Function

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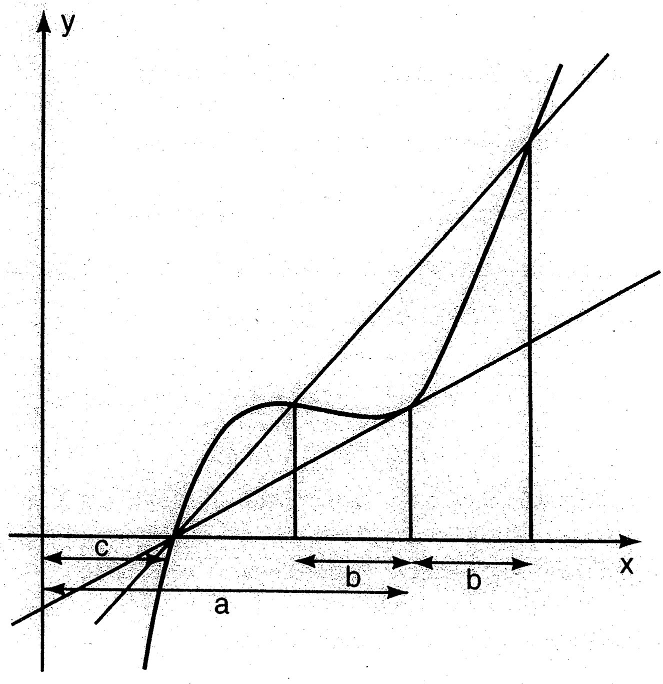

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EDITORIAL

We include in this issue of *Function* a variety of articles, letters, and problems which we hope you will enjoy.

G J Greenbury presents in the *Front Cover* article an interesting way of visualising the real and imaginary parts of the complex solutions to cubic equations. On the other hand, K R S Sastry's article looks at the positive real roots, particularly integers, of cubic polynomials, finding relationships between the roots, side-lengths of an associated triangle, its perimeter and area. The article includes several related problems which you can try. The second feature article also poses a puzzle for you to solve, which originates with a business venture.

Following the February *History of Mathematics* column on the armillary sphere, M Deakin presents another instrument used to read information such as the time of the day, the latitude, and the direction of travel: the *astrolabe*. Although more sophisticated instruments are used these days, an understanding of the three-dimensional mathematics involved is as relevant today as it was in the past.

The *Computers and Computing* column is devoted to the renowned *Mandelbrot set*; a computer program is included which gives you the elements to produce an unlimited number of stunning pictures.

Kim Dean sent us another of her puzzling letters giving a proof she received from the eccentric physicist and mathematician Dai Fwls ap Rhyll, which questions the concept of probability. See if you can solve her problem.

As usual, P Grossman has prepared a challenging *Problem Corner*, with solutions to the problems included in the second last issue, and a number of new problems. You will also find the problems presented to the participants at the 1996 Australian Mathematical Olympiad.

Happy reading!

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THE FRONT COVER

Complex Solutions

Garnet J Greenbury

Every high school student knows that the real solutions of a polynomial equation are also the x–intercepts of the graph of the corresponding polynomial function. The geometric meaning of the complex solutions is not so widely known; this article shows one of the possible interpretations[1].

Let us start with a quadratic equation. From the quadratic formula we find that the complex solutions[2] are of the form $a+bi$ and $a-bi$. Assuming, without loss of generality, that the coefficient of x^2 is 1, we can write the quadratic equation as

$$(x-a)^2 + b^2 = 0$$

The graph of the corresponding function $y = (x-a)^2 + b^2$ appears in Figure 1, indicating also the minimum point (a, b^2).

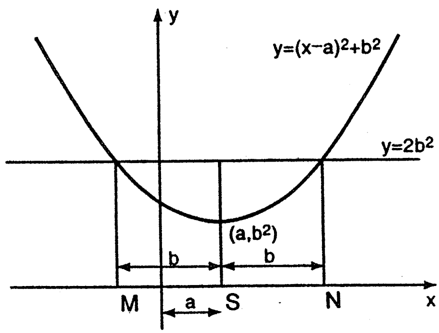

Figure 1

We could also interpret the meaning of the real and imaginary parts of the solutions in the following way:

[1] Another visualisation of complex roots was presented in *Function Vol* 15 *Part* 3.

[2] A short introduction to complex numbers is included in the *Computers and Computing* column.

Consider the points of intersection of the line $y = 2b^2$ and the parabola; we find them by equating the corresponding expressions:

$$2b^2 = (x-a)^2 + b^2$$

This leads to the solutions $x = a + b$ and $x = a - b$. Graphically, the line $y = 2b^2$ intersects the parabola at points with x−coordinates $a+b$ and $a-b$. In Figure 1, $OS = a$, the real part of the solutions, and $MS = SN = b$ correspond to the imaginary parts of the solutions.

Let us turn now to cubic equations. It is easy to verify that if $a + bi$ is a solution to a cubic equation, then its conjugate, $a - bi$, is also a solution. This means that complex solutions come in pairs; therefore a cubic under consideration has a real solution, say $x = c$, and two complex solutions $x = a + bi$ and $x = a - bi$. The equation then has the form

$$(x-c)[(x-a)^2 + b^2] = 0$$

The corresponding graph is drawn in Figure 2 and on the front cover.

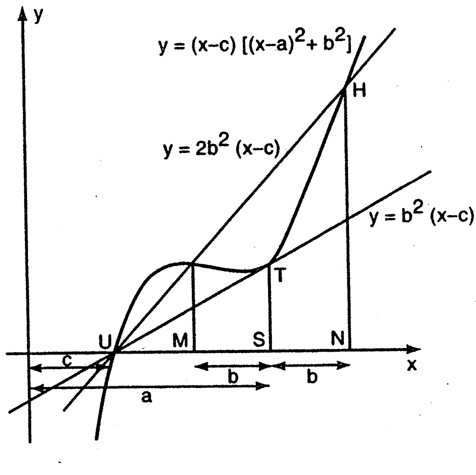

Figure 2

Consider the line $y = m(x - c)$ passing through U and tangent at T. The line intersects the cubic where

$$m(x-c) = (x-c)[(x-a)^2 + b^2]$$

which is equivalent to $x = c$ or $(x-a)^2 + b^2 - m = 0$. The line intersects the cubic only twice, so this last quadratic equation must have only one solution, which occurs when the discriminant is zero,

$$4a^2 - 4(a^2 + b^2 - m) = 0$$

Thus $m = b^2$. Then the x-coordinate of T is a.

Consider now the intersections of the line $y = 2b^2(x-c)$ (notice the 2 again) and the cubic; we solve the equation

$$2b^2(x-c) = (x-c)[(x-a)^2 + b^2]$$

which gives $x = a + b$, $x = a - b$, and also $x = c$.

Therefore the line $y = 2b^2(x-c)$ cuts the graph of the cubic function $y = (x-c)[(x-a)^2 + b^2]$ at three points. One of them corresponds to the real solution; the x-coordinates of the other two are $a + b$ and $a - b$. In Figure 2, $OS = a$, the real part of the complex roots, and $MS = SN = b$ corresponds to the imaginary parts.

We stop here with the question: is it possible to extend this idea to a polynomial equation of a higher degree?

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How to get rich quick!

Here is a simple mathematical way to convert one cent into one dollar. With just a few applications you can be rich in no time.

$$\begin{aligned} 1c &= \$0.01 \\ &= (\$0.1)^2 \\ &= (10c)^2 \\ &= 100c \\ &= \$1 \end{aligned}$$

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CUBIC POLYNOMIALS AND TRIANGLES

K R S Sastry

Does it surprise you to learn that the cubic polynomial $x^3+9x^2+20x+12$ yields the triangle whose side lengths are 8, 7 and 3 units, together with its area, the radius of the incircle (inradius), and the three exradii[1] – all almost effortlessly? (See Figure 1.) Additionally it yields the area of the triangle with side lengths $\sqrt{8}$, $\sqrt{7}$ and $\sqrt{3}$ units with the same ease!

