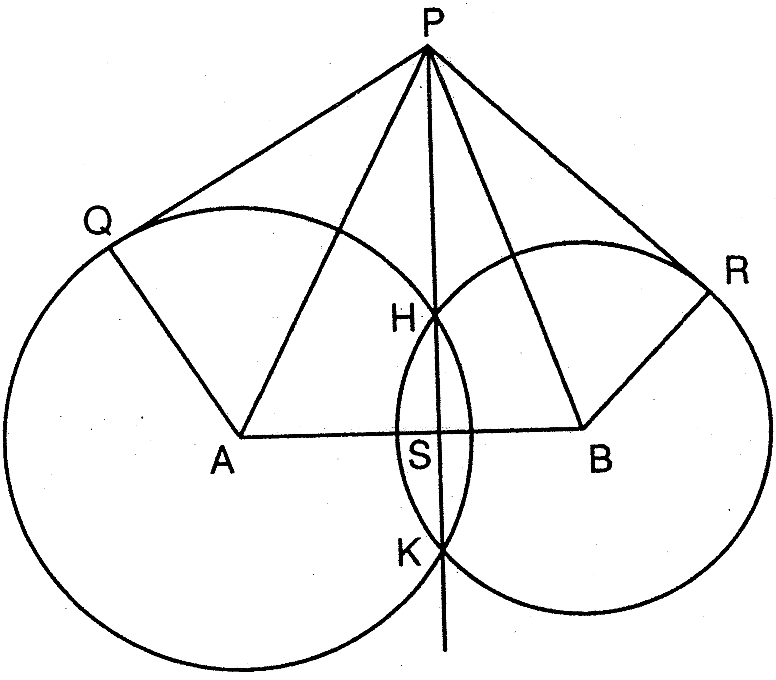
Function

Founder Editor G. B. Preston

Volume 15 Part 5

October 1991



A SCHOOL MATHEMATICS MAGAZINE

Reg. by Aust. Post Publ. No. VBH0171

FUNCTION is a mathematics magazine addressed principally to students in the upper forms of secondary schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. Recent issues have carried articles on advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

* * * * *

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
FUNCTION,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside back cover.

FUNCTION is published five times a year, appearing in February, April, June, August, October. Price for five issues (including postage): $16.00*; single issues $3.50. Payments should be sent to the Business Manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

*$8.00 for *bona fide* secondary or tertiary students.

* * * * *

Printed by MONASH UNIVERSITY PRINTING SERVICES

FUNCTION

Volume 15 *Part 5*

(Founder editor: G.B. Preston)

CONTENTS

* * * * *

FUNCTION welcomes submissions and queries from its readers. For details, see the information opposite.

Published by the department of Mathematics, Monash University

THE FRONT COVER

Michael A.B. Deakin, Monash University

Start by drawing two circles in a plane. A number of cases may arise. Figure 1 shows one such case and Figure 2, which is also our cover diagram, another. There are yet other cases you might like to find and explore for yourself.

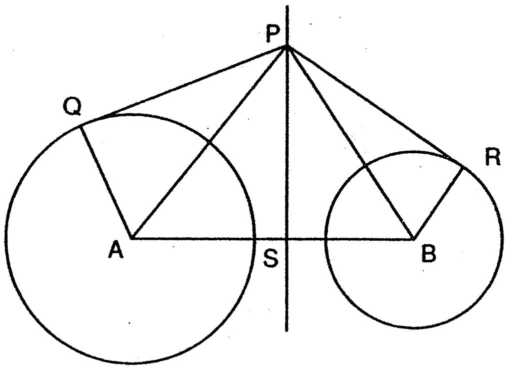


Figure 1

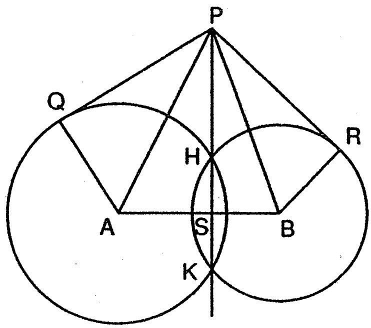


Figure 2

The circle centred on A has radius a, and that centred on B has radius b. From a point P we draw tangents PQ, PR to the circles and we so arrange matters that $PQ = PR$. We want to know

What is the set of all such points P?

To answer this, look at Figure 1. $PQ^2 = PR^2$ from the data. But $PQ^2 = PA^2 - AQ^2$ and $PR^2 = PB^2 - BR^2$. So

$$PA^2 - AQ^2 = PB^2 - BR^2.$$

From this we deduce

$$PS^2 + AS^2 - a^2 = PS^2 + SB^2 - b^2$$

if S is so chosen that $\angle PSB$ is a right angle.

Thus

$$AS^2 - SB^2 = a^2 - b^2.$$

This equation enables the point S to be uniquely determined. Thus, find this S and draw a line through S and perpendicular to AB. All points P on this line satisfy the requirement.

In the case of Figure 2, we must modify this slightly: all points *outside* the circles satisfy the requirement.

