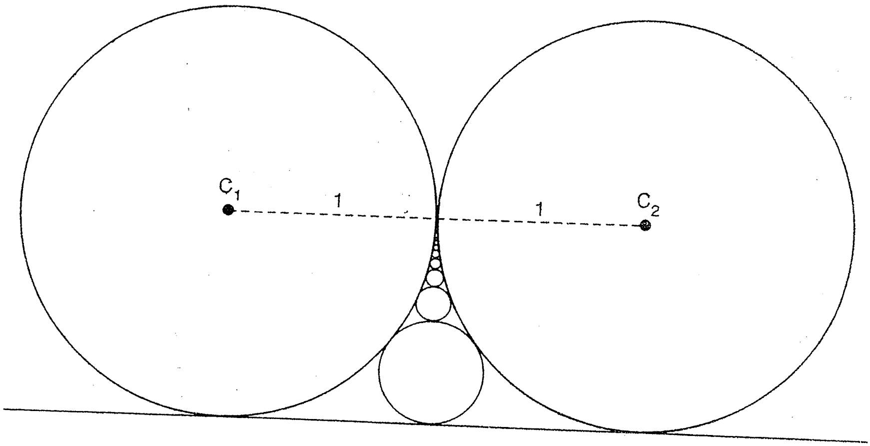
Function

A School Mathematics Magazine

Volume 20 Part 4

August 1996



Mathematics Department - Monash University

Reg. by Aust. Post Publ. No. PP338685/0015

Function is a mathematics magazine produced by the Department of Mathematics at Monash University. The magazine was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Function is published five times a year, appearing in February, April, June, August, and October. Price for five issues (including postage): \$17.00*; single issues \$4.00. Payments should be sent to: The Business Manager, *Function*, Mathematics Department, Monash University, Clayton VIC 3168; cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the Business Manager.

For more information about *Function* see the magazine home page at `http://www.maths.monash.edu.au/~cristina/function.html`.

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EDITORIAL

Welcome to the 99th issue of *Function*! We hope it brings something for every reader interested in mathematics.

The figure on the front cover depicts a sequence of converging circles determined by two touching circles of the same size; this sequence has a surprising connection with the sum of a well-known series.

The three feature articles included in this issue illustrate the use of mathematics in everyday situations. If you are a Tattslotto player and you tend to bet on a number that has not appeared for a long while, you should read M Clark's article; he analyses the problem of the time elapsed since a given number was drawn and calculates the related probabilities. If you are one of those newspaper readers who amuses yourself with solving anagrams, B Bolton explains some strategies based on the theory of groups. If you would like to calculate a cube root but your calculator can only handle square roots, M Deakin explains how you can still use your calculator to obtain a reasonable approximation.

The *History* column follows the article in the last issue about the way the Babylonians approximated the square root. It provides a closer look at and analysis of the mathematical calculations that appear in the clay tablets attributed to the Babylonians. In the *Computers and Computing* column you will find a program to draw the Pythagoras tree, which you could use as a starting point to draw your own trees.

If you couldn't solve the problems proposed in the April issue, or if you would like to compare your work, we include their solutions here. As usual, we also give you a few more problems to try. You are most welcome to send us your solutions.

The next issue of *Function* will be the 100th. We will celebrate it with an additional issue which will include a bit of the history of *Function*, a complete index of all articles published throughout the past 20 years, and more. Stay tuned!

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THE FRONT COVER

A Sequence of Circles

Cristina Varsavsky

The figure on the front cover depicts a sequence of touching circles inscribed in the region determined by two bigger touching unit circles and a common tangent line.

The radii of these circles form a sequence, $r_1, r_2, r_3, \ldots, r_n, \ldots$, which can be obtained by using the formula

$$r_n = \frac{1}{2n(n+1)}, \qquad n \geq 1. \tag{1}$$

Since the diameters of these touching circles add up to one, this sequence provides a proof for the infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \frac{1}{4\times 5} + \ldots = 1.$$

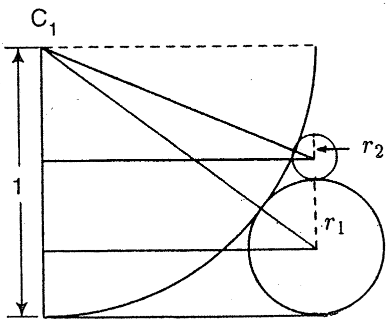

Figure 1

To prove equation (1) we refer to Figure 1. We draw a line from the centre of each of the inscribed circles to the centre of one of the bigger circles – say C_1. Applying Pythagoras's theorem to the larger right-angled triangle, we have

$$(1-r_1)^2 + 1 = (1+r_1)^2,$$

which, after expanding and collecting like terms, gives us

$$r_1 = \frac{1}{4}.$$

Applying now Pythagoras's theorem to the right-angled triangle determined by the circle with radius r_2, gives

$$(1 - 2r_1 - r_2)^2 + 1 = (1 + r_2)^2.$$

This we expand

$$(1 - 2r_1)^2 - 2(1 - 2r_1)r_2 + r_2^2 + 1 = 1 + 2r_2 + r_2^2$$

to obtain an expression for r_2 in terms of r_1, namely

$$r_2 = \frac{(1 - 2r_1)^2}{4(1 - r_1)}.$$

Similarly, we have for the triangle determined by the third circle (this is not drawn in Figure 1),

$$[1 - 2(r_1 + r_2) - r_3]^2 + 1 = (1 + r_3)^2.$$

This results in

$$r_3 = \frac{[1 - 2(r_1 + r_2)]^2}{4[1 - (r_1 + r_2)]}.$$

A pattern emerges here: each radius can be defined in terms of all the preceding radii according to the recursive formula

$$r_{n+1} = \frac{[1 - 2(r_1 + r_2 + \ldots + r_n)]^2}{4[1 - (r_1 + r_2 + \ldots + r_n)]}. \tag{2}$$

Now we only need to prove that (1) and (2) (together with $r_1 = 1/4$) define the same sequence. We do this by induction.

Equation (1) gives $r_1 = 1/4$, which coincides with our recursive definition. If we assume that $r_k = 1/2k(k+1)$ for all $k \leq n$, then we have to prove that equation (1) is also true for $k = n + 1$.

First we observe that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}, \quad k \geq 1$$

which is useful to simplify

$$\begin{aligned} r_1 + r_2 + \ldots + r_{n-1} + r_n &= \frac{1}{2}[(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \\ &\quad \ldots + (\frac{1}{n-1} - \frac{1}{n}) + (\frac{1}{n} - \frac{1}{n+1})] \\ &= \frac{1}{2}\left(1 - \frac{1}{n+1}\right) \\ &= \frac{n}{2(n+1)} \end{aligned}$$

Using this expression for the sum of the first n radii, we have

$$\begin{aligned} r_{n+1} &= \frac{[1 - 2(r_1 + r_2 + \ldots + r_n)]^2}{4[1 - (r_1 + r_2 + \ldots + r_n)]} \\ &= \frac{(1 - \frac{n}{n+1})^2}{4(1 - \frac{n}{2(n+1)})} \\ &= \frac{\left(\frac{1}{n+1}\right)^2}{2\frac{n+2}{n+1}} \\ &= \frac{1}{2(n+1)(n+2)}. \end{aligned}$$

Thus, by induction, (1) and (2) define the same sequence of radii.

* * * * *

How far is the horizon?

If one is reasonably near the surface of the earth, observing, say, from an elevation h, then the distance d to the horizon is proportional to $\sqrt{h}$, namely $d = k\sqrt{h}$. The value of k depends on the units of measurement involved, and it so happens that if d is measured in *miles* and h in *feet*, then k is the easily memorable number 1.23.

It is not so convenient in the metric system. If d is measured in kilometres and h in metres, then $k = 3.58$; or if d is measured in kilometres and h in millimetres, then $k = 0.113$.

* * * * *

ARE YOUR TATTSLOTTO NUMBERS OVERDUE?

Malcolm Clark

The lottery game Tattslotto is by far the most popular gambling activity in Victoria, with a recent survey showing that 66% of the adult population play Tattslotto at least once a year. In the present form of Tattslotto, each player makes at least four selections of 6 numbers from 1 to 45. These selections are marked on a machine-readable card that has space for up to 12 selections.

In each Tattslotto draw, the winning numbers are selected by a mechanical randomisation device, in which 45 numbered balls are mixed in a spherical container. Eight of these balls are selected without replacement, the first six designated as winning numbers, the last two as supplementary numbers. Those players who happen to select all six winning numbers share the First Division pool, winning around $200,000 to $300,000 on average. Lesser prizes are available under less stringent conditions; for example, a player with any 3 of the 6 winning numbers plus either supplementary number wins a 5th Division prize, typically around $20.