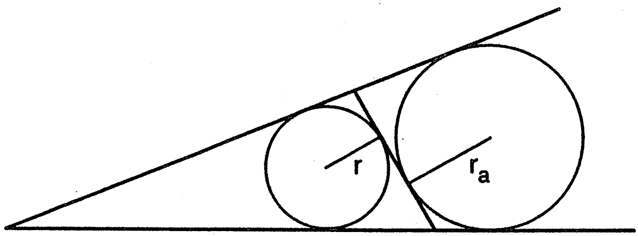

Figure 1

More generally, suppose α, β and γ are any positive real numbers, and let u, v and w be positive real numbers such that

$$x^3 + ux^2 + vx + w = (x+\alpha)(x+\beta)(x+\gamma).$$

Then there is a triangle with

T1. side lengths $(a,b,c) = (\beta+\gamma,\ \gamma+\alpha,\ \alpha+\beta)$

T2. semiperimeter $s = \frac{1}{2}(a+b+c) = u$

T3. area $A = \sqrt{uw}$

T4. inradius $r = \sqrt{\frac{w}{u}}$, and exradii $r_a = \frac{\sqrt{uw}}{\alpha}$, $r_b = \frac{\sqrt{uw}}{\beta}$, $r_c = \frac{\sqrt{uw}}{\gamma}$

and a triangle with

T5. side lengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$, and area $A' = \frac{1}{2}\sqrt{v}$.

[1]An *exradius* is the radius of a circle which is tangent to one of the sides of the triangle and to the extensions of the other two sides.

In what follows, we will use the results that the formula for the inradius of a triangle with area A, side lengths a, b, c, and semiperimeter s, is $r = \frac{A}{s}$, and the formulae for the exradii are $r_a = \frac{A}{s-a}$, $r_b = \frac{A}{s-b}$, $r_c = \frac{A}{s-c}$; see, for example, H S M Coxeter's *Introduction to Geometry.*

Proof of the claims T1 – T5

We begin with the fact that

$$\begin{aligned} x^3 + ux^2 + vx + w &= (x+\alpha)(x+\beta)(x+\gamma) \\ &= x^3 + (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x + \alpha\beta\gamma \end{aligned}$$

By equating the coefficients of like powers of x on both sides, we get

$$\alpha+\beta+\gamma = u,\ \alpha\beta+\beta\gamma+\gamma\alpha = v,\ \alpha\beta\gamma = w. \tag{1}$$

Let $a = \beta+\gamma$, $b = \gamma+\alpha$, and $c = \alpha+\beta$. Then $a+b = \alpha+\beta+2\gamma = c+2\gamma > c$, and similarly $b+c > a$ and $c+a > b$; so a, b, c are the side lengths of a triangle. This gives T1. T2 follows from the first of the equations (1), as you may readily check. We obtain T3 from Heron's formula for the area of a triangle:

$$A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{(\alpha+\beta+\gamma)\alpha\beta\gamma} = \sqrt{uw}.$$

T4 can be obtained by noting that $r = \frac{A}{s} = \frac{\sqrt{uw}}{u} = \sqrt{\frac{w}{u}}$, $r_a = \frac{A}{s-a} = \frac{\sqrt{uw}}{\alpha}$, and similarly for r_b and r_c.

To deduce T5 we need to do some algebra using Heron's formula. A triangle with side lengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$ has semiperimeter $\frac{1}{2}(\sqrt{a}+\sqrt{b}+\sqrt{c})$, so its area is

$$\begin{aligned} A' &= \sqrt{\frac{1}{2}(\sqrt{a}+\sqrt{b}+\sqrt{c})\frac{1}{2}(-\sqrt{a}+\sqrt{b}+\sqrt{c})\frac{1}{2}(\sqrt{a}-\sqrt{b}+\sqrt{c})\frac{1}{2}(\sqrt{a}+\sqrt{b}-\sqrt{c})} \\ &= \frac{1}{4}\sqrt{-a^2-b^2-c^2+2ab+2bc+2ca} \\ &= \frac{1}{4}\sqrt{4\alpha\beta+4\beta\gamma+4\gamma\alpha} \text{ on putting } a=\beta+\gamma, b=\gamma+\alpha, c=\alpha+\beta \\ &= \frac{1}{2}\sqrt{\alpha\beta+\beta\gamma+\gamma\alpha} \\ &= \frac{1}{2}\sqrt{v}. \end{aligned}$$

It is left as an exercise for you to show that a triangle with these side lengths always exists, i.e. that the triangle inequality is satisfied.

To illustrate these results numerically, let us take the cubic polynomial with which we opened our discussion and apply the formulae T1, ..., T5:

$$x^3 + 9x^2 + 20x + 12 = (x+1)(x+2)(x+6).$$

For this polynomial, $u = 9$, $v = 20$, $w = 12$, $\alpha = 1$, $\beta = 2$, $\gamma = 6$. Therefore we obtain:

T1. $(a, b, c) = (\beta + \gamma, \gamma + \alpha, \alpha + \beta) = (8, 7, 3)$

T2. $s = u = 9$

T3. $A = \sqrt{uw} = \sqrt{108} = 6\sqrt{3}$

T4. $r = \sqrt{\frac{w}{u}} = \sqrt{\frac{12}{9}} = \sqrt{\frac{4}{3}} = \frac{2\sqrt{3}}{3}$, $r_a = \frac{\sqrt{uw}}{\alpha} = \frac{6\sqrt{3}}{1} = 6\sqrt{3}$,
$r_b = \frac{\sqrt{uw}}{\beta} = \frac{6\sqrt{3}}{2} = 3\sqrt{3}$, $r_c = \frac{\sqrt{uw}}{\gamma} = \frac{6\sqrt{3}}{6} = \sqrt{3}$

T5. $A' = \frac{1}{2}\sqrt{v} = \frac{1}{2}\sqrt{20} = \sqrt{5}$.

We close this section with some problems for further discussion.

1. Find the cubic polynomial that can be associated with the right angled triangle of side lengths 5, 4, 3.
2. If a, b, c denote the side lengths of a triangle, show that $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are necessarily also the side lengths of a triangle. Give an example to show that if $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are side lengths of a triangle, then a, b, c are not necessarily side lengths of a triangle.

Some problems related to cubic polynomials

In this section, $\alpha, \beta, \gamma, u, v, w$ are assumed to be positive integers. The formulae T1, ..., T5 continue to hold in this section. As we shall now see, this new restriction on the parameters enables us to consider several interesting problems.

PROBLEM I. Determine all cubic polynomials of the form $x^3 + ux^2 + vx + 12$ subject to the above restrictions. Which of the triangles associated with these polynomials has the maximum area?

SOLUTION. For this polynomial, $w = 12 = \alpha\beta\gamma$ and α, β, γ are positive integers. So we consider all possible ways of factorising 12 into three factors, repetition of a factor being allowed:

$$12 = 1 \times 1 \times 12 = 1 \times 2 \times 6 = 1 \times 3 \times 4 = 2 \times 2 \times 3.$$

Hence there are four distinct solutions:

(i) $(x+1)(x+1)(x+12) = x^3 + 14x^2 + 25x + 12$

(ii) $(x+1)(x+2)(x+6) = x^3 + 9x^2 + 20x + 12$

(iii) $(x+1)(x+3)(x+4) = x^3 + 8x^2 + 19x + 12$

(iv) $(x+2)(x+2)(x+3) = x^3 + 7x^2 + 16x + 12$

The triangles associated with these polynomials have areas, respectively, $\sqrt{14 \times 12}$, $\sqrt{9 \times 12}$, $\sqrt{8 \times 12}$, $\sqrt{7 \times 12}$. Clearly, the maximum area, $\sqrt{14 \times 12} = 2\sqrt{42}$, is given by the isosceles triangle with side lengths 13, 13, 2.

PROBLEM II. Find the polynomials associated with triangles having the property that the area measure equals the perimeter measure. Hence determine all such triangles.

SOLUTION. For the area measure to equal the perimeter measure, we must have $\sqrt{uw} = 2u$, so $w = 4u$.

To determine these triangles, let

$$\begin{aligned} x^3 + ux^2 + vx + 4u &= (x+\alpha)(x+\beta)(x+\gamma) \\ &= x^3 + (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x + \alpha\beta\gamma. \end{aligned}$$

Then $\alpha + \beta + \gamma = u$, and

$$\alpha\beta\gamma = 4u = 4(\alpha + \beta + \gamma). \tag{2}$$

Solving equation (2) for γ, we obtain

$$\gamma = \frac{4(\alpha+\beta)}{\alpha\beta - 4}, \tag{3}$$

where $\alpha\beta > 4$ because α, β, γ are positive. Without loss of generality we may assume that $\alpha \le \beta \le \gamma$. Then $\alpha+\beta+\gamma \le 3\gamma$, so $\frac{1}{4}\alpha\beta\gamma \le 3\gamma$, from (2).