The line PS is called the *radical axis* of the two circles. Where there are three circles there are three radical axes and they all meet in a point O. [Can you prove this?] Figure 3 shows one of several possible cases.

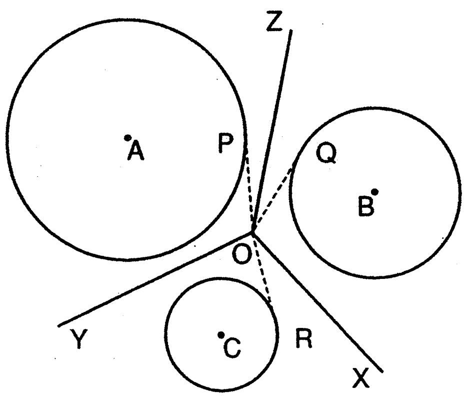


Figure 3

COMPUTER ALGEBRA

Pam Norton and Robyn Arianrhod, Monash University

Computers were initially developed to carry out long numerical computations. People quickly realised the potential of the new technology, and business and management applications soon became predominant as far as the volume of processing assigned to them is concerned. Scientific applications are still the most prestigious, and often require the most powerful computers.

It is now over 30 years since the computer was first used to perform algebraic calculations, although many people are still unaware of this capability. However, the possibility had been foreseen by Ada Augusta, Countess Lovelace, in the 19th century. The Countess helped Charles Babbage to develop his "Analytic Engine", one of the forerunners of computers. She wrote: "Many persons imagine that the nature of its processes must be arithmetical and numerical ... This is an error. The engine ... might bring out its results in algebraic notation, were provisions made accordingly."

Computer algebra packages have been used in a large number of different areas in science and engineering. The most extensive use has occurred in the fields where the algebraic calculations necessary are extremely tedious and time-consuming, such as general relativity, celestial mechanics and quantum mechanics. Before the advent of computer algebra, such hand calculations took many months to complete and were error-prone. Personal workstations can now perform much larger calculations without error in a matter of minutes.

General-purpose computer algebra systems deal with the manipulation of symbolic and algebraic expressions, such as polynomials and rational functions. In the traditional computer languages, such as FORTRAN, BASIC and PASCAL, a mathematical formula can only be evaluated numerically, once the variables and parameters have themselves been given numerical values, and then the result only has a relatively small number of significant digits (depending on the machine and the particular implementation), and may be inaccurate due to roundoff errors. In computer algebra systems, the same formula may be evaluated numerically with unlimited precision, and it can also have algebraic operations carried out on it, such as differentiation, integration, factorisation, expansion and series approximation.

Computer algebra systems are interactive and incremental. That is, the user enters a command, and if it is not syntactically correct, the package will indicate that an error has occurred, and usually give an indication of where the error occurred and what it was. If the command is correct, the package will execute it and return the result to the user. The user can use previous expressions and results to build new expressions, and hence solve problems in a step-by-step fashion.

Integers can have unlimited size, and fractions are represented exactly as quotients of such integers. Real numbers with as many digits as required are usually available. There are pre-defined identifiers with pre-assigned values and properties to represent i (= $\sqrt{-1}$), e, π, etc.

General-purpose computer algebra systems are usually designed to meet the following two requirements:

– to provide a set of basic commands which carry out routine manipulations;

- to offer a programming language which allows the user to build procedures to perform various required tasks.

Most of the current systems have been written in various dialects of the list processing language LISP, although two of the more recent packages (MATHEMATICA and MAPLE) have been written in the language *C*.

Computer algebra systems of various sizes and complexities have been available for some years. There have been considerable developments, however, in recent years in terms of the development of systems suitable for use in the teaching of mathematics. Computer algebra systems demand very large amounts of computer time and memory, and it is only the steady fall in hardware costs that has made their use for undergraduate teaching possible.

The main general-purpose computer algebra systems available today are SCRATCHPAD, MACSYMA, REDUCE, MATHEMATICA, MAPLE and DERIVE (a successor to muMath). SCRATCHPAD is a fairly recent IBM research system, requiring a large IBM mainframe to run. MACSYMA is a well-established scientific and engineering workhorse, with many libraries of routines available. It has often been termed an 'expert mathematical system', and has been developed on DEC and VAX mainframes and minicomputers. MATHEMATICA is widely available on a range of machines, from the CRAY to the Macintosh and IBM-PC (80386). It incorporates a significant graphics package along with both numerical and algebraic computations. REDUCE is also available on a wide range of machines, including IBM-PCs. It has been used especially by high-energy physics and relativity researchers, but it does not have a graphics capability. MAPLE was designed as a teaching system, and is now available for a wide range of machines, including Macintosh and IBM-PC (80386). It offers graphics, particularly on workstation and microcomputer versions, as well as numerical and algebraic computations. DERIVE is a very simple, menu-driven (and hence user-friendly) system with graphics capability, designed especially for education purposes, and is only available for IBM-PCs or compatibles.

To demonstrate how easy these computer algebra systems are to use, we now give a few sample commands using the language MAPLE, together with the computer's response and some comments on the output.