Some Tattslotto players select the same numbers week after week, often based on the birthdays or ages of close family members, or other numbers perceived to be "lucky". This raises the question: how long (how many draws) will it take before such a player wins, say, a 5th Division prize?

This question is too difficult to answer here. Instead, we consider a simpler but related question: how long will it take for one *specified* number to be drawn by the Tattslotto machine?

We will assume, for the rest of this article, that the Tattslotto machine is a perfect randomisation device, in that at each draw, each of the $\binom{45}{8} = 215\,553\,195$ possible subsets of 8 numbers from 45 has equal probability of being chosen, *regardless of the outcome of all previous draws.*

To be definite, suppose that our specific number of interest is number 13. First, what is the probability that number 13 will be "drawn", i.e. will be one of the 8 chosen numbers? Since all possible selections of 8 numbers from 1 to 45 are equally likely, we simply need to work out how many of those selections contain number 13. To do this, imagine that we put the

8 selected numbers into 8 boxes. We know that one of the boxes must contain number 13; the contents of the remaining 7 boxes must be chosen from the remaining 44 numbers (see Figure 1).

$$\boxed{?}\quad\boxed{?}\quad\boxed{13}\quad\boxed{?}\quad\boxed{?}\quad\boxed{?}\quad\boxed{?}\quad\boxed{?}$$

Figure 1

Hence the number of selections which contain number 13 is $\binom{44}{7}$. Therefore the probability, p, that number 13 will be drawn is

$$\begin{aligned} p &= \frac{\text{number of selections containing 13}}{\text{Total number of selections}} \\ &= \frac{\binom{44}{7}}{\binom{45}{8}} = \frac{44!\,37!\,8!}{37!\,7!\,45!} = \frac{8}{45}. \end{aligned} \tag{1}$$

Now let X denote the number of draws until number 13 appears as one of the 8 drawn numbers. At each successive draw, number 13 will appear with probability p, and will fail to appear with probability $q = 1 - p$, independently of what happens in other draws. Hence the probability that it will take x draws until number 13 is drawn is

$$P(X = x) = pq^{x-1} = \frac{8}{45}\left(\frac{37}{45}\right)^{x-1}, \; x = 1, 2, 3, \ldots . \tag{2}$$

This is simply the probability that number 13 fails to appear in the first $x - 1$ draws, and that it does appear in the x-th draw. We multiply the probabilities because we are assuming that the Tattslotto draws are independent in the probability sense.

The probability distribution given by (2) is known as the *geometric distribution.* Although there are infinitely many values of x, the probabilities nevertheless add up to 1. To see this, note that

$$\begin{aligned} p + pq + pq^2 + pq^3 + \ldots &= p(1 + q + q^2 + q^3 + \ldots) \\ &= \frac{p}{1-q} = \frac{p}{p} = 1, \end{aligned}$$

using the *geometric series*

$$1 + q + q^2 + q^3 + \ldots = \frac{1}{1-q}, \text{ for } |q| < 1. \tag{3}$$

The dots here indicate that the summation continues indefinitely.

On average, how many draws does it take until number 13 first appears? To compute this, we find the *mean* or *expected value* μ of the *random variable* X by multiplying each possible value of X by its probability, and summing up. In symbols,

$$\begin{aligned}\mu &= p + 2pq + 3pq^2 + 4pq^3 + \ldots \\ &= p(1 + 2q + 3q^2 + 4q^3 + \ldots)\end{aligned}$$

Even though there are infinitely many terms in this summation, the mean value μ is a finite number. To see this, we differentiate the geometric series (equation (3)) with respect to q, obtaining[1]

$$1 + 2q + 3q^2 + 4q^3 + \ldots = (1-q)^{-2} = \frac{1}{p^2}. \tag{4}$$

Hence

$$\mu = p \cdot \frac{1}{p^2} = \frac{1}{p} = \frac{45}{8} = 5.625.$$

This result agrees with intuition. Since 8 of the 45 numbers are drawn each week, we would expect to wait 45/8 weeks on average for any specific number (e.g. number 13) to turn up.

Similarly, we can easily compute the probability $P(X \leq x)$ that number 13 will be drawn in no more than x draws. The complementary event is that number 13 *fails* to be drawn in the next x draws; this has probability q^x. By the rule for complementary events,

$$P(X \leq x) = 1 - q^x, \; x = 1, 2, 3, \ldots \tag{5}$$

Two important points should be noted. First, the preceding arguments apply for any *specific* number, e.g. number 1, number 2, number 3 and so on, not just number 13. Secondly, the same result holds if we count time backwards rather than forwards. For example, we could equally well ask: how many draws is it since number 13 was last drawn? The answer is given by the same probability distribution (2) by exactly the same argument. Note that $x = 1$ corresponds to number 13 being drawn in the last (most recent) draw.

[1]The *term-by-term differentiation* used to derive (4) from (3) appears intuitively obvious, but it requires some advanced mathematical theory to justify it. See also the front cover article in *Function, Vol* 20, *Part* 3 for this type of series.

This leads us into a more interesting question. Each week, Tattslotto publishes a table giving the past history of Tattslotto results. In particular, it gives, for each number from 1 to 45, the number of weeks (draws) since that number was last drawn. Each count starts from zero, rather than 1, as we did in deriving (2) and (5). Hence, eight of those entries are zero, corresponding to the 8 numbers drawn in the last draw. But some entries in this table are surprisingly large. For example, after Draw 1511 (13 January 1996), number 1 had not been drawn for 23 weeks of Saturday draws (see Figure 2). This was the most "overdue" number out of the 45, with the *maximum* number of "weeks since last drawn". Since this is much larger than the average μ, this might suggest, at first glance, that the Tattslotto machine is faulty. Alternatively, could such an extreme result happen just by chance? The answer is "Yes", as we now show.

SATURDAY TATTSLOTTO FIRST DRAW RESULTS HISTORY AT 13 JANUARY 1996 - DRAW 1511

NO. OF WEEKS SINCE EACH NO. DRAWN								NO. OF TIMES EACH NO.DRAWN SINCE 413							
1: 23	2: 0	3: 0	4: 11	5: 0	6: 1	7: 1	8: 1	1: 110	2: 101	3: 110	4: 99	5: 96	6: 89	7: 104	8: 119
9: 0	10: 11	11: 0	12: 8	13: 5	14: 4	15: 2	16: 0	9: 104	10: 88	11: 101	12: 92	13: 99	14: 99	15: 102	16: 95
17: 12	18: 2	19: 2	20: 14	21: 1	22: 1	23: 3	24: 18	17: 90	18: 96	19: 108	20: 100	21: 92	22: 91	23: 107	24: 83
25: 5	26: 2	27: 3	28: 0	29: 0	30: 6	31: 7	32: 4	25: 101	26: 98	27: 84	28: 94	29: 95	30: 88	31: 104	32: 84
33: 1	34: 2	35: 6	36: 11	37: 5	38: 8	39: 13	40: 2	33: 99	34: 101	35: 105	36: 95	37: 93	38: 96	39: 77	40: 104
41: 13	42: 4	43: 1	44: 8	45: 2				41: 108	42: 101	43: 108	44: 95	45: 95			

Figure 2

We need some extra notation first. For each integer j from 1 to 45, let X_j denote the number of draws since number j was last drawn. For example, "$X_{13} = 5$" means that it is 5 weeks since number 13 was last drawn. Also, let $Y = \max(X_1, X_2, \ldots, X_{45})$, so that Y is the number of weeks since the "most overdue" number was last drawn. In Figure 2, $Y = 23$. What is the probability of this happening by chance, i.e. assuming that the Tattslotto machine is perfect?

Since all the X_j's are random, so too is their maximum Y. It turns out that it is easier to compute the cumulative probabilities for Y, rather than the direct probabilities such as $P(Y = 23)$. The argument is quite ingenious, and goes as follows.

For any positive integer y, the maximum, Y, of the 45 X_j's is less than or equal to y if *each* of the X_j's is less than or equal to y, and *vice versa*. Hence both "events" must have the same probability. In symbols,

$$\begin{aligned} P(Y \le y) &= P(X_1 \le y, X_2 \le y, \ldots, X_{45} \le y) \\ &= P(X_1 \le y)P(X_2 \le y)\ldots P(X_{45} \le y) \end{aligned} \tag{6}$$

assuming that the X_j's are all independent of one another. Each of the 45 factors on the right-hand side of (6) is given by (5), with x replaced by $y + 1$. So, *assuming* that the X_j's are independent, we find

$$P(Y \le y) = (1 - q^{y+1})^{45}, \ y = 0, 1, 2, \ldots \tag{7}$$

Hence

$$\begin{aligned} P(Y = y) &= P(Y \le y) - P(Y \le y - 1) \\ &= (1 - q^{y+1})^{45} - (1 - q^{y})^{45}, \ y = 1, 2, \ldots \\ \text{and } P(Y = 0) &= p^{45}. \end{aligned} \tag{8}$$

In fact, the X_j's are *not* independent. As already noted, eight of the X_j's must always be zero (not the same eight each time!). In contrast, our derivation from (6) to (7) does not rule out the possibility that *all* 45 X_j's could be simultaneously zero. So equation (7) must be regarded as an approximation.