Thus $\alpha\beta \le 12$, so $\alpha^2 \le 12$ and therefore $\alpha \le \sqrt{12}$. Hence α equals 1, 2 or 3 because α is a positive integer. Also, since $\beta \le \gamma$, we obtain from (3)

$$\beta \le \frac{4(\alpha+\beta)}{\alpha\beta - 4}.$$

Rearranging this inequality gives

$$\alpha\beta^2 - 8\beta - 4\alpha \le 0.$$

Hence β must lie between the two roots of the corresponding quadratic equation; in particular, β is less than or equal to the larger root:

$$\beta \le \frac{4 + 2\sqrt{\alpha^2+4}}{\alpha}. \tag{4}$$

Case 1. $\alpha = 1$. From (4), $\beta \le 4 + 2\sqrt{5}$. Because β is a positive integer and $\alpha\beta > 4$, we have $5 \le \beta \le 8$. From (3),

$$\gamma = \frac{4(\alpha+\beta)}{\alpha\beta - 4} = \frac{4(1+\beta)}{\beta - 4}$$

which is a positive integer if β equals 5, 6 or 8. This analysis yields $(\alpha, \beta, \gamma) = (1, 5, 24)$; $(1, 6, 14)$; $(1, 8, 9)$. In turn this yields $(a, b, c) = (\beta+\gamma,\ \gamma+\alpha,\ \alpha+\beta) =$ (29, 25, 6); (20, 15, 7); (17, 10, 9).

Case 2. $\alpha = 2$. Applying the steps used in the above analysis we obtain

$$\begin{aligned} (\alpha, \beta, \gamma) &= (2,3,10); (2,4,6) \\ (a, b, c) &= (13,12,5); (10,8,6) \end{aligned}$$

Case 3. This case does not yield any solutions for γ in positive integers.

In summary, we have five triangles in which the area measure equals the perimeter measure:

$$(29,25,6); (20,15,7); (17,10,9); (13,12,5); (10,8,6)$$

the last two being right angled triangles.

Problems for further discussion

1. Give a geometric interpretation to the polynomial $x^3 + ux^2 + vx + m^2u$, in the manner of Problem II in this section. Show that for every positive integer m there is at least one triangle with integer valued sides.

2. It is possible to interpret u, v, w, α, β and γ in other ways. For example, we may interpret α, β, γ in

$$x^3 + ux^2 + vx + w = (x + \alpha)(x + \beta)(x + \gamma)$$

as the edge lengths of a rectangular box. Then what interpretation can be given to u, v, w? What box problem have we solved simultaneously with Problem II? What rectangular box-related interpretation can be given to Problem I which was solved earlier in this section? Answer this last question with regard to problem 1 above.

3. A triangle is called a *Heron* triangle if the sides and area are all positive integers. For example, all the five triangles determined in Problem II are Heron triangles. A Heron triangle will be generated by the polynomial $x^3 + ux^2 + vx + w$ if $uw = m^2$ for some natural number m. Describe a construction to generate Heron triangles in the manner of our discussion.

Reference

Coxeter, H S M *Introduction to Geometry*, 1969, New York, Wiley.

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Readers of Function *will be familiar with the name K R S Sastry. He has taught mathematics in India and Ethiopia. His problem proposals and articles appear in a number of mathematics journals. He has photography as a hobby.*

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It is unworthy of excellent persons to lose hours like slaves in the labour of calculations.

– Gottfried Wilhelm von Leibnitz

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WHOLESALE PROFIT

Michael A B Deakin, Monash University

About a year ago, we were driving through western Victoria and stopped for a cup of coffee at a bright-looking café-*cum*-craftshop. It turned out to be run by a couple who had been forced off the land by the drought and who had started this venture to revive their flagging fortunes. Much of the craftwork they were selling on behalf of various individuals who used their business as an outlet to the public.

The wife told us that she had different business arrangements with these various suppliers. Some specified that the price to the public was to be so much and of this they expected such-and-such a percentage; others stipulated a figure they expected to receive and left the mark-up to the retailer to decide. She had noticed a curious thing, she told us.

In order to explain it better, I will use some simple mathematical notation, although she did not. But let us call the **Wholesale Price** W and the **Retail Price** R. Then the **Markup** (the shop's **Profit**) M, say, will be given by

$$M = R - W.$$

The first type of supplier specified both R and also M (expressed as a percentage of R). The second type specified W and left it to the storekeepers to fix their own value of M.

What she had noticed was that if a supplier of the first type specified 25% (i.e. $1/4$) of R as many did, this corresponded to her imposing a markup of 33% (more accurately $1/3$) of W in the case of a supplier of the second type. (As an example, suppose the supplier wanted a stuffed toy koala sold for \$100 and awarded her 25% of this amount, this was the same as if another supplier had wanted \$75 for an equivalent stuffed koala and she had marked it up by $1/3$ (i.e. \$25) and so ended up selling both koalas at \$100.)

Her question was this: why is there this relation between the $1/4$ and the $1/3$?

Can you supply the answer? Can you also supply a more general rule of which this is just a special case? And if you can, can you show why it works?

Over to you!

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HISTORY OF MATHEMATICS

The Astrolabe

Michael A B Deakin

My last article was devoted to an astronomical instrument known as the armillary sphere. The armillary sphere was, in essence, a metal replica of the heavens and it could be used to mimic their motions and so to facilitate astronomical calculations. To be at all accurate, an armillary sphere had to be very large and so a usable instrument was rather unwieldy and certainly not portable.

However, the discovery was made that the same ends could be served by a smaller, flat instrument: the subject of this month's column. This advance may have been due to the early astronomer Hipparchus (2nd century BC) and most certainly was known to the later Ptolemy.

Both the armillary sphere and the smaller, practical instrument were referred to by the Greeks as "astrolabes", the latter being specifically called the "little astrolabe". Nowadays, however, we reserve the term "astrolabe" for the smaller instrument.

Its main purpose was to observe the sky, and, from the positions of the sun or the stars, tell the time of day or night and determine one's latitude and direction of travel. It could also be used as a basic surveying device to find, say, the angle between the sun and zenith or to determine the height of a mountain.

The principle on which it depended was based on a perspective rendering of the heavens onto a flat surface which goes by the name "stereographic projection". The theory of stereographic projection was given by Ptolemy in his *Planisphaerium*, which shows how to project the various 3-dimensional components of the armillary sphere onto a flat surface.

Much of this theory was passed on to the later astronomers Theon and his daughter Hypatia and through her to her pupil Synesius. Synesius, with Hypatia's help, designed an astrolabe and had it made up by an expert silversmith. He presented it to an official of the Roman Empire, a man with whom he wished to curry favour. We still have the covering letter that he wrote to accompany his gift. It may be, as some have suggested, that he also produced a longer monograph on the theory, a more technical work than the surviving letter, but if so this must have become lost.

Let us now see how the theory of stereographic projection worked. Because Ptolemy, Theon, Hypatia and Synesius all lived and worked in Alexandria, we shall look at matters from their point of view, rather than from our own here in Melbourne. (Almost all astrolabes ever built were designed for use in the northern, rather than the southern, hemisphere.)

As noted in the previous column, the celestial sphere, the apparent shape of the heavens with its fixed pattern of stars, may be analysed exactly as a replica of the terrestrial globe. As we saw, this appears to rotate once over the course of each 24 hours, each star, therefore, tracing out a circle in the observer's sky. This circle will be a circle of latitude on the celestial sphere.

This is one way of looking at the matter. Another is to take the sky as seen by the observer at any one time. Directly overhead is the point of zenith, and below this is a hemisphere of the celestial sphere: the visible part lying above the observer's horizon. Between these extremes are circles of what we might call "observer latitude": 80° above the horizon is an 80° circle, etc. (The technical name for such a circle of "observer latitude" is an *almucantar.*) The observer's celestial sphere will be tilted with respect to the standard one and *vice versa.* For an observer at, say, 31° north latitude (the approximate latitude of Alexandria), the north celestial pole will be 31° above the horizon, or 59° below the zenith.

Equivalently, we may adopt a standard orientation and thus seek to accommodate the peculiar features of the observer's latitude. Take the observer's celestial sphere and so orient it that the north celestial pole is at the top (and the south celestial pole at the bottom). Figure 1 shows in cross-section the geometry involved. The north celestial pole will still be on the circle 31° above the observer's horizon. This 31° circle is one of a number that could be drawn on the, now tilted, observer's celestial sphere.