Please note that this is "working output". It could be printed out more neatly, but this misses the point. We want to use MAPLE like a sophisticated scratch-pad and this is an example of such use.

```
# find the prime factors of an integer;
> ifactor(123454321);
                                   2      2
                               (41)   (271)

#factor a poynomial over the rationals;
> factor(x^4-4);
                               2          2
                             (x   - 2) (x   + 2)

# solve an equation;
> roots := solve(x^3+x^2=5*x+2,x);
                                          1/2               1/2
              roots := 2, - 3/2 + 1/2 5     , - 3/2 - 1/2 5
```

The roots are $2, \frac{-3\pm\sqrt{5}}{2}$. They solve $x^3 + x^2 = 5x + 2.$

```
> roots[1];
                                       2

> roots[2];
                                                1/2
                              - 3/2 + 1/2 5

> roots[3];
                                                1/2
                              - 3/2 - 1/2 5

# change the precision;
> Digits := 20;
                                Digits := 20

> evalf(roots[2]);
                            -.38196601125010515180
```

This is $\frac{-3\pm\sqrt{5}}{2}$ to 20 decimal places.

```
> Digits:=10;
                                Digits := 10

> evalf(roots[2]);
                                -.381966011

# expand a polynomial;
> expand((12*x)^5);
                                            5
                                  248832 x

#define a function;
> f := x/((x+1)*(x+2));
                                            x
                            f := ---------------
                                  (x + 1) (x + 2)

# express f as a partial fraction decomposition;
> convert(f,parfrac,x);
                                 1           1
                             - ----- + 2 -----
                               x + 1     x + 2

# evaluate limits;
> limit(sin(x)/x,x=0);
bytes used=410936, alloc=163840, time=2.240
                                     1
```

The numerical evaluation took 2.24 units of CPU time.

```
# calculate sums;
> sum(r^2,r=1..n);
                          3                  2
          1/3 (n + 1)  - 1/2 (n + 1)  + 1/6 n + 1/6
```

$$1^2 + 2^2 + ... + n^2 = \frac{(n+1)^3}{3} - \frac{(n+1)^2}{2} + \frac{n+1}{6}.$$

```
# differentiate a function;
> diff(sin(a*x^2),x);

                                     2
                             2 cos(a x ) a x
```

```
# indefinite integration and definite integration;
> int(ln(x)/x,x);
                                          2
                                1/2 ln(x)
```

$$\int (\ln x)\frac{dx}{x} = \frac{1}{2}(\ln x)^2 \text{ [+ const].}$$

```
> int(ln(x)/x,x=1..2);
bytes used=813416, alloc=286720, time=4.730
                                           2
                                1/2 ln(2)

# load the linear algebra package for matrix calculations;
> with(linalg);
> a:=array([[1,2,3],[0,1,4],[1,1,1]]);
                                [ 1  2  3 ]
                                [         ]
                           a := [ 0  1  4 ]
                                [         ]
                                [ 1  1  1 ]

> det(a);
                                    2

> inverse(a);
                           [ - 3/2  1/2  5/2 ]
                           [                 ]
                           [   2    -1   -2  ]
                           [                 ]
                           [ - 1/2  1/2  1/2 ]

# solve a system of linear equations ax=b;
> b:=array([1,2,3]);
                        b := array(1 .. 3,[1,2,3])

> linsolve(a,b);
bytes used=1213700, alloc=385024, time=7.770
                             array(1 .. 3,[7,-6,2])

> writeto(terminal);
```

Computer Algebra and Einstein's Equations

To get a feel for the immense importance of computer algebra in scientific research, consider Einstein's theory of gravity. This is based upon the replacing of our ordinary concepts of space and time by a four-dimensional "space-time" which is warped by the presence of matter. The complicated geometry of this "space-time" is described by Einstein's equations. Their *appearance* is deceptively simple. They are written

$$R_{\mu\nu} = 0. \tag{1}$$

However, μ is a shorthand symbol and may take the values 1, 2, 3, 4 and the same is true of ν. Thus, Equation (1) is not just one equation, but a set of sixteen equations†, i.e.

---

† Actually ten, because R_{21} is the same as R_{12}, etc..

$$R_{11} = 0,\ R_{12} = 0,\ \ldots,\ R_{14} = 0,\ \ldots,\ R_{44} = 0.$$

But now each of *these* equations is much more complicated than it looks. For example, R_{11} is the sum of four separate terms†

$$R_{11} = R^1{}_{111} + R^2{}_{121} + R^3{}_{131} + R^4{}_{141} \tag{2}$$

and each of *these* terms is itself the sum of more terms. E.g.

$$\begin{aligned} R^2{}_{121} = {} & \Gamma^2{}_{11,2} - \Gamma^2{}_{12,1} + \Gamma^2{}_{12}\Gamma^1{}_{11} + \Gamma^2{}_{22}\Gamma^2{}_{21} \\ & + \Gamma^2{}_{32}\Gamma^3{}_{21} + \Gamma^2{}_{42}\Gamma^4{}_{21} - \Gamma^2{}_{11}\Gamma^1{}_{12} - \Gamma^2{}_{21}\Gamma^2{}_{12} - \Gamma^2{}_{31}\Gamma^3{}_{32} + \Gamma^2{}_{31}\Gamma^4{}_{42}. \end{aligned} \tag{3}$$