Nevertheless, it is a very good approximation. To check the accuracy of (7) and (8), I wrote a computer program which simulated a total of 50 000 Tattslotto draws (roughly equal to 1000 years of Saturday draws!). After each draw, the program automatically computed the number of draws since each number was last drawn. After every 50th draw, the program recorded the actual value of Y (the maximum number of draws since last drawn). From these 1000 simulated values of Y, it was then possible to estimate the probabilities of various values of Y, without making any assumptions about the X_j's.

Figure 3 compares the approximate probability distribution (8) (solid curve) with the simulated one (dashed curve). As can be seen, both graphs have the same shape, with the approximate distribution shifted slightly to the left relative to the simulated distribution.

The probability that the most overdue number has not been drawn for 23 weeks is 0.0567 by (8), or 0.066 by the simulated probabilities. However, this does not answer the real question: how *unusual* is such an outcome? To answer this, we compute the probability of Y being 23 or *greater*. This turns out to be 0.395 by (8) or 0.427 by simulation. Since the total probability of all possible Y's is 1 (represented by the area under the smooth curve in Figure 3), our observed value of $Y = 23$ must be near the centre of the distribution of Y-values, as can be confirmed by the vertical dotted line in Figure 3. In other words, the observed result could well have arisen by chance alone, and there is no reason to suspect that the Tattslotto machine is faulty.

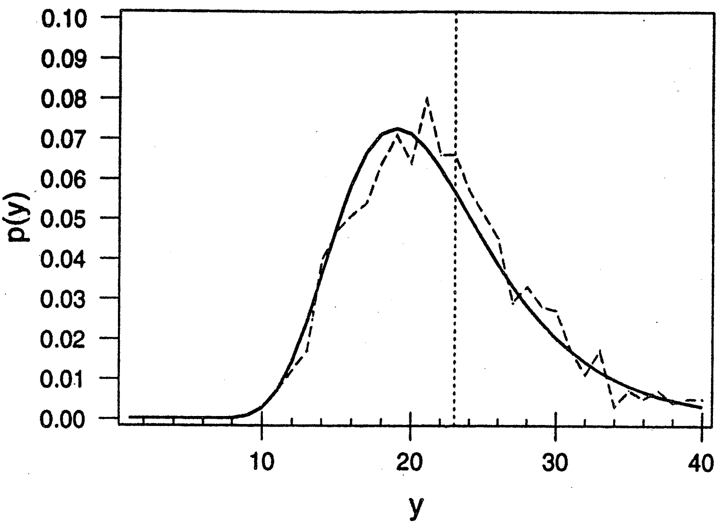

Figure 3: Approximate and simulated probabilities

Some Tattslotto players believe that the most overdue number is more likely to come up in the next draw than other numbers. This is a false belief. Unlike the players, the Tattslotto machine has no memory, and the most overdue number has exactly the same probability of being drawn as all other numbers. Being a maximum, the number of weeks since the most overdue number was last drawn is bound to be large. Figure 3 and equations (7) and (8) indicate how large it can be, by chance alone.

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ANAGRAM GROUPS: A "CRYPTIC" PIECE OF MATHEMATICS

Bert Bolton, University of Melbourne

The crossword puzzles of the daily newspapers in Australia, such as *The Age* and *The Australian*, are popular with readers. They seem at first sight to attract readers who are more literary than mathematical or scientific, but there are some mathematical problems concealed in them. Here is one which uses a knowledge of group theory.

Groups are concerned with a set of objects $\{a, b, c, \ldots\}$ and there can be a finite or an infinite number of the objects. In the problems that we will discuss there is a finite number of objects, and each is an "operator" which produces changes in the order of letters in a word.

A set G of elements is a *group* if and only if it is possible to define a binary operation which associates with every pair of elements a, b of G an element ab so that

1. ab is an element of G (it is said that G is *closed* under the binary operation),
2. $a(bc) = (ab)c$ for all elements a, b, c (associative law),
3. G contains an identity operator E such that for each element a of G, $Ea = aE = a$,
4. for each element a of G, G contains an *inverse* a^{-1} such that $a^{-1}a = aa^{-1} = E$.

The binary operation is not necessarily commutative, that is, in general, ab is not equal to ba.

We now return to the crossword puzzle.

Each word that has to be inserted into the crossword grid is found from its clue describing the word in a more or less hidden way; the harder puzzles are called *cryptic*. One of the familiar devices for hiding the word is to use an anagram, which is a transposition of letters of a word or phrase, to form a new word or phrase. A recent example of an anagram was given in *The Australian* crossword, which is taken from *The Times* newspaper in the United Kingdom and is traditionally one of the hard puzzles. The word has an odd number of letters and we concentrate on odd-numbered anagrams. Words with even numbers of letters are not so readily treated.

The clue from the puzzle is as follows:

In time apt perhaps to get irritable (9).

A nine-lettered word is sought and the word "perhaps" in the clue gives a lead. The first three words of the clue have 9 letters and "perhaps" suggests that the order of these 9 letters should be rearranged to make a word meaning "irritable". This is so; the answer was "impatient". As the puzzle proceeds, some letters are known from completed clues at right angles. If, for example, it is already known that the 9-lettered word ends in –E–T, the solver can be tempted to try the ending MENT or IENT and the "light can dawn". One of the difficulties of an anagram is to avoid the distracting pattern of letters already in the clue and there is a classic method of doing this. The letters of the anagram are arranged in a circle as in Figure 1. This helps the solver to start with any letter. There are many arrangements of 9 letters; if all letters are different from each other, the number of arrangements is the factorial of 9, that is $9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362,880$. But the phrase "in time apt" has two letters i and t that are repeated. Interchanging only the two letters i or the two letters t leaves the arrangement unchanged and then the number of arrangements is $9!/2!2! = 90,720$. Not all of them make a recognisable word.

Anagrams of 15 letters are sometimes given; the grid of squares in the crossword is 15×15. The first circular arrangement such as in Figure 1 often does not yield the answer required. New patterns of letters can certainly be found by random interchanges but one systematic method is to rewrite the circular pattern as follows. Take the first letter, omit the second, take the third, omit the fourth, take the fifth, etc. round the circle twice, and the pattern is given in Figure 2. We call this result the action of a "hop operator", and because the first letter is placed beside its second neighbour, we call it the "hop two" operator. The idea can be extended to a "hop three" operator and so on, but for the moment we restrict the discussion to hop two operators. This method of rearranging the letters can be repeated starting with Figure 2.

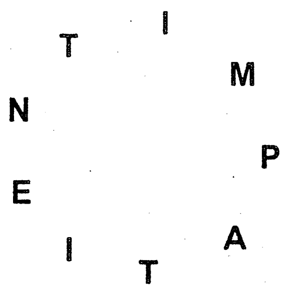

Figure 1

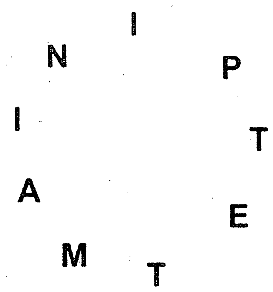

Figure 2

We now examine this procedure in more detail. We choose a five-lettered 'word' $ABCDE$ to make the diagrams easier. They are given in Figures 3 and 4, equivalent to Figures 1 and 2 respectively. We denote by O the hop two operator which changes the circular pattern from Figure 3 to Figure 4. The letter O will be used for the equivalent hop two operator in further patterns. There is no need to draw the letters in a circle so they will be used in a straight line, remembering that the 5 letters are repeated to represent the action of going twice around the circle. The consequence of the operator on $ABCDE$ is then

$$O(ABCDE) = ACEBD.$$

Using the operator O again gives OO which is written O^2, and

$$O^2(ABCDE) = O(ACEBD) = AEDCB.$$

Further,

$$O^3(ABCDE) = O(AEDCB) = ADBEC$$

and finally

$$O^4(ABCDE) = O(ADBEC) = ABCDE$$

which is the starting pattern. Further uses of O will only repeat what has already been found. We see that the operator O^4 restores the starting pattern. If we keep the concept of an operator, we can define the identity operator E that preserves the pattern on which it operates, and we can summarise the above by saying that for a word of 5 letters, there are 4 operators E, O, O^2, O^3, with $O^4 = E$. The letter E comes from the German word *die Einheit*, unity. Confusion between E as an operator and E as a letter can be minimised by noting the context.

