitted to exceed 100.) You are given the choice of going first or second. Which should you choose, and what is the winning strategy?

PROBLEM 21.2.5 (from *Mathematics and Informatics Quarterly*, 2/96)

(a) At least two of these statements, apart from this one, are true.

(b) At least two of these statements, apart from this one, are false.

(c) At least one of these statements is false.

(d) x of these statements are true.

Given that, if you knew the value of x, you could determine uniquely which statements are true and which are false, determine the value of x.

PROBLEM 21.2.6

The minute and hour hands on a watch are interchanged. Prove that the resulting arrangement does not correspond to a valid time unless the positions of the two hands coincide.

* * * * *

The 1997 Australian Mathematical Olympiad

The contest was held in Australian schools on February 4 and 5. On each day students had to sit a paper consisting of four problems, for which they were given four hours. About 90 students in years 9 to 12 sat the examinations. In March, top scorers are to participate in the Asian Pacific Mathematics Olympiad (APMO), a major international competition for students from Pacific Rim countries as well as from Argentina, South Africa and Trinidad & Tobago. Here are the problems of the Australian contest:

1. Let ABC be a triangle with $AB = AC$ and $\angle BAC < 120°$. Let D be the midpoint of BC. Choose point E on AD such that $\angle AEB = 120°$. Let E' be any point on AD distinct from E. Prove that

$$EA + EB + EC < E'A + E'B + E'C.$$

2. Let $a_1, a_2, \ldots, a_k$ be real numbers satisfying the following two conditions:

   (i) $0 \le a_1 \le a_2 \le \ldots \le a_k$;

   (ii) $a_1 + a_2 + \ldots + a_k = 1$.

   Prove that $\dfrac{a_1 + a_2 + \ldots + a_n}{n} \le \dfrac{1}{k}$ for $n = 1, 2, \ldots, k$.

3. Determine all functions f defined for all real numbers and taking real numbers as values that satisfy the inequality

$$|f(x+h) - f(x)| \le h^2$$

   for all real numbers x and h.

4. A staircase is a sequence of ordered pairs of non-negative integers (x_1, y_1), $(x_1, y_2), (x_2, y_2)$, (x_2, y_3), (x_3, y_3), (x_3, y_4), $(x_4, y_4), \ldots$ in which $0 \le x_1 < x_2 < x_3 < \ldots$ and $0 \le y_1 < y_2 < y_3 < \ldots$. Prove that if each point (x, y) in the coordinate plane, with x and y being non-negative integers, is coloured either red or blue, then it is possible to find an infinite staircase such that all its points are the same colour.

5. For each positive integer n let $p(n)$ be the product of all positive integers that divide n. Prove that if a and b are positive integers and $p(a) = p(b)$, then $a = b$.

6. For any real number x, let $[x]$ denote the largest integer not exceeding x. Prove that if n is a positive integer, then

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}].$$

7. Let m and n be integers greater than 1. Prove that

$$\frac{1}{\sqrt[m]{n+1}} + \frac{1}{\sqrt[n]{m+1}} > 1.$$

8. Let ABC be a triangle with $\angle ABC = 60°$ and $\angle BAC = 40°$. Let P be a point on AB such that $\angle BCP = 70°$ and let Q be a point on AC such that $\angle CBQ = 40°$. Let BQ intersect CP at R. Prove that AR (extended) is perpendicular to BC.

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