As if this were not enough, each of the symbols $\Gamma^2{}_{42}$, $\Gamma^2{}_{11,2}$ etc. is itself the sum of more terms. E.g.

$$\begin{aligned} 2\Gamma^3{}_{21} = {} & g^{31}(g_{12,1}+g_{11,2}-g_{21,1}) + g^{32}(g_{22,1}+g_{21,2}-g_{21,2}) + g^{33}(g_{32,1}+g_{31,2}-g_{21,3}) \\ & + g^{34}(g_{42,1}+g_{41,2}-g_{21,4}) \end{aligned} \tag{4}$$

All in all, there are about a thousand separate terms in the expression for R_{11} alone!

It is the various quantities represented by the different symbols involving g that we wish to find. These represent the curvatures of the "space-time" as it is buckled by the influence of matter.

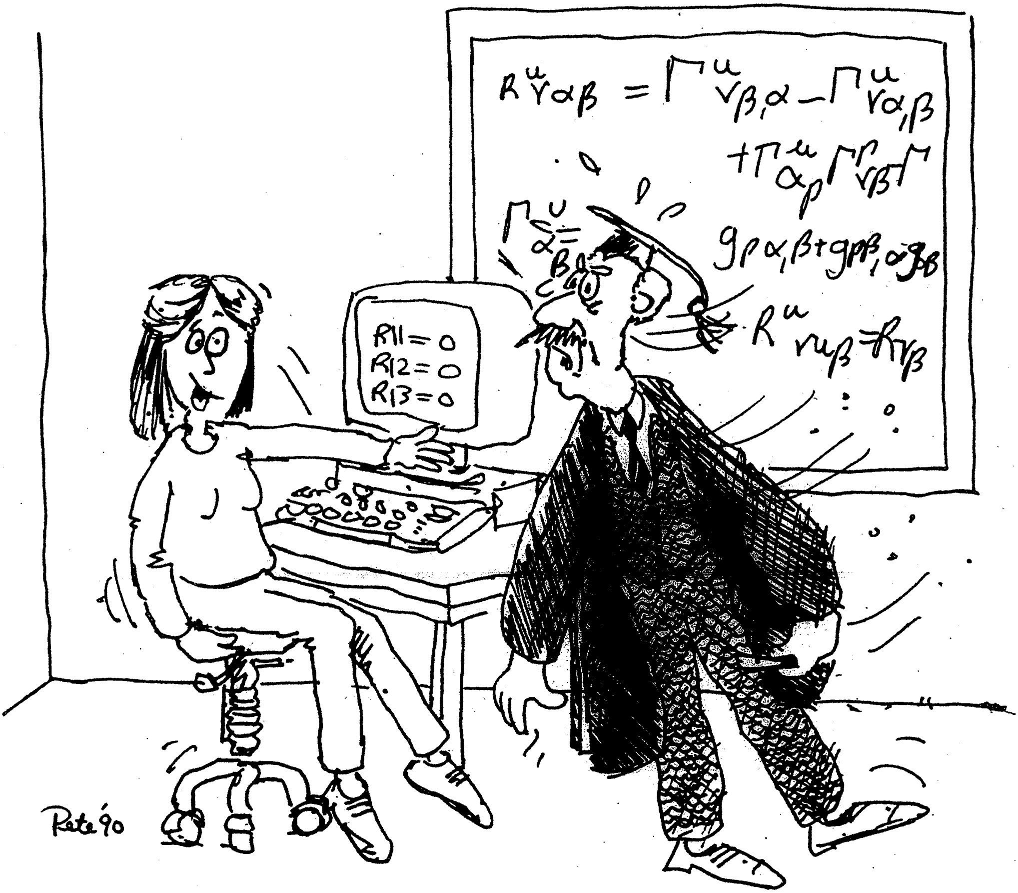
Without going into any more detail, we hope you can see that dealing with Einstein's equations is a very tedious, time-consuming and complicated process in which it is extremely likely that one will make "careless" mistakes! The development of computer algebra packages which can execute – in a matter of minutes – all the sums, products and derivatives of the *symbols* involved in Einstein's equations makes life incomparably easier and more pleasant for relativity workers. It also enables us to tackle new problems that would have been impossible without symbolic computation packages. Thus, the development and application of such packages is at present an exciting and major research area.

* * * * *

---

† Don't worry about the *meanings* of all these different symbols. Just appreciate the enormous value of a computer algebra program which does so much of the hard work!

Computer algebra as a research tool in general relativity



Our thanks to Pete Biran for this cartoon.