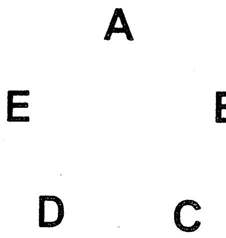

Figure 3

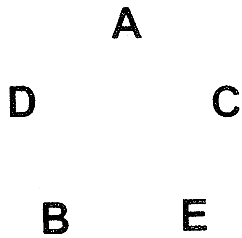

Figure 4

In general, let n be the number of letters and m be the number of operators; Table 1 shows the relationship between m and n for a few values of n.

n	3	5	7	9	11	13	15	17
m	2	4	3	6	10	12	4	8

Table 1

The set of m operators for n letters is a group and the number m is called the *order* of the group. As with so many branches of mathematics that look "pure" when discovered, group theory has proved to be very powerful when applied to new problems. Groups have helped with problems in the geometry of 2 and 3 dimensions and also with problems in advanced ideas in the physics of fundamental particles.

We need to find O^{-1}, the inverse operator of O in the group. We find it by noting that O is telling us how to advance through the pattern of n letters; the operator O^{-1} defines a retreat through the pattern. These ideas of advancing and retreating suggest the interpretation of OO^{-1}; a retreat followed by an advance should leave the pattern unchanged, or in the new language

$$OO^{-1} = E.$$

Recalling that $OO^3 = E$ for the discussion of $n = 5$, we can guess what O^{-1} must be. We have

$$O^3(ABCDE) = ADBEC$$

and the letters A, B, C, D and E are now arranged just one letter apart from the initial pattern. For instance, A and B are separated by D which could be achieved by starting at A and omitting the 2 letters B and C to put D next to A. Once the pattern $AD\ldots$ is started, omit the next 2 letters EA to get B and the pattern builds up $ADB\ldots$ and so on. Then

$$O(ADBEC) = ABCDE = E(ABCDE)$$

$$\text{or } OO^{-1} = E.$$

It is straightforward to confirm that $O^{-1}O = E$.

The generalisation of this method of defining O^{-1} for any n letters is to note that

$$O^{-1}(ABC\ldots;n) = A(\text{omit } (n-1)/2)\alpha(\text{omit } (n-1)/2)\beta\ldots$$

where α is the letter separated from A by $(n-1)/2$ letters and β is the letter separated from α by $(n-1)/2$ letters, etc. Then

$$OO^{-1}(ABC\ldots;n) = A\gamma\ldots.$$

The letter γ is separated from A by $(n-1)/2 + 1 + (n-1)/2$ letters which is n letters and γ is then B, independent of n.

The simplest way to see how geometry is brought within the scope of groups is through the representations of the operators. For the group $\{E, O, O^2, O^3\}$ we draw a unit circle and label the ends of two orthogonal diameters as in Figure 5. The operator E is represented by a point on the circle. The operator O is represented by a 90° rotation from E to the end of the vertical diameter marked O. Then O^2 must involve a rotation of 180°, O^3 a rotation of 270° and $O^4 = E$, a rotation of 360° which is just the starting point E. Such a group is known as *cyclic*.

The group in question is the cyclic group of order 4, and for general n, there is a cyclic group of order m.

The discussion of Figure 5 is equivalent to the more familiar expressions either that the figure has four-fold symmetry or that rotating the pattern by 90 degrees leaves it invariant. Examples are found in many places. The benzene molecule C_6H_6 has a carbon atom at each vertex of a regular hexagon with a hydrogen atom radially outwards from each vertex. The chemical properties of the benzene molecule relate to the properties of a group. Church and decorative windows are often found with 3, 4, 5, 6 and

8-fold symmetries. The swastika, made so unpopular in recent history, has four-fold rotational symmetry.

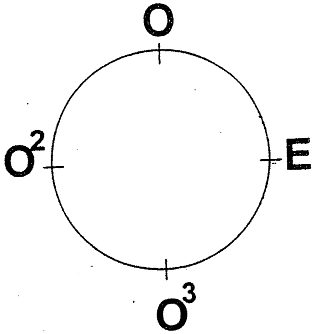

Figure 5

The grids of crossword puzzles have symmetries. The grids can have 2-fold or 4-fold symmetries and sometimes they have a further reflection symmetry about a horizontal or vertical axis. This is a problem distinct from the cyclic groups arising from the anagrams. Occasionally the printer makes the mistake of putting the wrong grid with the verbal clues and then the solver has to construct the pattern of black and white squares from the symmetry given by the clues, before attempting to find the answer to each clue.

So far we have confined the representations to two-dimensional patterns, but symmetrical patterns also occur in three dimensions. The crystals and their symmetries that are found in nature or grown in laboratories have been classified according to group theory and group representation. The quartz crystal, whose vibrations regulate an "electronic" watch or clock, was grown under careful conditions and such crystals always have the same symmetries and properties.

* * * * *

I have traveled the length and breadth of this country and talked with the best people, and I can assure you that data processing is a fad that won't last out the year.

– From the editor in charge of business books
for Prentice Hall, 1957

* * * * *

HOW TO CALCULATE CUBE ROOTS WITH SQUARE ROOTS

Michael A B Deakin

Of course, if you have any sort of sophisticated calculator at all, then it is a simple matter to calculate the cube root of any positive number N by raising it to the power $\frac{1}{3}$. However, if you don't have this then other measures are needed. Here is one approach that I learned from a calculator salesman about 25 years ago. I can still use it on a small pocket Tandy that I bought some time back then. This is very basic, but it does possess a $\surd$ (**square root**) button and a single memory. For the method to work, the first is essential, the second helpful.

The key to the method is to take an approximation, x_0 let us call it, and to improve on this. The improved approximation, x_1 say, is the fourth root of the product of x_0 and N. This can be calculated as $\surd\surd(x_0 N)$. Once x_1 is found, then we can use it to produce an even better approximation x_2, and so on.

Here is how I got a value for the cube root of 2 on my little Tandy. I indicate the Memory button by **M** and the **R**ecall button by **R**.

$2\mathbf{M}\surd$	Enters 2 into the memory and uses $\sqrt{2}$ as x_0, giving	$x_0 = 1.4142135$
$\times\mathbf{R} = \surd\surd$	Multiplies x_0 by the 2 in memory and takes the fourth root, giving	$x_1 = 1.2968395$
$\times\mathbf{R} = \surd\surd$	Repeats the previous step, giving	$x_2 = 1.2690509$
	$\vdots$	$\vdots$
		$x_{12} = 1.2599210$

and this is the best approximation this calculator can give.

First up, let's see how the method works. We have

$$x_1 = (x_0N)^{\frac{1}{4}},\ x_2 = [(x_0N)^{\frac{1}{4}}N]^{\frac{1}{4}},\ x_3 = \{[(x_0N)^{\frac{1}{4}}N]^{\frac{1}{4}}N\}^{\frac{1}{4}}, \text{ and so on.}$$

Each successive approximation is the product of a power of x_0 and a power of N.

The exponents of x_0 are successively $\frac{1}{4}, (\frac{1}{4})^2, (\frac{1}{4})^3, \ldots, (\frac{1}{4})^n$, and so on. These exponents approach the limit 0 as n tends to infinity, and $x_0{}^0 = 1$. (Start with any positive number and take successive fourth roots, or for that matter square roots, and you will find that the numbers approach 1, although it can sometimes happen with some calculators that this limit is not exactly achieved.) So, whatever positive number we begin with, we end up with a limiting value of 1 for this part of the answer.

The exponents of N are successively $\frac{1}{4}, \frac{1}{4} + (\frac{1}{4})^2, \frac{1}{4} + (\frac{1}{4})^2 + (\frac{1}{4})^3, \ldots, \frac{1}{4} + (\frac{1}{4})^2 + \ldots + (\frac{1}{4})^n$, and so on. These are the successive sums of a geometric series with first term and common ratio both equal to $\frac{1}{4}$. The limit of these sums as n tends to infinity is $\frac{1}{3}$, as you can easily check. So the limit of the powers of N is $N^{\frac{1}{3}}$, and this will be the limit of the product because the other factor approaches 1.