Passing through this tilted sphere is the plane of the celestial equator. Consider now the set of lines connecting the south celestial pole to one of these almucantars, say the 59° circle that just touches the horizon. Each of these lines will pass through the plane of the celestial equator, and together they define a curve upon it. This curve is in fact a circle. See Figure 2, which shows the 59° almucantar. Indeed, each of the almucantars gives rise to such a circle. We may make a diagram of the various circles set up by so projecting each of the almucantars onto the plane of the celestial equator.

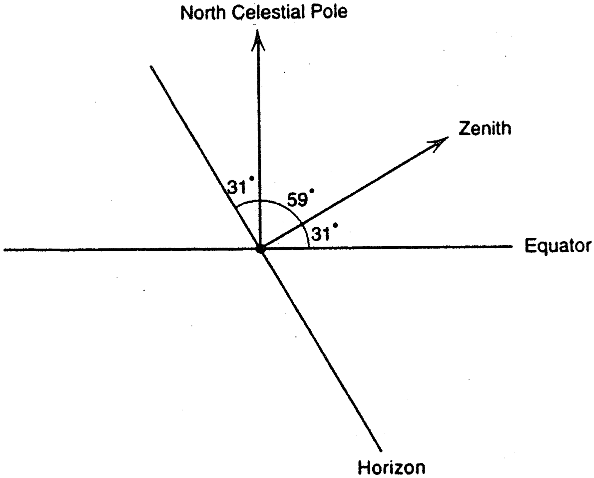


Figure 1. A cross-section of the "tilt" as applicable to Alexandria. The celestial equator is shown as horizontal, and zenith is 31° above this. The observer's horizon is thus shown inclined. The observer's experience, of course, is that zenith is vertically above. To recover this point of view, rotate the diagram.

The projections of the almucantars were engraved onto a circular plate known as the *mater* and forming one component of the astrolabe. The rim of the mater was usually taken to be the circle representing the Tropic of Capricorn. Thus almucantars which intersected this were not shown in full, but only as circular arcs.

Furthermore, lines of equal azimuth, which are lines of "observer longitude" (as the almucantars are lines of "observer latitude"), project by this same means into circles, and these too are engraved onto the mater and the various "climates".

A third set of lines was engraved on the mater of the instrument as well. These were curves representing the various "hour angles"; the details of these will here be omitted.

As the earlier discussion implies, the pattern of lines on the mater is specific to one particular latitude. If the instrument was to be used elsewhere, the mater was overlaid with the pattern for that latitude. Such overlays were called *climates*.

As well as the mater and the climates, the astrolabe had other components. The next we need to consider is the *rete* (the word means "net" and it somewhat resembles a net; it was also called the "spider" for similar reasons). The rete was designed to represent the various major stars: most notably those of the constellations of the zodiac. These are those parts of the sky through which the sun appears to pass in the course of a year, as ascertained by its points of rising and setting. These lie about another circle in the celestial sphere – known as the *ecliptic*.

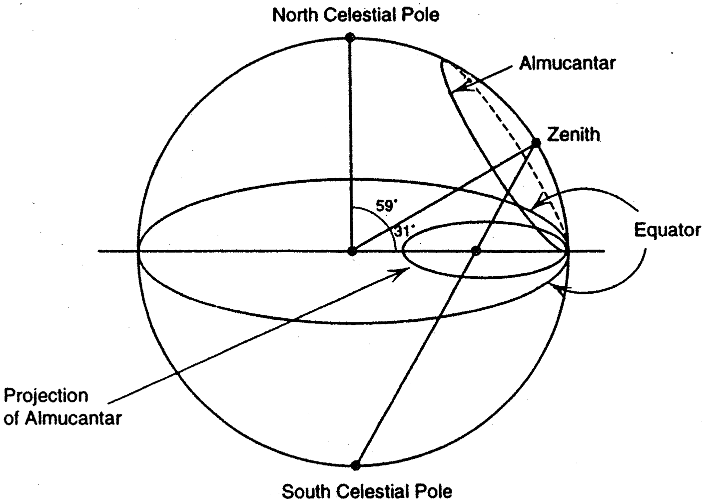


Figure 2. The projection of the 59° almucantar, shown in a perspective rendering. Lines through the south celestial pole and through the oblique circle (the 59° almucantar) intersect the plane of the equator in another circle. Each other almucantar does likewise (although the various circles so produced are not concentric).

The ecliptic also projects, by the same means, into a circle and this was moulded in metal to form the principal feature of the rete. The various stars occupy fixed positions with respect to the ecliptic. The rete was a highly intricate patterned device, whose shape incorporated the circular arc of the ecliptic and various points which represented the major fixed stars.

Thus the rete gives the positions of the fixed stars in the celestial sphere, and does so moreover in a two-dimensional format. This remains unchanged whatever the position of the observer. The mater (and the climates), by contrast, show those features of the situation that depend upon the position (i.e., the latitude) of the observer. Thus in applications to navigation, one rete suffices (as long as the various stars used in it remain visible), but the appropriate climate must be inserted to replace the mater.

The third, and simplest, component is the *alidade.* This is essentially a sighting device. By aligning the line of sight to some major star along the alidade, and reading off the angle from a scale engraved on the back of the mater, the star's angle of elevation could be determined.

From simple observations of this type and by the use of rotations of the rete relative to the mater (or the appropriate climate), as well as by use of the various tables that might be engraved on the back of the mater, the astrolabe enabled the mechanical, or partly mechanical, computation of time, direction and other such quantities.

Each of the computations depended on the correct placement of some point of the rete relative to the relevant climate, and this in turn duplicated the corresponding operation on an armillary sphere. The portability of the instrument, its small size (typically some 25 cm across), and its ease of handling made it popular and practical. It was known from ancient times and continued in use till the seventeenth century AD.

The armillary sphere is, in essence, a scale model of the heavens, so that computations relating to the real-life situation may be carried out by means of simulations performed on the model. The theory of stereographic projection allows the use, instead of the sphere, of a less obvious replica, but one in which all the essential features are preserved and which has the advantages of portability and practicality.

For over a thousand years the astrolabe was in widespread use. In our Western tradition of navigation it came to be replaced by the sextant, but in Arab tradition it continued and even today has a small place in Islamic ritual.

Figure 3 shows the front and Figure 4 the back of an astrolabe lent to us by Robin Turner, formerly of the Monash University department of Physics. It is a small one, being some 10 cm across, and is described in an accompanying flyer as "Issued and Authenticated by the [US] National Maritime Historical Society [and] Crafted by the Franklin Mint". It is almost certainly a replica of a French instrument from about 1580.

The base-plate with its set of engraved lines in Figure 3 (at the very back) is the mater. Forward of this and able to rotate with respect to it is an elaborate and ornate rete incorporating the ecliptic circle marked with the signs of the zodiac. The prominent linear structure running from top left to bottom right is the rule (an aid to reading the instrument) and forward of this is a locking pin (running from lower left to upper right).

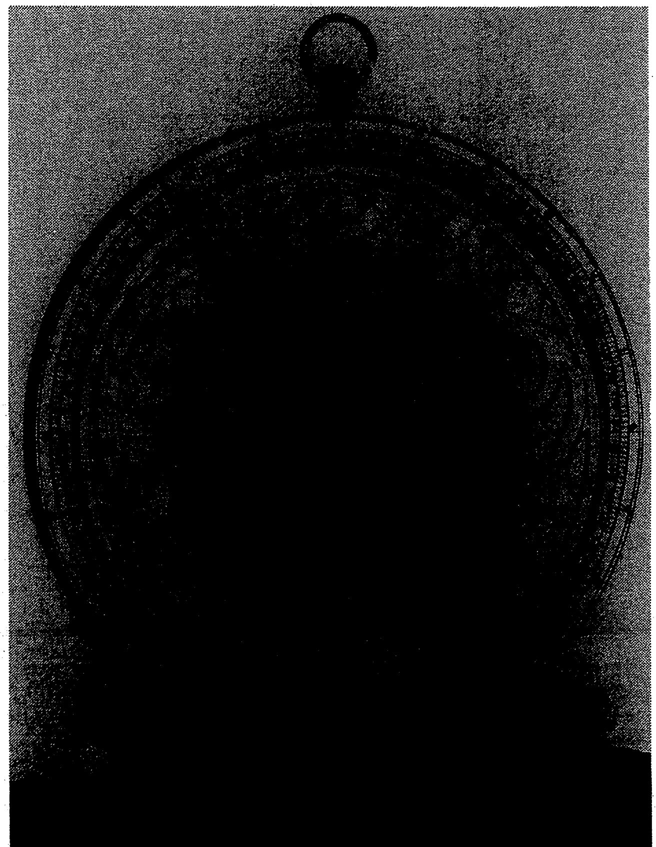


Figure 3. The front of an astrolabe

Figure 4 shows the ecliptic (with zodiacal signs) around the rim and the alidade. Just visible in this latter are the sight-holes that enable the elevations of stars to be measured. For this purpose, the instrument was suspended by the ring at the top, thus ensuring that it was aligned vertically.