HYPERSPHERES

Karl Spiteri, Student, University Of Melbourne

A sphere of radius a and centred on the origin has the equation

$$x^2 + y^2 + z^2 = a^2 \tag{1}$$

and from this it may be determined that its volume is given by

$$V = \frac{4}{3}\pi a^3 . \tag{2}$$

The sphere is a three-dimensional object which has a two–dimensional analogue, the circle. A circle of radius a, centred on the origin, has the equation

$$x^2 + y^2 = a^2 \tag{3}$$

and its area, the two-dimensional equivalent of volume, is given by

$$A = \pi a^2 . \tag{4}$$

We may imagine a one-dimensional analogue of Equations (1), (3):

$$x^2 = a^2 . \tag{5}$$

which is a line-segment of length (the one-dimensional analogue of volume)

$$L = 2a . \tag{6}$$

What happens if we go up to 4, 5,... dimensions?

In four dimensions, we would have, analogously to Equations (1), (3), (5),

$$x^2 + y^2 + z^2 + w^2 = a^2 \tag{7}$$

and this equation defines a four-dimensional *hypersphere*. We can use calculus to find a four-dimensional *hypervolume* (H_4) for this hypersphere. The result is

$$H_4 = \frac{\pi^2}{2} a^4 . \tag{8}$$

Similarly in five dimensions we find

$$H_5 = \frac{8\pi^2}{15} a^5 . \tag{9}$$

In general we find

$$H_n = V_n a^n$$

where H_n is the hypervolume in n-dimensional space, and V_n is some number depending on n.

Here is a table for V_n:

n	V_n	
1	2	$= 2$
2	π	$= 3.14 \ldots$
3	$\frac{4\pi}{3}$	$= 4.18 \ldots$
4	$\frac{\pi^2}{2}$	$= 4.93 \ldots$
5	$\frac{8\pi^2}{15}$	$= 5.26 \ldots$
6	$\frac{\pi^3}{6}$	$= 5.16 \ldots$
7	$\frac{16\pi^3}{105}$	$= 4.72 \ldots$
8	$\frac{\pi^4}{24}$	$= 4.05 \ldots$
9	$\frac{32\pi^4}{945}$	$= 3.29 \ldots$
10	$\frac{\pi^5}{120}$	$= 2.55 \ldots$

It is interesting to note that V_n at first increases, reaches a maximum at $n = 5$ and decreases thereafter. See the graph overleaf.

I was interested to get a general formula for V_n. We may use calculus to prove the formula

$$V_n = \left(\frac{2\pi}{n}\right) V_{n-2} \tag{11}$$

and from this and the result $V_2 = \pi$ we may find that when n is even

$$V_n = \frac{[\pi^{n/2}]}{(n/2)!} \tag{12}$$

We may also use Equation (11) to give a formula for V_0, which is rather hard to imagine. We find however that $V_0 = 1$.

When n is odd, $n = 2m - 1$, for some m. Looking at the odd entries in the table we see that they suggest the formula

$$V_n = \frac{2^m \pi^{m-1}}{1.3.5 \cdots\cdots n}. \tag{13}$$

Since $n = 2m - 1$, $m = \frac{1}{2}(n+1)$. So

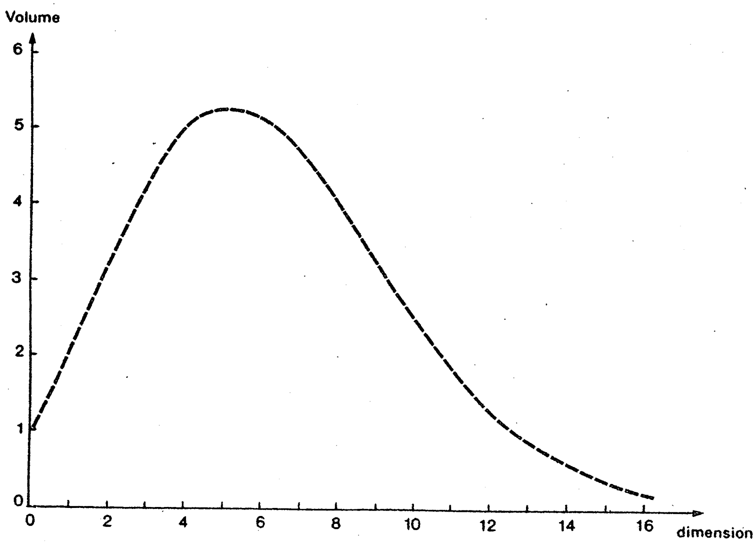
$$V_n = 2^{\frac{1}{2}(n+1)} \pi^{\frac{1}{2}(n+1)} \;/\; 1.3.5.\cdots.n \tag{14}$$

This formula may also be rigorously proved, but I will not do this here.

Equations (12), (14) allow us to show that $V_n \longrightarrow 0$ as $n \longrightarrow \infty$ and this feature is shown in the graph above. We can check this numerically also: $V_{100} = \pi^{50}/50!$ and this figure works out to be less than 2.4×10^{-40}. (In 100 dimensions, a hypersphere occupies only a miniscule fraction of the hypercube that surrounds it!)

The graph suggests that it may be possible to investigate V_n for fractional n, but I have not looked closely at this. However, I did investigate S_n, the generalisation of surface area. S_n (for integral n) is maximised when $n = 7$ and it also has a limit of zero as $n \longrightarrow \infty$.

Strange and interesting things happen in multidimensional worlds: very often contrary to our "3-D intuition".