We can also find out how the successive approximations improve on each other. Suppose the exact value of the cube root is x and that one of our approximations gives a value $x + \varepsilon$, where ε is a small error. Now because x is the exact value of the cube root, we have $N = x^3$. Thus the next approximation is $[x^3(x + \varepsilon)]^{\frac{1}{4}}$. That is to say, $[x^4(1 + \varepsilon/x)]^{\frac{1}{4}}$, and this works out to be $x(1 + \varepsilon/x)^{\frac{1}{4}}$. Now if h is small and n not too large, then $(1 + h)^n$ is very close to $1 + nh$. We can apply this result here to find that the new value is approximately $x + \frac{1}{4}\varepsilon$. In other words, each step in the calculation reduces the error by a factor of 4.

As approximations go, this is a rather slow approach to the true value, and for serious numerical work other methods would be employed. Nonetheless, the simplicity of the method and its ease of application make it a nice example to explore.

It is often particularly instructive to explore cases where you already know the answer. For example, if $N = 8$, we know that the answer we seek is 2. If we start with $x_0 = \sqrt{8}$, we find $x_5 = 2.0006769$, and successive approximations after this are 2.0001691, 2.0000422, 2.0000105, etc. The rule of $\frac{1}{4}$ is quite evident.

For large numbers, $\sqrt{N}$ is not a particularly good initial approximation. Nevertheless, this is not a major problem. As an example, I looked at the case $N = 1\,000\,000$. The true value of the cube root is of course 100, and if we start with the approximation $\sqrt{N}$, then we have $x_0 = 1000$, which is 10 times too big. However, things soon settle down; the next approximations are in order 177.82794100, 115.47819847, 103.66329284, and already we are only a bit more than 3% out. After this, we get 100.90350448, 100.22511483, 100.05623126, 100.01405485, and so on. Notice the "$\frac{1}{4}$ rule" here again. It becomes established quite quickly.

* * * * *

Mathematician at work

An engineer and a mathematician are in a room with one door. Both of them are asked (one at a time) to go into the next room, where they will find a pot of water and a stove, and boil the water.

The engineer goes into the next room, sees the pot of water sitting on a table, puts it on the stove, turns on the stove, and waits until it boils.

A little bit later (the room is reset for the mathematician), the mathematician goes into the next room, sees the pot sitting on the table, puts it on the stove, turns on the stove, and waits until it boils.

Now both of them are asked to do it again. However, this time the pot of water is on the floor. The engineer sees the pot of water on the floor, puts it on the stove, and then waits until it boils.

The mathematician, however, sees the pot of water on the floor, puts it on the table and proclaims "The problem is now reduced to one which has been previously solved."

* * * * *

Where a calculator on the ENIAC is equipped with 10 000 vacuum tubes and weighs 30 tons, computers in the future may have only 1 000 tubes and weigh only one and a half tons.

– From *Popular Mechanics*, 1949

* * * * *

HISTORY OF MATHEMATICS

More on Babylon

Michael A B Deakin

In our last issue, Benito Hernández-Bermejo wrote about the Babylonian tablet YBC 7289 and the inference that the ancient Babylonians calculated square roots by means of an averaging process. In an earlier issue (*Volume* 15, *Part* 3), we also encountered the ancient Babylonians: in this case their listing of Pythagorean triples. Quite coincidentally, as the editorial board was looking over the article on square roots, a flurry of email arrived directly on these matters.

The "bible" on ancient Babylonian mathematics may be said to be the work of Neugebauer and Sachs. In English there is their co-authored book *Mathematical Cuneiform Texts* and it was from this that we took the illustration that the editors appended to Hernández-Bermejo's article. However, this book was published in 1945, and so is quite old by now. The same point therefore applies with even more force to Neugebauer's earlier *Mathematische Keilschrift-Texte*, a three-volume effort published over the period 1935-1937. Although this pioneering work was of an exceptionally high quality, modern researchers have gone beyond it to discover more. And, as always, to raise more questions.

Contemporary workers like Jens Høyrup and Eleanor Robson have re-evaluated much of the older work, especially in the light of more recent evidence. The rest of this article depends on some recent postings by Eleanor Robson to a select newsgroup on the history of mathematics, together with some very pertinent comments by David Fowler from the same newsgroup.

In the first place, we are not just talking of Babylon, the city, but using the term rather loosely to apply to quite a large area of what is now Iraq, Syria and Iran. Indeed, we have no archaeological evidence of Babylon itself during the relevant (early) period (around 2000-1600 BC); excavation of the relevant levels is both technically and politically impossible. The early tablets (those described in works up to about 1945) were dug up (rather than scientifically excavated) and often collected by tourists who bought them at street markets and the like. They made their way to Europe and eventually into the museums where they are held today, but they lack what scholars call "provenance" – because of the manner in which they made their way to their present locations, we have no direct evidence as to where they originally came from and when they were made.[1]

However, we now have thousands of clay tablets with mathematical calculations and tables from this general area and time. They are classified into various types. The largest group is that of the "table texts", which give products, reciprocals, squares, cubes and other powers, as well as listing conversions between the two systems of number representation that were in use.[2] Then there were "problem texts", which in essence were teaching aids. The next category are student "workbooks". These three categories account for nearly all of the known tablets. Three further but much smaller categories (which will not concern us here) make up the remainder.

The tablet known as YBC 7289 (and briefly discussed in Hernández-Bermejo's article) is an example of a student workbook tablet. It is usually described as depicting a square of side 30 with its two diagonals drawn. Along one diagonal is the cuneiform number which today we would write as $(1;24,51,10)$. This has to be interpreted as

$$1+\frac{24}{60}+\frac{51}{60^2}+\frac{10}{60^3},$$

which works out to be 1.414212963, a number very close to $\sqrt{2}$, whose value is more exactly 1.414213562. However, $(1;24,51,10)$ is the best approximation to be had using three places after the semicolon in base sixty.

Underneath this number is another, which is normally interpreted as $(42;25,35)$, and this is the product of 30 and $(1;24,51,10)$. So the usual interpretation of the tablet is that the student, having been given an approximate value of $\sqrt{2}$, was asked to calculate the diagonal of a square of side 30.

This is one view of the matter, but it is complicated by the fact that the tablet nowhere employs any equivalent of the semicolon – the way in which we today distinguish the integral part from the fractional part. (In base ten this function is performed by the decimal point.) Thus, David Fowler suggests that the 30 may in fact be $(0;30)$, which is the Babylonian equivalent to our $\frac{1}{2}$. On this interpretation, the $(42;25,35)$ is not to be so interpreted but rather as $(0;42,25,35)$, the Babylonian equivalent of

[1]This is why the date of the tablet known as Plimpton 322 and described in *Function*, *Vol* 15, *Part* 3 is so uncertain.

[2]Here we will only be concerned with the so-called "scientific" system; the one using base sixty.

$1/\sqrt{2}$. This has the merit of making the two numbers $(1;24,51,10)$ and $(0;42,25,35)$ a reciprocal pair: their product is 1. It is known that the Babylonians were very interested in reciprocal pairs. (They were needed for division.) So Fowler's reading is very attractive, although we have no way of proving it definitively.

It was on the basis of YBC 7289 that Neugebauer and Sachs suggested that the Babylonians found square roots by using the averaging algorithm – the one described in Hernández-Bermejo's article. Apart from the numerical value itself, there was very little evidence for the suggestion, and there are other algorithms that could have been used to produce the same value. Thus it was only a hypothesis that this was how they proceeded. However, we now have some further evidence that makes this hypothesis much more likely, although that evidence remains a little short of being fully convincing.

There is another tablet, known as VAT 6598, which supplies much of the missing evidence.[3] VAT 6598 is actually only a fragment of the original; a large part of it has broken off. However, a few years ago, one of the curators of the British Museum collection realised that one of their fragments (known as BM 96957) was in fact part of the missing piece.[4] It is known that this tablet is over 3500 years old and thought very likely that it comes from Sippar, near Baghdad. In any case, we are now able to read much of the tablet and to follow the calculations on it.

Eleanor Robson has translated it into English, and here is her version.

> (problem xviii) A gate, of height $\frac{1}{2}$ ⟨ a rod ⟩, 2 cubits and breadth 2 cubits. What is its diagonal? You: square $0;10$, the breadth. You will see $0;01\ 40$, the base. Take the reciprocal of $0;40$ (cubits), the height; multiply by $0;01\ 40$, the base. You will see $0;02\ 30$. Break in half $0;02\ 30$. You will see $0;01\ 15$. Add $0;01\ 15$ to $0;40$, the height. You will see $0;41\ 15$. Its diagonal is $0;41\ 15$. The method.

> (problem xix) If a gate has height $0;40$ (cubits)[5] and diagonal $0;41\ 15$, what is its breadth? You: take $0;40$, the height, from

[3]The tablet is in the National Museum in Berlin and the VAT stands for Vorderasiatische Tontafelsammlung (i.e. Near Eastern Clay Tablet Collection). 6598 is its number in their catalogue.