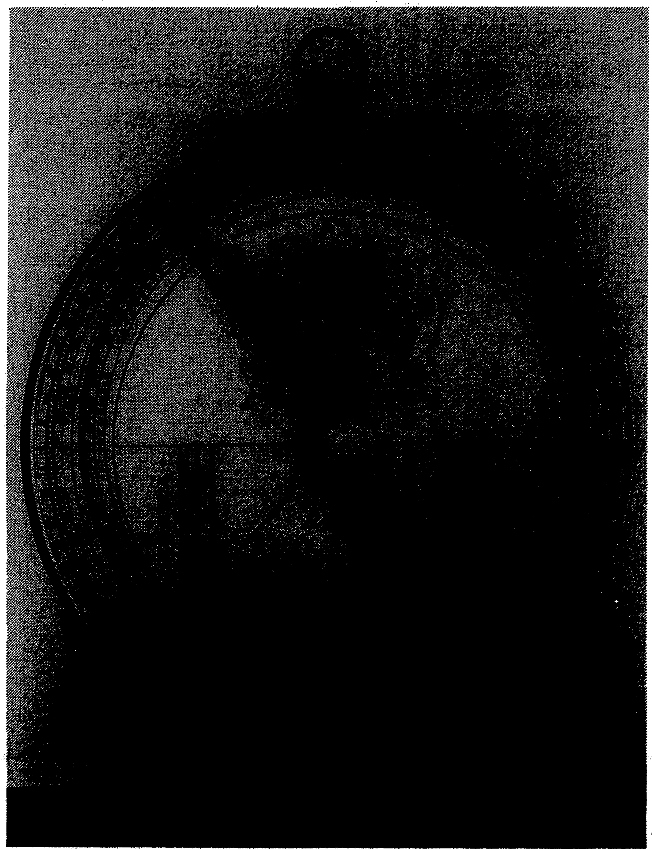


Figure 4. The back of an astrolabe

Further Reading

A good and reasonably accessible account of the astrolabe is given by J D North in his article in *Scientific American* (January 1974). Regrettably it contains two minor errors that readers should know about. The first is that North states that Hipparchus visited Alexandria. This was once widely believed, but it is now known not to have been the case. The second occurs in the caption to the figure on North's p.100. This figure is a diagram similar to Figure 2 of this article. It states that it applies for a latitude of 40° North; this should be 50°. On p.130 of the same issue of *Scientific American* is a list of further reading matter.

COMPUTERS AND COMPUTING

A Colourful Map of the Complex Plane

Cristina Varsavsky

One of the great advantages of the availability of computer power is the opportunity it provides for the graphical display of information, adding another dimension to the analysis of many mathematical situations and problems.

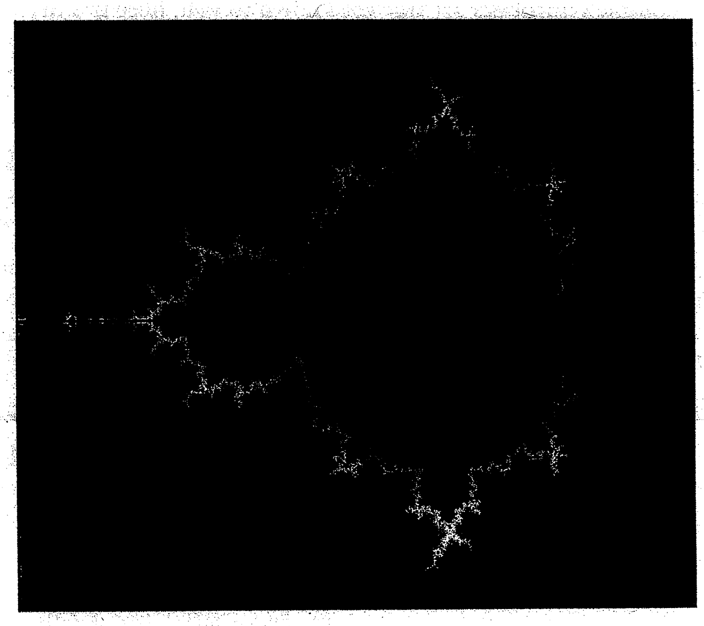


Figure 1

You must have already seen the picture depicted in Figure 1, most likely displayed with stunning colours, which unfortunately we cannot reproduce in this journal. You probably also know its name: it is known as the *Mandelbrot set.* It was defined in 1905 by the French mathematician Pierre Fatou but its visual representation was made possible only with the advent of computers. In this article, we will present the basic principles behind Figure 1, and also a simple computer program to produce it on your screen.

We often hear that a picture is worth a thousand words, and this is certainly true about Figure 1. The Mandelbrot set is not just a pretty picture, it actually displays the intricate behaviour of complex numbers under a particular and very simple iteration, namely

$$z_{n+1} = z_n^2 + c \tag{1}$$

That is, a complex number, z_{n+1}, is generated from the complex number z_n; the number c is a complex parameter. Figure 1 is a map displaying the behaviour of each complex number c as the generator of the iteration (1).

If complex numbers are not yet known to you, here is a brief introduction which should be enough for the purpose of this article. Complex numbers are numbers of the form

$$z = a + bi$$

where a is a real number called the *real part*, b is a real number called the *imaginary part*, and i is an imaginary number such that $i^2 = -1$. Note that if $b = 0$, the corresponding complex number is a *real* number. We represent complex numbers as points in the *complex plane*, where the horizontal axis is for the real part and the vertical axis for the imaginary part of the complex number; that is, the point (a, b) represents the complex number $a+bi$. The magnitude of the complex number z is defined as $|z| = \sqrt{a^2 + b^2}$, which coincides with the notion of distance to the origin.

The algebra of complex numbers is an extension of the algebra of real numbers. We have $(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$. For the product, we expand in the usual way and recall that $i^2 = -1$:

$$\begin{aligned}(a + bi) \times (c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

Now let us go back to Figure 1 to see how it is produced with the iterative formula (1). For each complex number c an iteration is started using formula (1) with the starting value of $z_0 = 0$, generating a sequence z_0, z_1, z_2, $z_3, \ldots$, called the *orbit* of the complex number c. Some orbits may remain bounded, while others escape towards infinity. According to this behaviour, the complex number c is classified either as a *prisoner* or as an *escapee*. For example, take $c = 0$; then $z_0 = 0$ and as we square it we obtain zero again, so only zeros are generated with this formula, therefore $c = 0$ is a prisoner.

Now take a more "complex" example, $c = -2 + 1.5i$. We have $z_0 = 0$. Then

$$z_1 = z_0^2 + (-2 + 1.5i) = -2 + 1.5i$$

Apply (1) again to obtain

$$z_2 = z_1^2 + c = (-2 + 1.5i)^2 + (-2 + 1.5i) = -0.25 - 4.5i$$

Then $z_3 = -22.19 + 3.75i$, $z_4 = 476.22 - 164.91i$, $z_5 = 1.99 \times 10^5 - 1.57 \times 10^5 i, \ldots$; the sequence rapidly tends towards infinity, which means that $-2 + 1.5i$ is an escapee.

The Mandelbrot set is the set of all prisoners; it appears in the middle of Figure 1 and it is also shown in Figure 2. The rest of Figure 1 is shaded according to the speed with which the sequence escapes toward infinity.