LIFE IS NOT MEANT TO BE ALWAYS PALINDROMIC

Hans Lausch, Monash University

Spelling words from back to front may be fun, when one is very young. If that fun persists for an intolerably lengthy period, it is called dyslexia. This phenomenon is deeply rooted in history as a concomitant of many cultures. In the days of the Roman Empire, it was noticed that ROMA was read AMOR by dyslectics, and vice versa. Almost certainly, numerous Romans struggled with the pairings IV - VI, IX - XI, XL - LX etc. Of course, we have our own problem when confronting the unlucky number 13 and having to read it as "thirteen" instead of "ten-three"; reading "24" is much easier! Some comfort might be derived from observing that school students speaking German are worse off. Even "24" has to be read by them as "vierundzwanzig", literally "four-and-twenty" which is truly Shakespearean.

Living in 1991 is a veritable blessing. Back-to-front is as good as front-to-front in the reading of 1991. Morevoer, 1991 is the product of the primes 11 and 181, which are equally good in this regard. When it comes to words, would anyone dispute the symmetries in "madam"? And there are whole sentences of that sort, too. My colleague John Stillwell has kindly written out for me a couple of sentences which offer no fresh information to readers, should they run over them again, backwards and letter-by-letter: the programmatic "A man, a plan, a canal - Panama!" and the dedicated "Norma is as selfless as I am, Ron."

"Running back again" is one meaning of the English word "palindrome", related through classical Greek to "palingenesis", re-creation, and to "hippodrome", horse run. Feeling familiar with palindromes, be they numbers, words or sentences, we wish to generate them from non-palindromes. *Prima facie*, numbers are excellent material for such experiments. The smallest interesting number for our experiments is 12. Turned around, it becomes 21. Adding these two numbers creates "compensation" and, not surprisingly, we get as sum the palindromic number 33. Our scheme fails when we reach 19, as 19 + 91 = 110; this is 011 read backwards. Continuing, we add once more, 110 + 011, and note, with satisfaction, the result: the palindromic number 121. Try 59: 59 + 95 = 154; keep going: 154 + 451 = 605; finally 605 + 506 = 1111, what a palindrome! After a series of trials, we may ask whether *every* number sooner or later gives rise to a palindrome in this way. I do not know the answer.

So far we have worked in the decimal system. Switching to the simplest number system, the binary, the question remains valid. The only digits in the binary system are 0 and 1, and binary addition is easily learned. Just as in the decimal system, one number is placed beneath the other; the work starts at the right-hand end, and attention must be given to the transfer of occasional "carries", i.e. of ones. The example

$$\begin{array}{r} 111011 \\ +\quad 1110 \\ \hline 1001001 \end{array}$$

includes all points which require attention.

Let us try to generate palindromes as before, this time in binary. Commencing with 1101, say, leads to the following sequence of additions:

$$\begin{array}{rr} & 1101 \\ + & 1011 \\ \hline & 11000 \\ + & 00011 \\ \hline & 11011 \end{array} \text{, a palindrome!}$$

Or, if we choose 11001 as our "seed", our calculation, although not as short, is still terminated by a palindrome:

$$\begin{array}{rr} & 11001 \\ + & 10011 \\ \hline & 101100 \\ + & 001101 \\ \hline & 111001 \\ + & 100111 \\ \hline & 1100000 \\ + & 0000011 \\ \hline & 1100011. \end{array}$$

Nonetheless, there is a number that does not cause a binary palindrome, however often we repeat our procedure. It is 10110, whose decimal equivalent ironically - is palindromic 22. How can we see this? The initial four additions are:

$$\begin{array}{rr} & 10110 \\ + & 01101 \\ \hline & 100011 \\ + & 110001 \\ \hline & 1010100 \\ + & 0010101 \\ \hline & 1101001 \\ + & 1001011 \\ \hline & 10110100. \end{array}$$

So far, so bad. We shall write a sequence of n ones as $1_{(n)}$ and a sequence of n zeros as $0_{(n)}$; in this new notation, our latest sum reads $101_{(2)}010_{(2)}$.

We now demonstrate that four further steps applied to $101_{(n)}010_{(n)}$, where n is any integer greater than 1, will produce $101_{(n+1)}010_{(n+1)}$ with no palindrome arising intermediately. Our procedure in this case will therefore create four patterns of binary numbers which get ever longer and never palindromic.

To facilitate the subsequent additions, we present $101_{(n)}010_{(n)}$ in the equivalent form $101_{(n-2)}11010_{(n-2)}00$. The first two steps are:

$$\begin{array}{rr} & 101_{(n\text{-}2)}11010_{(n\text{-}2)}00 \\ + & 000_{(n\text{-}2)}10111_{(n\text{-}2)}01 \\ \hline & 110_{(n\text{-}2)}10001_{(n\text{-}2)}01 \\ + & 101_{(n\text{-}2)}00010_{(n\text{-}2)}11 \\ \hline & 1011_{(n\text{-}2)}10100_{(n\text{-}2)}00 \end{array}$$

or, re-written, $101_{(n\text{-}2)}110100_{(n\text{-}2)}00$.