[4]Sadly, about a third of the original is still missing!

[5]This actually means $(0;40)$ rods, or 8 cubits. The expression has been rather confusingly abridged.

0; 41 15, the diagonal. 0; 01 15 is the remainder. Double 0; 01 15. You will see 0; 02 30. Multiply 0; 40, the length, by 0; 02 30, the factor that you saw. You will see 0; 01 40. What is the square root? 0; 10 is the square root. The method.

(problem xx) The breadth is 2 cubits, the diagonal is 0; 41 15. What is the height? You: no (solution).

Let us now see quite what all this means. The cubit was an old unit for measuring length. It was based on the length of the human forearm and so its actual size varied quite considerably; however, a rough modern equivalent is 50 cm. 12 cubits made a rod. So if we look at problem xviii, we find that the height of the gate is $\frac{1}{2}$ a rod plus 2 cubits, a total of 8 cubits, which is $\frac{2}{3}$ of a rod. We now have $\frac{2}{3} = \frac{40}{60}$, so the height in rods is (0; 40), about 4 metres. Similarly the breadth in rods is (0; 10), about 1 metre.

In what follows, I'll avoid using the actual numbers involved in the problems (all dealing with the same rectangle, as we shall see), but rather use algebraic symbols. This is so as to assist our 20th Century understanding, of course; it wasn't how the ancient Babylonians would have worked.

So for the height of the gate I will write h. For its breadth I will put b and for its diagonal I will have d. So in the first problem (xviii) the number we seek is d, and the value will be (exactly) $\sqrt{b^2 + h^2}$. Now let us follow the instructions on the tablet, and as we do so translate them into modern mathematics.

See also the figure (based on a diagram actually shown on the tablet, but here we use more modern pictorial conventions).

Square the breadth	b^2	(called the "base")
Take the reciprocal of the height	$\frac{1}{h}$	
Multiply by the base	$\frac{b^2}{h}$	
Break in half	$\frac{b^2}{2h}$	
Add this to the height	$h + \frac{b^2}{2h}$.	

This is given as the value of d.

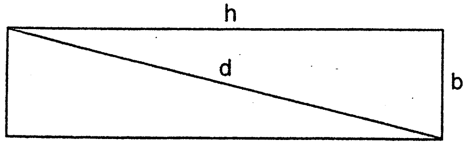

Figure 1: A modern version of the right-angled triangle of VAT 6598 (viewed side-on as in the original).

Now let us compare this value with that obtained by the method explained by Hernández-Bermejo in his article. Take h as the first approximation to the value of d. This will be an *underestimate* (quite clearly). Now consider $\frac{b^2+h^2}{h}$. This will be an *overestimate* of $\sqrt{b^2+h^2}$, the value we seek. So take the average of the underestimate and the overestimate, $\frac{1}{2}\left(h+\frac{b^2+h^2}{h}\right)$, which simplifies to $h+\frac{b^2}{2h}$, the value found on the tablet.

We can readily check how accurate the approximation is. In our notation $b=\frac{1}{6}$ and $h=\frac{2}{3}$, so the actual value of d is $\frac{1}{6}\sqrt{17}$, or about 0.6872. The value given on the tablet is $(0;41,15)$, which is 0.6875, so we have quite a good approximation, produced very easily.

Now let us look at problem xix. Clearly this (and also the aborted problem xx) refer to the same rectangle. Here the value sought is that of b, and the true value is $\sqrt{d^2-h^2}$. We follow the steps as in the previous case.

Take the height from the diagonal	$d-h$	(the "remainder")
Double the remainder	$2(d-h)$	
Multiply this by the length (height)	$2h(d-h)$	
Take the square root	$\sqrt{2h(d-h)}$.	

This is actually a reasonable approximation to the true value, because b is considerably less than h. Numerically, we have $d=\frac{11}{16}$ (this is the value in our modern notation of $(0;41,15)$ in base sixty) and $h=\frac{2}{3}$. Thus the correct value for b is about 0.16796, where the value given is $\frac{1}{6}$, or 0.16666... . The solution on the tablet has $\sqrt{2h(d-h)}$, or $\sqrt{(h+h)(d-h)}$, whereas correct is $\sqrt{(d+h)(d-h)}$. It happens that the value found for $2h(d-h)$ is an exact square (Babylonian $(0;01,40)$ is our $\frac{1}{36}$), and so the square root is exact. It would appear that it must have been either memorised or found from a table of exact square roots, as no method is given for finding it.[6]

The evidence for the "averaging algorithm" therefore depends on problem xviii. It is not quite as strong as we might like. In the first place, the averaging is not direct, although the *answer* that is produced is the same answer that is obtained by averaging. But all we really have is that $h+\frac{b^2}{2h}$ is a good approximation to $\sqrt{b^2+h^2}$. This could in fact have been produced in several ways. One quite good one is to use the "binomial approximation" $(1+k)^\alpha \approx 1+\alpha k$, where k is small and α not overly large. It works like this:

$$\sqrt{b^2+h^2} = \sqrt{h^2\left(1+\frac{b^2}{h^2}\right)} = h\left(1+\frac{b^2}{h^2}\right)^{1/2} \approx h\left(1+\frac{b^2}{2h^2}\right) = h+\frac{b^2}{2h}.$$

In the context of this problem, we have $k=\frac{b^2}{h^2}=\frac{1}{16}$, which is reasonably small, and so the method works, but it may not be how the Babylonians got the approximation.

Another possibility is to take x as an approximation to $\sqrt{N}$, and seek to improve it. Suppose the better approximation is $x+\varepsilon$. But then we have $N=(x+\varepsilon)^2 \approx x^2+2x\varepsilon$, as ε is small and thus ε^2 is very small. So $\varepsilon=\frac{N-x^2}{2x}$, as used in problem xviii. Mathematically, this method, the binomial approximation and the averaging technique are all equivalent to one another[7], but *psychologically* they are not, and so *historically* they are not. They use *different insights* although they reach *the same answer.*

The second objection comes from one of David Fowler's postings, and it is this: that there is no hint of the approximation's being improved by applying the method a second time. In Hernández-Bermejo's article, there was the very clear idea of *iteration*, that is to say of using a process over and over again to improve accuracy. There is no suggestion of this in the present case. In the three algorithms considered above, the averaging technique and the third approach lend themselves readily enough to iteration but this is not quite so apparent with the binomial approximation.

[6]It will be noted that the value discovered for b is in fact the exact value given in problem xviii, whereas the formula that produced it is only an approximation. This is because the value of d is not itself exact. It may also be noted that if the same method were to be applied to problem xx, the error would be much larger. (It is one sexagesimal digit.)

[7]And to other techniques as well.

So it would seem that we are not entirely sure that the Babylonians actually used the algorithm that is often attributed to them. That they did is one hypothesis – an attractive one, but the evidence for it is hardly compelling.

In my next column, I will look again at another tablet, Plimpton 322. It is a pleasure to thank Dr Robson for her constructive contributions to both these articles.

* * * * *

A precise statement

It is often imagined that the love of precise detail in mathematics makes it cold and inhuman, but this need not be so.

The town of Ross lies almost in the centre of Tasmania and is one of the more historic towns of that island state. It is pretty enough but cold and windswept; its cemetery and the few remains of a women's prison bespeak a harsh past.

Perhaps one notices most the graves of children, tiny plots saying "died aged 3 months" and so on. Two children in one family both died on the same day. Infant mortality seems to have been high.

But surely the most moving headstone of all records the passing of Martha Bacon on the 23rd of May, 1862, aged 14 months and one day.

Someone cared enough to be precise.

* * * * *

Music is the pleasure the human soul experiences from counting without being aware that it is counting.

– Gottfried Wilhelm Leibniz

* * * * *

CbXpP

COMPUTERS AND COMPUTING

The Pythagoras Tree

Cristina Varsavsky

In this column we have previously showed several programs to draw fractals on the screen. This time we present the Pythagoras tree. The basic step in its construction uses the representation of Pythagoras's theorem for right-angled triangles. This is depicted in Figure 1: the sum of the areas of the two smaller squares generated by the two adjacent sides equals the area of the larger square determined by the hypotenuse.