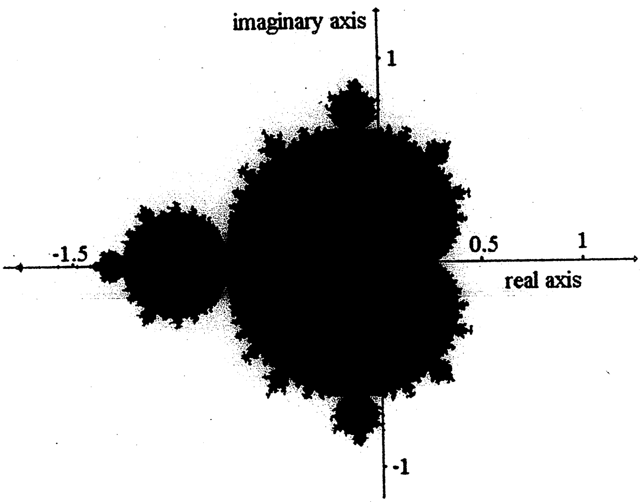


Figure 2

The computer program is now very simple. For each pixel on a 640×480 screen with coordinates (a, b) we do the following:

Step 1: Obtain the corresponding complex number c with real part c_1 and imaginary part c_2. This transformation depends on what part of the plane we want to have on the screen. For the part of the plane with $-2 \leq x \leq 2$, $-1.5 \leq y \leq 1.5$, the corresponding transformations are $c_1 = (a * 4/640) - 2$ and $c_2 = b * (-3)/480 + 1.5$.

Step 2: Apply iteration formula (1), using c obtained in Step 1 and $z_0 = 0$. We use the pairs (u, v) and (x, y) for the old and new terms of the sequence respectively. Using formula (1) we have

$$\begin{aligned} x + yi &= (u + vi)^2 + (c_1 + c_2 i) \\ &= (u^2 - v^2 + 2uvi) + (c_1 + c_2 i) \\ &= (u^2 - v^2 + c_1) + (2uv + c_2)i \end{aligned}$$

Step 3: Decide whether c is a prisoner or an escapee. As it is difficult to write code for testing convergence or divergence of the sequence, we make a practical approximation: if after 100 iterations the sequence remains in the circle centred at the origin with radius 2, we decide this is a prisoner and we paint it black. If it escapes that circle, it is an escapee and we paint it according to the number of iterations it takes to go outside that circle. The definition of the picture will improve if the upper bound of the number of iterations is increased.

Here is the QuickBasic code for the program:

```
SCREEN 12
FOR b = 1 TO 480
  FOR a = 1 TO 640
    u = 0: v = 0
    c_1 = (a * 4 / 640) - 2
    c_2 = b * (-3) / 480 + 1.5

    x = u * u - v * v + c_1
    y = 2 * u * v + c_2

    count = 1
    check = x^2 + y^2

    WHILE ((check <= 4) AND (count <= 100))
      u = x: v = y
      x = u * u - v * v + c_1
      y = 2 * u * v + c_2
      count = count + 1
      check = x^2 + y^2
    WEND
```

```
    IF count >= 30 THEN COLOR 7
    ELSEIF count >= 5 THEN
      COLOR 5
      ELSEIF count >= 4 THEN
        COLOR 4
        ELSEIF count >= 3 THEN
          COLOR 3
          ELSEIF count >= 2 THEN
            COLOR 2
            ELSE
              COLOR 1
    ENDIF

    PSET (a, b)

  NEXT a
NEXT b
```

QuickBasic is not the best language for this sort of task; you may actually need to leave your computer working for quite a while to produce the picture. A more powerful language such as C is recommended here; the translation should not present any difficulties.

Figure 3

You can produce many other stunning pictures by slightly modifying this program. Zooming into regions close to the boundary of the Mandelbrot set will provide more detail, revealing greater beauty and complexity. For this you need only to change the formulas in Step 1. Figure 3, for example, depicts the behaviour of complex numbers iterated through equation (1) in the region

$$-1.475 \leq \text{real part} \leq -1.225 \ , \ \ -0.09375 \leq \text{imaginary part} \leq 0.09375$$

Another interesting approach is to change formula (1), or even the way formula (1) is used to define a set. For example, for a fixed constant c we can paint the pixels according to the behaviour of the corresponding complex number as the starting point z_0 for formula (1). You will then be exploring what are known as *Julia sets.*

Further reading

There are numerous books about fractals which include the Mandelbrot set. The following are good sources to learn more about them. In the reference [3] there is an interesting foreword by Benoit Mandelbrot titled *Fractals and the Rebirth of Experimental Mathematics.*

1. Gleick J, *Chaos – Making a New Science*, 1987, Viking.
2. Mandelbrot B, *The Fractal Geometry of Nature*, 1982, San Francisco, Freeman.
3. Peitgen H, Jürgens H, Saupe D, *Fractals for the Classroom*, 1992, Springer–Verlag.

* * * * *

Nature exhibits not simply a higher degree but an altogether different level of complexity. The number of distinct scales of length of natural patterns is for all practical purposes infinite.

– Benoit Mandelbrot, *The Fractal Geometry of Nature*

* * * * *

LETTER TO THE EDITOR

It had been some time since I had heard from my erratic correspondent, the eccentric Welsh physicist and mathematician Dai Fwls ap Rhyll. In fact, though a year ago I surmised that he had put a rather interesting posting on the Internet, I had had no direct communication since 1991. I was beginning again to wonder what he was up to and why he had not bothered to write.

However, I have got used to his little ways and so was pleased and relieved to get a letter from him. It seems that he has been concerned with the foundations of probability theory – a difficult topic that I can well imagine occupying his time for the bulk of five years. He sees this branch of mathematics as badly in need of an overhaul. I give below one of his most telling examples.

The problem is to draw a chord at random in a fixed circle. If the circle has radius 1 (for example, though this makes no real difference), what is the probability that the length of the chord exceeds $\sqrt{3}$?

He shows that this question has at least three different answers, and so we learn either that $\frac{1}{2} = \frac{1}{3} = \frac{1}{4}$ or else that probability is not a viable concept!

All three approaches use the fact that $\sqrt{3}$ is the length of the side of an equilateral triangle inscribed in the circle. (You may care to verify this for yourselves.)

Now he takes such a triangle (Figure 1) and allows one of its sides, BC, to be bisected at M. Through M he draws a diameter AK. Let O be the centre of the circle and let N be the point on AK such that $OM = ON$. We find that $MN = \frac{1}{2}AK$. (Again I leave this as an exercise in simple geometry.) But now if we take any chord parallel to BC, then either its centre will lie *inside* MN or else outside it, each with probability $\frac{1}{2}$. Furthermore, this calculation will apply however we place the side BC (whatever its orientation). Thus the required probability is $\frac{1}{2}$.

Next he considers Figure 2. This time he takes the vertex A and considers the set of chords passing through A. To assist with the analysis, draw the tangent EAD touching the circle at A. Suppose the chord to make an angle θ with this tangent. If θ lies between 60° and 120°, then the chord has length greater than $\sqrt{3}$; otherwise not. But the probability that θ indeed lies in this range is $\frac{1}{3}$. Again the calculation will apply however we place the side BC (whatever its orientation). Thus the required probability is $\frac{1}{3}$.

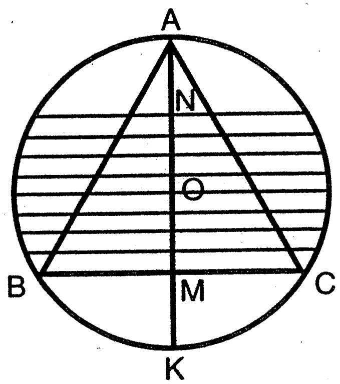


Figure 1

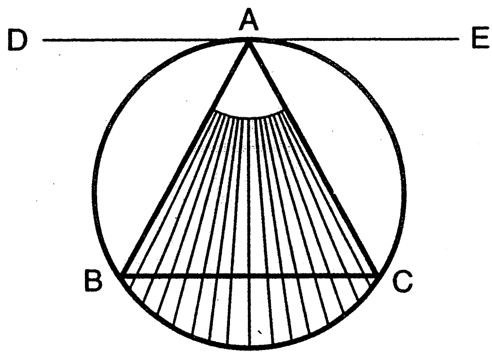


Figure 2

Finally he takes Figure 3. Inside the equilateral triangle ABC he inscribes a circle, centre O and touching the side BC at M. By an extension of his first argument, he readily shows that the chord has length greater than $\sqrt{3}$ if its midpoint lies *inside* this new circle. But it is quite easy to see that the radius of the small circle is $\frac{1}{2}$. (Again you may prove this as an easy exercise.) So the required probability is the probability that the midpoint of the chord lies inside the small circle. This probability is precisely the ratio of the area of the small circle to that of the large. That is to say, $\frac{1}{4}$, a figure you may readily deduce.