The third and fourth steps yield:

$$\begin{array}{rr} & 101_{(n\text{-}2)}110100_{(n\text{-}2)}00 \\ + & 000_{(n\text{-}2)}010111_{(n\text{-}2)}01 \\ \hline & 110_{(n\text{-}2)}001011_{(n\text{-}2)}01 \\ + & 101_{(n\text{-}2)}101000_{(n\text{-}2)}11 \\ \hline & 1011_{(n\text{-}2)}110100_{(n\text{-}2)}00, \end{array}$$

which, simplified, reads $101_{(n+1)}010_{(n+1)}$. And this is the number we have been after.

Thus life is not meant to be always palindromic - in binary at least.

* * * * *

COMPUTER SECTION

EDITOR: R.T. WORLEY

This computer section contains an article submitted by Karl Spiteri, and he dedicates it to Dr J. Upton, one of his first-year lecturers at Melbourne University, who died earlier this year. Karl's program requires arithmetic to be done to a higher precision than that available with normal computer languages such as standard BASIC, so he treats a number as an array of digits, storing each digit in a separate array location. Arithmetic is done by mimicking the standard long multiplication/long division routines.

The idea for this work was completed in year 12 (1989), yet the program was modified late last year. In year 12 I came across a book by Martin Gardner, in which was contained a problem which I solved by this program. The problem concerned factorials, namely:-

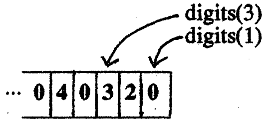
"How many trailing zeros are there in 1000! ?"

Martin Gardner's book contained printouts of all the digits in various factorials. I wrote a program to do this, and from this arose other ideas.

The program is written in BASIC, and can easily be modified to suit any machine. As submitted it runs on an IBM PC. The fundamental idea is to use arrays to store individual digits of numbers in to allow manipulation. There is room for modification and improvements, such as

- Storing more than just one digit per array location.
- Graphing the number distribution.
- Printing out the result in square, hexagon, triangle or diamond pattern (as in Gardner's book).

The fundamental method used in the program is as follows. The program calculates in turn 2!, 3!, 4!,...,(N-1)!, N! by using the property that $I! = I \times (I-1)!$. The digits of a number are stored in an array. For example, 8! = 40320 is stored as



The number of digits (5, in the case of 40320) is stored in the variable LENGTH. To progress to $9! = 9 \times 8!$ this number is multiplied by 9 in the standard "long multiplication’ way.

... | 0 | 4 | 0 | 3 | 2 | 0 |

3 0 2 1 ← Carries

× 9

... | 3 | 6 | 2 | 8 | 8 | 0 |

Multiplication is performed by

NEXDIG = DIGIT(PO) * 9 + CARRY

where CARRY is the amount carried forward from the previous digit multiply. This is broken down into the digit

DIGIT(PO) = NEXDIG MOD 10

and the carry forward to the next digit

CARRY = NEXDIG \ 10

(note that '\' in IBM BASIC is integer division - for example 10 \ 3 = 3). There may be carry out of the top digit, in which case the length of the number increases.

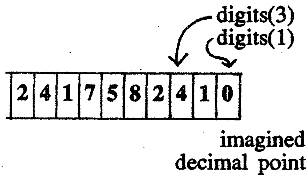
The number of trailing zeros could be determined easily by counting their number. However there is a simple formula which it is quicker to use. This is based on repeated division by 5. For example 999! has 246 trailing zeros since

$$999/5 = 199.8$$
$$199/5 = 39.8$$
$$39/5 = 7.8$$
$$7/5 = 1.4 \qquad \text{(we stop here as } 1 < 5)$$

and

$$199 + 39 + 7 + 1 = 246.$$

The factorial program can easily be modified to calculate 1/(n!) – instead of long multiplying the digits of a large integer by n we long divide the digits of a decimal number by n. However while an integer has an obvious length a decimal number such as 1/7 = 0.142857142... has no obvious length. In this case we need to specify the number of digits to be used. For example, if we represent 1/7 using 10 digits we would use a representation similar to the one factorial program used, except that we read the digits from from right to left with an implied decimal point after digit(1), and that there is a fixed length.