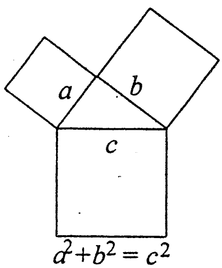

Figure 1

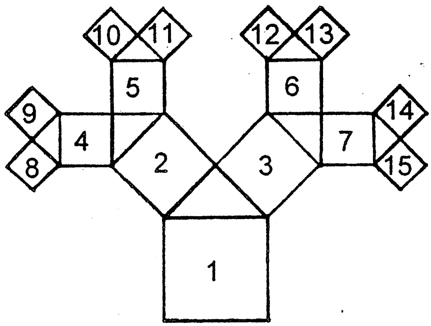

Figure 2

The representation of Pythagoras's theorem for an isosceles right-angled triangle is the basic repeating motif – on an ever reduced scale – of the Pythagoras tree. Figure 2 shows the first four steps of its construction. We start with the square labelled as 1, then we construct the two squares 2 and 3 so that they meet at 90 degrees. The third step consists of repeating the same procedure for the square 2, to obtain squares 4 and 5; and to the square 3, to obtain the squares 6 and 7. The process is repeated for each of these four new squares to obtain the squares 8, 9, 10, 11, 12, 13, 14, and 15. The Pythagoras tree is the figure obtained as this iterative process is performed infinitely many times.

Here is how we write a computer program to draw this tree – or rather its first few branches. First we observe that the number of squares drawn at each successive stage is 1, 2, 4, 8, 16, ... respectively; then at the

p-th stage we draw 2^p squares (note that square 1 corresponds to $p = 0$). Each square is one of the two squares supporting the same triangle, one to its left, the other to its right. From Figure 2 we observe that all even numbered squares are to the the left of the corresponding triangle, and the odd numbered squares are to the right. Since the Pythagoras tree is a binary tree, we can use the binary representation of the numbers to draw the corresponding square. For example, the binary representation of 13 is 1101, and this sequence of ones and zeros tells us how to traverse the tree to find the position of the square 13. A "1" simply means "turn right", and a "0", "turn left". So we start from square 1, then we traverse the tree to the right (second 1 in the binary representation), then left (the 0), and then right again (the last 1).

The QuickBasic program that follows uses this binary representation of the position of each square within the tree. Figure 3 shows the corresponding output.

```
REM Pythagoras Tree

SCREEN 9: WINDOW (-8, -3)-(8, 9)
pi = 3.1416
order = 8: DIM array(order)
x = 0: y = 0: z = 1: w = 1

FOR i = 0 TO order
   FOR j = 2 ^ i TO 2 ^ (i + 1) - 1
       s = j
       x = 0: y = 0
       length = 1: angle = 0
       FOR m = 0 TO i - 1
          array(i - m) = s MOD 2
          s = INT(s / 2)
       NEXT m
       x = 0: y = 0
       FOR m = 1 TO i
       IF array(m) = 0 THEN
           x = x - length * (COS(angle) + 2 * SIN(angle))
           y = y + length * (2 * COS(angle) - SIN(angle))
           angle = angle + pi / 4: length = length / SQR(2)
```

```
        ELSE
            x = x + length * (COS(angle) - 2 * SIN(angle))
            y = y + length * (2 * COS(angle) + SIN(angle))
            angle = angle - pi / 4: length = length / SQR(2)
        END IF
        NEXT m
        z = length * (COS(angle) + SIN(angle))
        w = length * (COS(angle) - SIN(angle))

        LINE (x - w, y - z)-(x + z, y - w)
        LINE -(x + w, y + z)
        LINE -(x - z, y + w)
        LINE -(x - w, y - z)
    NEXT j
NEXT i
b$ = INPUT$(1)
END
```

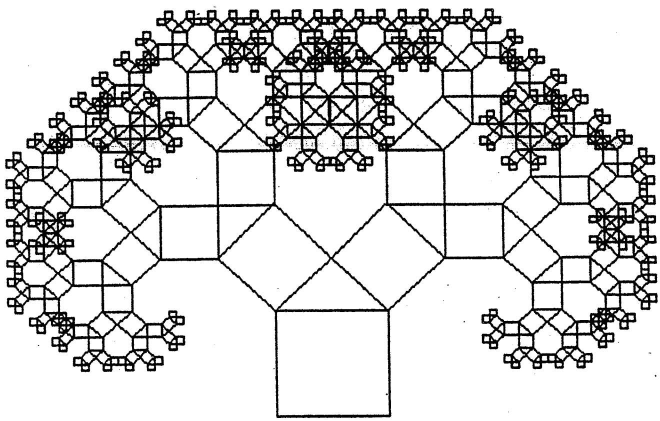

Figure 3

There are several ways you can modify the program to obtain more sophisticated trees. One such way is to start with a right-angled triangle with unequal sides.

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 20.2.1 (modified from a problem in *Alpha*, a German mathematics magazine, September 1995)

It is known that the teachers for classes 5A to 5E will be Mr Brown, Mrs Green, Mr Black, Mr Gray and Ms White, but it has not yet been announced which teacher will be in charge of which class. The table below shows the predictions by two students. The first student made two correct guesses associating teacher and class, and the second made three correct guesses. Who is the teacher for each class?

Class	5A	5B	5C	5D	5E
1st student's guess:	Black	Green	White	Gray	Brown
2nd student's guess:	Brown	Black	Gray	White	Green

SOLUTION

The two students did not agree on any teacher. Thus, a correct guess by the first student means that the second was wrong twice, first by giving the particular teacher the wrong class, and then by choosing the wrong teacher for the particular class. If the other correct guess by the first student did not refer to the teacher and class already shown to be wrongly assigned by the second, the second student would have made more than two wrong guesses. The first student must therefore have picked correctly the teachers for classes 5C and 5D, which the second student had interchanged while guessing all the others correctly.

PROBLEM 20.2.2 (modified from a problem in *Alpha*, June 1995)

"It's curious," says Karen. "I decided to select the PIN for my Bankcard by dividing my year of birth by the street number of our house and choosing the last four digits shown on my calculator. They turn out to be 1996."

"How many digits does your street number have?" asks Melissa.

"Two."

"In that case you made a mistake or your calculator is not operating properly. You cannot get the digits 1, 9, 9, 6 in sequence among the decimal places when you divide an integer by a two-digit integer."

Prove that Melissa is right.

SOLUTION

Let a and b denote respectively Karen's year of birth and street number. Then we can write the decimal expansion of $\frac{a}{b}$ as $N.x_1x_2\dots x_n1996x_{n+5}\dots$. Now multiply by 10^{n+1}:

$$\frac{10^{n+1}a}{b} = Nx_1x_2\dots x_n1.996x_{n+5}\dots \tag{1}$$

where the left-hand side is written with the usual conventions of algebra, and the right-hand side is a decimal expansion.

The integer part of the right-hand side of equation (1) is $Nx_1x_2\dots x_n1$; call this number k, and subtract it from both sides:

$$\frac{10^{n+1}a - bk}{b} = 0.996x_{n+5}\dots \tag{2}$$

The left-hand side is a fraction whose denominator is less than 100. The right-hand side is just less than 1. But the largest fraction less than 1 with a denominator less than 100 is $\frac{98}{99}$, which has a decimal expansion of $0.989898\dots$, clearly less than the right-hand side of equation (2). Therefore the sequence 1, 9, 9, 6 cannot occur among the decimal places when an integer is divided by a two-digit integer. (In fact, essentially the same argument shows that the sequence 9, 9 cannot occur.)

PROBLEM 20.2.3 (from *Alpha*, June 1995)

Let $ABCS$ be a regular pyramid with base $\triangle ABC$ and apex S. Let the angle at S in each side face be α. Let M be a point on a side face at a distance l from the apex. Determine the length of the shortest closed path through M and enclosing S, if it exists.

SOLUTION

The simplest way to solve the problem is to unfold the pyramid! Figure 1 shows the three side faces of the pyramid after they have been flattened out, together with a second copy (shown dashed) of the face containing M, with the copy of M denoted by M'. Provided that $\alpha < 60°$, the line $\overline{MM'}$ corresponds to a closed path on the pyramid, through M and enclosing S.

From the diagram, we see that the length of this line is $2l \sin \frac{3\alpha}{2}$.

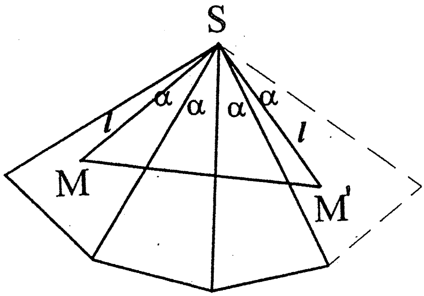

Figure 1

If $\alpha \geq 60°$, the problem has no solution, as all closed paths through M and enclosing S have length greater than $2l$, and paths of length arbitrarily close to $2l$ can be found.