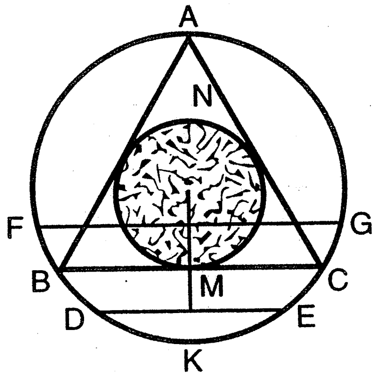


Figure 3

As so often happens, I can see no flaw in Dr Fwls's reasoning, and so must agree with him that there are indeed problems with the notion of probability (for the possibility that $\frac{1}{2} = \frac{1}{3} = \frac{1}{4}$ is just too painful to contemplate)!

Kim Dean
Union College
Windsor

* * * * *

[There is a] computer disease that anybody who works with computers knows about. It's a very serious disease and interferes completely with the work. The trouble with computers is that you *play* with them!

– Richard Feynman

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 19.5.1 (Garnet J Greenbury, Upper Mt Gravatt, Qld)

Prove that, in a right triangle,

$$2\tan^{-1}\left(\frac{\text{hypotenuse} - \text{base}}{\text{height}}\right) = \tan^{-1}\left(\frac{\text{height}}{\text{base}}\right)$$

SOLUTION by Benito Hernández-Bermejo (Universidad Nacional de Educación a Distancia, Madrid, Spain)

Let h be the hypotenuse, x the base and y the height. Since the triangle is right-angled, we have $h^2 - x^2 = (h-x)(h+x) = y^2$, so

$$\frac{h-x}{y} = \frac{y}{h+x}.$$

Then we have equivalently to prove that

$$2\tan^{-1}\left(\frac{y}{h+x}\right) = \tan^{-1}\left(\frac{y}{x}\right),$$

or, briefly, $2\beta = \alpha$ (see Figure 1, in which BCD is the original triangle).

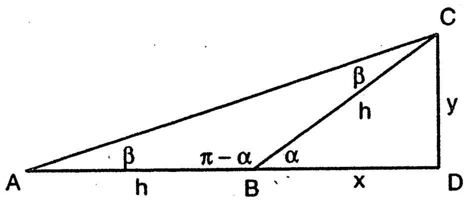


Figure 1

But notice that the triangle ABC is isosceles, since two of its sides are of length h. Then the three angles of ABC are β, β and $\pi - \alpha$. Since the sum of the angles of ABC must be equal to π, we have

$$2\beta + \pi - \alpha = \pi,$$

or $2\beta = \alpha$. This proves the result.

Also solved by Sani Susanto (Monash University), Keith Anker (Monash University), and the proposer.

PROBLEM 19.5.2

Let ABC be a triangle, and let O be any interior point of ABC. Prove that $AB + AC > OB + OC$.

SOLUTION

Produce $\overline{CO}$ to meet $\overline{AB}$ in K (see Figure 2). Then

$$\begin{aligned} AB + AC &= BK + KA + AC \\ &> BK + KC \\ &= BK + KO + OC \\ &> BO + OC \\ &= OB + OC \end{aligned}$$

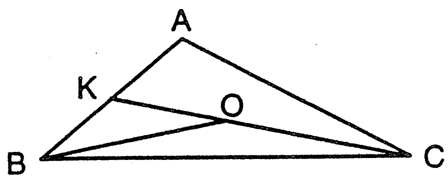


Figure 2

Also solved by Keith Anker (Monash University).

PROBLEM 19.5.3

There is a unique number such that its square and its cube together use each of the 10 digits exactly once. Find the number.

SOLUTION (composite of solutions by Claudio Arconcher (São Paulo, Brazil), Keith Anker (Monash University), and the editors)

Let n denote the number required. Then n^2 and n^3 together use each of the 10 digits exactly once. If $n < 47$ then n^2 has at most 4 digits and n^3 has at most 5, so this is impossible. Similarly, if $n > 99$ then n^2 has at least 5 digits and n^3 has at least 7, so this too is impossible. Therefore $47 \leq n \leq 99$.

The last digit of n cannot be 0, 1, 5 or 6, since then n^2 and n^3 would both end in the same digit.

At this point, the number of cases to check is small enough to be manageable with a calculator. Nevertheless, we can get a bit further using general arguments. By the "casting out nines" rule (equivalently, using arithmetic modulo 9), $n^2 + n^3$ must be divisible by 9 because the sum of the digits from 0 to 9 is divisible by 9. Therefore $9 \mid n^2(n+1)$, so $9 \mid n^2$ or $9 \mid n+1$, since n^2 and $n+1$ are coprime. Hence $3 \mid n$ or $9 \mid n+1$.

The numbers still under consideration are therefore

$$48, 53, 54, 57, 62, 63, 69, 72, 78, 84, 87, 89, 93, 98, 99.$$

Now using a calculator, the cubes of all but three of these numbers (69, 84 and 93) are found to contain repeated digits. On checking these three numbers, we find that 69 is the unique number satisfying the conditions: $69^2 = 4761$ and $69^3 = 328509$.

PROBLEM 19.5.4

Positive integers are to be expressed using the digits 1, 2, 3 and 4 exactly once each, together with the binary operations $+, -, \times, /$, the unary operations of negation and square root, and logarithm to a base, where the base must be supplied from the available digits. For example, 4 can be expressed as:

$$\sqrt{1+3}\log_2 4.$$

Which positive integers can be expressed in this way?

SOLUTION

All of them! The number n is given by the expression

$$-\log_2 \log_{3-1} \underbrace{\sqrt{\sqrt{\ldots\sqrt{4}}}}_{n+1}$$

Solution to an earlier problem

The solution to the following problem from the June 1992 issue has not previously appeared in *Function*.

PROBLEM 16.3.2 (Juan-Bosco Romero Márquez, Valladolid, Spain)

Let ABC and $A'B'C'$ be two right-angled triangles with sides a, b, c and a', b', c' respectively. Suppose that $a > b \geq c$ and $a' > b' \geq c'$ and that

$\angle ABC > \angle A'B'C'$. Let $A''B''C''$ be the triangle with sides a'', b'', c'' such that $a'' = aa'$, $b'' = bb' + cc'$ and $c'' = bc' - b'c$. Prove that $\triangle A''B''C''$ is right-angled, and evaluate its area, circumradius and inradius as well as $\angle A''B''C''$.

SOLUTION

The hypotenuses of ABC and $A'B'C'$ are the sides with lengths a and a' respectively, so $a^2 = b^2 + c^2$ and $a'^2 = b'^2 + c'^2$. Therefore

$$\begin{aligned}
b''^2 + c''^2 &= (bb' + cc')^2 + (bc' - b'c)^2 \\
&= b^2b'^2 + 2bb'cc' + c^2c'^2 + b^2c'^2 - 2bb'cc' + b'^2c^2 \\
&= b^2b'^2 + c^2c'^2 + b^2c'^2 + b'^2c^2 \\
&= (b^2 + c^2)(b'^2 + c'^2) \\
&= a^2a'^2 \\
&= (aa')^2 \\
&= a''^2
\end{aligned}$$

Hence $\triangle A''B''C''$ is right-angled, with hypotenuse of length a''. Its area is clearly $\frac{1}{2}b''c''$, and its circumradius is $\frac{1}{2}a''$ (since the circumradius of a right-angled triangle equals half the length of the hypotenuse).

Let r be the inradius of $\triangle A''B''C''$. Let D be the incentre, and let E, F and G denote the points of tangency of the incircle with the sides $\overline{A''B''}$, $\overline{A''C''}$ and $\overline{B''C''}$ respectively. Then the right-angled triangles $\triangle B''DE$ and $\triangle B''DG$ are congruent, so $B''E = B''G$, and similarly $C''F = C''G$. Hence $a'' = B''C'' = B''G + C''G = B''E + C''F = c'' - r + b'' - r$, so $r = \frac{b''+c''-a''}{2}$.

$$\begin{aligned}
\angle A''B''C'' &= \operatorname{artan}\frac{b''}{c''} \\
&= \operatorname{artan}\frac{bb' + cc'}{bc' - b'c} \\
&= \operatorname{artan}\frac{\frac{b'}{c'} + \frac{c}{b}}{1 - \frac{b'c}{bc'}} \\
&= \operatorname{artan}\frac{\frac{b'}{c'} + \frac{c}{b}}{1 - \frac{b'}{c'}\frac{c}{b}} \\
&= \operatorname{artan}\frac{b'}{c'} + \operatorname{artan}\frac{c}{b} \\
&= \angle A'B'C' + \angle ACB
\end{aligned}$$

PROBLEMS

Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received in sufficient time will be acknowledged in the next issue but one, and the best solutions will be published.

PROBLEM 20.2.1 (modified from a problem in *Alpha*, a German mathematics magazine, September 1995)

It is known that the teachers for classes 5A to 5E will be Mr Brown, Mrs Green, Mr Black, Mr Gray and Ms White, but it has not yet been announced which teacher will be in charge of which class. The table below shows the predictions by two students. The first student made two correct guesses associating teacher and class, and the second made three correct guesses. Who is the teacher for each class?

Class	5A	5B	5C	5D	5E
1st student's guess:	Black	Green	White	Gray	Brown
2nd student's guess:	Brown	Black	Gray	White	Green

PROBLEM 20.2.2 (modified from a problem in *Alpha*, June 1995)

"It's curious", says Karen. "I decided to select the PIN for my Bankcard by dividing my year of birth by the street number of our house and choosing the last four digits shown on my calculator. They turn out to be 1996."

"How many digits does your street number have?" asks Melissa.

"Two."

"In that case you made a mistake or your calculator is not operating properly. You cannot get the digits 1, 9, 9, 6 in sequence among the decimal places when you divide an integer by a two-digit integer."

Prove that Melissa is right.

PROBLEM 20.2.3 (from *Alpha*, June 1995)

Let $ABCS$ be a regular pyramid with base $\triangle ABC$ and apex S. Let the angle at S in each side face be α. Let M be a point on a side face at a distance l from the apex. Determine the length of the shortest closed path through M and enclosing S, if it exists.

PROBLEM 20.2.4

Let P be a point inside a triangle ABC. Let D, E and F be on $\overline{AB}, \overline{AC}$ and $\overline{BC}$ respectively, such that $\overline{PD} \perp \overline{AB}$, $\overline{PE} \perp \overline{AC}$, and $\overline{PF} \perp \overline{BC}$ (see Figure 3). It is required to choose P so as to minimise $PD + PE + PF$. Show that if $\triangle ABC$ is not isosceles then P must be situated at a vertex. What happens if $\triangle ABC$ is (a) isosceles; (b) equilateral?

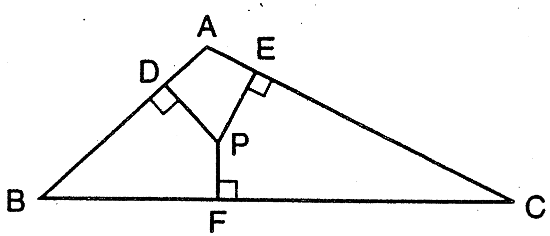


Figure 3

PROBLEM 20.2.5 (from *Mathematical Digest*, July 1995, University of Cape Town)

Exactly one of the following five statements is true. Which one?

(1) All of the following.

(2) None of the following.

(3) Some of the following.

(4) All of the above.

(5) None of the above.

PROBLEM 20.2.6 (from *Mathematical Mayhem*, Vol 8, Issue 3, University of Toronto)

Show that the sum of any 1996 consecutive integers cannot be a power of an integer with exponent greater than one.

* * * * *

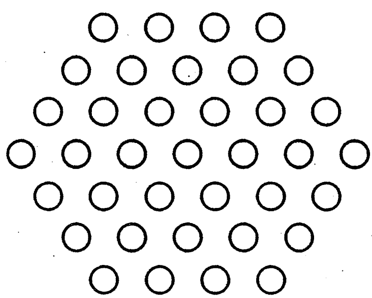
The 1996 Australian Mathematical Olympiad

The contest was held in Australian schools on February 6 and 7. On each day students had to sit a paper consisting of four problems, for which they were given four hours. About 120 students in years 9 to 12 sat the examinations. On March 12, top scorers participated in the Asian Pacific Mathematics Olympiad (APMO), a major international competition for which twenty Pacific Rim countries registered. In addition, students from Argentina, South Africa, and Trinidad and Tobago have tested their skills on the APMO problems.

1. Let $ABCDE$ be a convex pentagon such that $BC = CD = DE$ and each diagonal of the pentagon is parallel to one of its sides.
   Prove that all the angles in the pentagon are equal, and that all sides are equal.

2. Let $p(x)$ be a cubic polynomial with roots r_1, r_2, r_3. Suppose that
   $$\frac{p(\frac{1}{2}) + p(-\frac{1}{2})}{p(0)} = 1000.$$
   Find the value of $\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}$.

3. A number of tubes are bundled together into a hexagonal form.
   The number of tubes in the bundle can be 1, 7, 19, 37 (as shown in the figure below), 61, 91, ... If this sequence is continued, it will be noticed that the total number of tubes is often a number ending in 69.
   What is the 69th number in the sequence which ends in 69?



4. For which positive integers n can we rearrange the sequence $1, 2, \ldots, n$ to $a_1, a_2, \ldots, a_n$ in such a way that $|a_k - k| = |a_1 - 1| \neq 0$ for $k = 2, 3, \ldots, n$?

5. Let $a_1, a_2, \ldots, a_n$ be real numbers and s a non-negative real number such that

   (i) $a_1 \leq a_2 \leq \ldots \leq a_n$;

   (ii) $a_1 + a_2 + \ldots + a_n = 0$;

   (iii) $|a_1| + |a_2| + \ldots + |a_n| = s$.

   Prove that
   $$a_n - a_1 \geq \frac{2s}{n}.$$

6. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD respectively such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ.
   Show that M is the midpoint of PQ.

7. For each positive integer n, let $\sigma(n)$ denote the sum of all positive integers that divide n. Let k be a positive integer and $n_1 < n_2 < \ldots$ be an infinite sequence of positive integers with the property that $\sigma(n_i) - n_i = k$ for $i = 1, 2, \ldots$.
   Prove that n_i is a prime for $i = 1, 2, \ldots$.

8. Let f be a function that is defined for all integers and takes only the values 0 and 1. Suppose f has the following properties:

   (i) $f(n + 1996) = f(n)$ for all integers n ;

   (ii) $f(1) + f(2) + \ldots + f(1996) = 45$.

   Prove that there exists an integer t such that $f(n + t) = 0$ for all n for which $f(n) = 1$ holds.

Most of the students who participated in the Australian Mathematical Olympiad had gathered experience from last year's Senior Contest of the Australian Mathematical Olympiad Committee. Students had been given four hours to solve the following five problems:

1. Let ABC be a triangle having area 1, and let x be a number with $0 < x \leq 1$. Let A', B' and C' be points on BC, CA and AB respectively such that $BA' : A'C = CB' : B'A = AC' : C'B = (1 - x) : x$.
   Express the area of the triangle $A'B'C'$ in terms of x.

2. The digits 1234567891011...19941995 are written on the blackboard forming the number N_1. The digits of N_1 at even places are wiped off the blackboard. Let N_2 denote the number that is left over. Now the digits of N_2 at odd places are wiped off the blackboard. Let N_3 denote the number that is left over. The digits of N_3 at even places are wiped off the blackboard. Let N_4 denote the number that is left over and so on. This process continues until only one digit remains on the blackboard. Find this digit.

   (**Note:** places are counted from the left, e.g. in the number 12345, the digit 1 is at the first place, the digit 2 at the second, etc.)

3. Determine all quadruples (p_1, p_2, p_3, p_4) of primes that satisfy

   (i) $p_1 < p_2 < p_3 < p_4$;
   (ii) $p_1p_2 + p_2p_3 + p_3p_4 + p_4p_1 = 882$.

4. Determine all polynomials $p(x)$ with real coefficients such that

$$tp(t-1) = (t-2)p(t)$$

   for all real numbers t.

5. Let Δ be a right-angled triangle with the following properties:

   (i) both sides enclosing the right angle are of integer length;
   (ii) if the perimeter of Δ is, say, n centimetres, then its area is n square centimetres.

   Determine the side lengths of Δ.

* * * * *

Someone once asked [New Zealand-born] A C Aitken, professor at Edinburgh University, to make 4 divided by 47 into a decimal. After four seconds he started and gave another digit every three-quarters of a second: "point 08510638297872340425531914". He stopped (after twenty-four seconds), discussed the problem for one minute, and then restarted: "191489" – five-second pause – "361702127659574468. Now that's the repeating point. It starts again with 085. So if that's forty-six places, I'm right." To many of us such a man is from another planet, particularly in his final comment.

from Anthony Smith, *The Mind*, 1984, London, Hodder & Stoughton

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Function is a mathematics magazine produced by the Department of Mathematics at Monash University. The magazine was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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