PROBLEM 20.2.4

Let P be a point inside a triangle ABC. Let D, E and F be on $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ respectively, such that $\overline{PD} \perp \overline{AB}$, $\overline{PE} \perp \overline{AC}$, and $\overline{PF} \perp \overline{BC}$ (see Figure 2). It is required to choose P so as to minimise $PD + PE + PF$. Show that if $\triangle ABC$ is not isosceles then P must be situated at a vertex. What happens if $\triangle ABC$ is (a) isosceles; (b) equilateral?

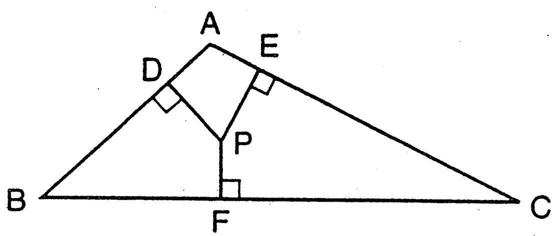

Figure 2

SOLUTION

Begin by observing that the area of the triangle with vertices at A, B and P is $\frac{1}{2}AB \cdot PD$, the area of the triangle with vertices at A, C and P is $\frac{1}{2}AC \cdot PE$, and the area of the triangle with vertices at B, C and P is $\frac{1}{2}BC \cdot PF$. Thus the total area of the triangle ABC is $\frac{1}{2}(AB \cdot PD + AC \cdot PE + BC \cdot PF)$, irrespective of how P is chosen.

Now suppose that $\triangle ABC$ is not isosceles, and that $AB > AC > BC$. Then:

$$\frac{1}{2}AB(PD + PE + PF) > \frac{1}{2}(AB \cdot PD + AC \cdot PE + BC \cdot PF) \quad (1)$$

unless PE and PF are both zero, in which case the two sides are equal. Therefore $PD + PE + PF$ is minimised if $PE = PF = 0$, so P coincides with C, the vertex opposite the longest side.

If $\triangle ABC$ is isosceles with $AB > AC = BC$, the same argument applies, and P is again the vertex opposite the longest side. If, on the other hand, $\triangle ABC$ is isosceles with $AB = AC > BC$, then both sides of (1) are equal whenever $PF = 0$, so P may be any point on the side $\overline{BC}$.

Finally, if $\triangle ABC$ is equilateral then $AB = AC = BC$, so the area of the triangle is $\frac{1}{2}AB(PD + PE + PF)$. Hence P may lie anywhere on or inside the triangle.

PROBLEM 20.2.5 (from *Mathematical Digest*, July 1995, University of Cape Town)

Exactly one of the following five statements is true. Which one?

(1) All of the following.

(2) None of the following.

(3) Some of the following.

(4) All of the above.

(5) None of the above.

SOLUTION

(1) and (4) are clearly false. If (3) were true then (5) would be true, which would lead to a contradiction. Hence (3) is false. If (5) were true

then (2) would be false, and this would yield a contradiction. Therefore (5) is false. This leaves (2) as the true statement. It is easy to check that this assignment of truth values ((2) is true, the other four statements are false) is consistent.

PROBLEM 20.2.6 (from *Mathematical Mayhem*, Vol 8, Issue 3, University of Toronto)

Show that the sum of any 1996 consecutive integers cannot be a power of an integer with exponent greater than one.

SOLUTION by Derek Garson

Let S denote such a sum. Let $T(n)$ denote the sum of the first n consecutive integers. Then S can be written in the form $T(n+1996)-T(n)$ for some value of n. It is well known that $T(n) = \frac{1}{2}n(n+1)$. Hence

$$\begin{aligned} S &= \frac{1}{2}(n+1996)(n+1997) - \frac{1}{2}n(n+1) \\ &= \frac{1}{2}(n^2+1996n+1997n+1996\times 1997 - n^2 - n) \\ &= \frac{1}{2}(1996\times 2n + 1996\times 1997) \\ &= 998(2n+1997) \\ &= 2\times 499(2n+1997) \end{aligned}$$

[The same result can be obtained in other ways. In particular, it follows directly from the formula for the sum of an arithmetic series. Eds.]

Since 499 and $2n+1997$ are both odd, 2 appears exactly once in the prime factorisation of S, so it is clear that the claim is true. (More formally, if $S = m^t$, then, since $2 \mid S$, we must have $2 \mid m$. If $t \geq 2$ then this would imply that $4 \mid S$, so $2 \mid 499(2n+1997)$, which is clearly false.)

Solution to an earlier problem

The problem below, by K R S Sastry, appeared in the April 1991 issue. (We have reworded it slightly for clarity.) A solution has not previously appeared in *Function.*

PROBLEM 15.2.3

A triangle is called *self-median* if two of its sides are in proportion to two of its medians. More precisely, if $\overline{AD}$ and $\overline{BE}$ are medians of $\triangle ABC$ then the triangle is self-median if $\frac{AC}{AD} = \frac{BC}{BE}$. Let $\overline{AD}$ and $\overline{BE}$ be medians and G

the centroid of $\triangle ABC$. Prove that if $\angle DGC = \angle BAC$ and $\angle CGE = \angle ABC$ then $\triangle ABC$ is self-median.

SOLUTION

The situation is depicted in Figure 3.

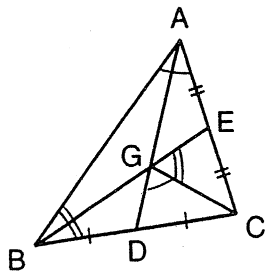

Figure 3

$$\angle DGC = \angle BAC \text{ and } \angle CGE = \angle ABC \quad \text{(given)}$$

$$\angle DGE = \angle BAC + \angle ABC \quad \text{(adding the angles)}$$

$$\angle DGE + \angle ACB = \angle BAC + \angle ABC + \angle ACB = 180°$$

$$\text{(sum of the angles of } \triangle ABC\text{)}$$

$$\angle CDG + \angle CEG = 180°$$

$$\text{(since the angles of the quadrilateral } CDGE \text{ sum to } 360°\text{)}$$

$$\angle CDG + \angle BDG = 180° \text{ and } \angle CEG + \angle AEG = 180° \quad \text{(straight angles)}$$

$$\angle CEG = \angle BDG \text{ and } \angle CDG = \angle AEG$$

$$\triangle ABE \text{ and } \triangle CDG \text{ are similar} \quad \text{(2 common angles)}$$

$$\text{and } \triangle ABD \text{ and } \triangle CEG \text{ are similar} \quad \text{(2 common angles)}$$

$$\frac{CG}{AB} = \frac{CD}{BE} \quad \text{and} \quad \frac{CG}{AB} = \frac{EC}{AD}$$

$$\frac{CD}{BE} = \frac{EC}{AD}$$

$$\frac{\frac{1}{2}BC}{BE} = \frac{\frac{1}{2}AC}{AD}$$

$$\frac{AC}{AD} = \frac{BC}{BE}.$$

Therefore $\triangle ABC$ is self-median.

PROBLEMS

Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received in sufficient time will be acknowledged in the next issue but one, and the best solutions will be published.

PROBLEM 20.4.1 (K R S Sastry, Dodballapur, India)

Show that the graph of the polynomial $p(x) = x^4 - 2x^2 + 2x + 2$ has a common tangent line at two distinct points on it.

PROBLEM 20.4.2 (from *Yidiot Achronot* newspaper, Israel; posted on the Internet by Greg Barron)

Use each of the numbers 1, 5, 6 and 7 exactly once, together with the four basic operations of arithmetic $(+, -, \times, /)$ and parentheses, to obtain an expression equal to 21. The numbers may be used in any order, and there is no restriction on the number of times each operator may be used.

PROBLEM 20.4.3 (Claudio Arconcher, São Paulo, Brazil)

Let $ABCD$ be a convex quadrilateral. Let P be a point inside $ABCD$. From P, draw perpendiculars to the sides of $ABCD$, extending them outside the quadrilateral: $p_1 \perp \overline{AB}$, $p_2 \perp \overline{BC}$, $p_3 \perp \overline{CD}$, and $p_4 \perp \overline{DA}$. Let Q_1 be a point on p_1 outside $ABCD$. Construct Q_2 on p_2 with $BQ_1 = BQ_2$, Q_3 on p_3 with $CQ_2 = CQ_3$, and Q_4 on p_4 with $DQ_3 = DQ_4$.

(a) When this construction is possible, is it always true that $AQ_4 = AQ_1$?

(b) Investigate what happens if Q_1 moves along p_1 towards $\overline{AB}$ until one of Q_1, Q_2, Q_3 and Q_4 reaches the quadrilateral. Under what conditions on the quadrilateral do all four points reach the quadrilateral simultaneously?

PROBLEM 20.4.4 (Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain)

Find positive integer solutions to the equation $x + y + xy = x^2 + y^2$.

Published by Monash University Mathematics Department