The second program calculates the sum

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!}$$

which is a close approximation to Euler's number e if n is large. The reciprocals of the factorials are calculated as above, and added. The length used for the numbers is four more than the number of digits of e that are required to be printed. Analysis of the error caused by truncating at this many digits shows that the last digit of the sum printed will normally be correct, and can only be 1 too small if it is not correct. This assumes that the limit of 1142 on the number of terms is enforced.

```
10 REM +++++++++++++++++++++++++++++++++++++++++++++
20 REM ++ FACTORIALS ++++ by Karl Spiteri ++++++++++
30 REM +++++++++++++++++++++++++++++++++++++++++++++
40 REM ++ modified for FUNCTION by R.T. Worley +++++
50 REM +++++++++++++++++++++++++++++++++++++++++++++
60 CLS
70 PRINT"FACTORIALS.                    By Karl Spiteri"
80 DIM DIGIT(3000): REM allows up to 1142!
90 PRINT "This program calculates the factorial of a number you enter"
100 PRINT "(n! = n(n-1)(n-2)...3.2.1 )": PRINT
110 INPUT"Enter the number";NUMBER
120 IF NUMBER<0 THEN PRINT"Only numbers >= 0, please": END
130 IF INT(NUMBER)<>NUMBER THEN PRINT"Only whole numbers, please": END
140 IF NUMBER> 1142 THEN PRINT"Sorry - numbers > 1142 not allowed": END
145 TIME$="00:00:00"
150 REM Initialise (BASIC sets all array elements to zero - assume this)
160 DIGIT(1)=1
170 LENGTH = 1
180 REM Calculate factorial
190 FOR I=2 TO NUMBER
200 GOSUB 1000
210 NEXT I
220 REM print number, number of digits and trailing zeros
230 GOSUB 2000
240 REM print occurrences of each digit
250 FOR I=0 TO 9
260 GOSUB 3000
270 NEXT I
```

```
275 PRINT "elapsed time:";TIME$
280 REM done
290 END
999 REM multiply digit array by i (assumes i*10+carry < max integer)
1000 CARRY = 0
1010 FOR PO=1 TO LENGTH
1020 NEXDIG=DIGIT(PO)*I+CARRY
1030 DIGIT(PO)=NEXDIG MOD 10
1040 CARRY = NEXDIG\10
1050 NEXT PO
1060 REM Carry out of last digit increases length
1070 IF CARRY=0 THEN GOTO 1200
1080 LENGTH=LENGTH+1
1090 DIGIT(LENGTH)=CARRY MOD 10
1100 CARRY = CARRY\10
1110 GOTO 1060
1200 RETURN
1999 REM print number and calculate number of trailing zeros
2000 TZ=0
2005 PRINT"factorial(";NUMBER;") is
2006 GOSUB 4000
2010 N=NUMBER\5
2020 TZ = TZ+N
2030 N=N\5
2040 IF N>1 THEN GOTO 2020
2050 PRINT NUMBER;"! has";LENGTH;"digits and";TZ;"trailing zeros."
2060 RETURN
2999 REM calculate and print number of instances of digit i
3000 ND = 0
3010 FOR PO=1 TO LENGTH
3020 IF DIGIT(PO) = I THEN ND = ND+1
3030 NEXT PO
3040 PRINT"there were";ND;"occurrences of digit";I
3050 RETURN
3999 REM print number itself
4000 D$="0123456789"
4005 FOR PO = LENGTH TO 1 STEP -1
4010 PRINT MID$(D$,DIGIT(PO)+1,1);
4020 NEXT PO
4025 PRINT
4030 RETURN
```

```
10 REM +++++++++++++++++++++++++++++++++++++++++++++
20 REM ++ EULER'S NUMBER ++++ by Karl Spiteri ++++++
30 REM +++++++++++++++++++++++++++++++++++++++++++++
40 REM ++ modified for FUNCTION by R.T. Worley +++++
50 REM +++++++++++++++++++++++++++++++++++++++++++++
60 CLS
65 PRINT"Euler's Number [e].                    By Karl Spiteri"
90 PRINT "This program calculates the approximation to e given by"
95 PRINT"1 + 1/(1!) + 1/(2!) + 1/(3!) + ... + 1/(n!)"
100 PRINT: INPUT"What value of n is to be taken";NTERMS
115 IF NTERMS < 2 THEN PRINT"At least 2, please":END
120 INPUT"How many decimal places";PLACE
130 PRINT"Working..."
135 TIME$ = "00:00:00"
```

```
140 PLACE = PLACE+5
150 IF PLACE > 1142 THEN PRINT"Sorry - places > 1137 not allowed": END
155 IF PLACE < 6 THEN PRINT"at least one place needed":END
160 DIM,TERM(PLACE),E(PLACE)
170 REM Initialise (BASIC sets all array elements to zero - assume this)
175 E(1)=2: TERM(1)=1
180 REM Calculate terms and add on to e
190 FOR DIVISOR=2 TO NTERMS
200 GOSUB 1000: GOSUB 2000
210 NEXT DIVISOR
220 REM print result
230 GOSUB 3000
275 PRINT "elapsed time:";TIME$
280 REM done
290 END
999 REM divide digit array by divisor (assumes digit array is number<10)
1000 CARRY = 0
1010 FOR PO=1 TO PLACE
1020 TOTAL = CARRY*10 + TERM(PO)
1030 TERM(PO) = TOTAL\DIVISOR
1040 CARRY = TOTAL MOD DIVISOR
1050 NEXT PO
1060 RETURN
1999 REM add number less than 10 in term() to number < 10 in e(). Result < 10
2000 CARRY = 0
2010 FOR PO=PLACE TO 1 STEP -1
2020 TOTAL = E(PO) + CARRY + TERM(PO)
2030 E(PO) = TOTAL MOD 10
2040 CARRY = TOTAL \ 10
2050 NEXT PO
2060 IF CARRY>0 THEN PRINT"Error - result overflowed": END
2070 RETURN
2999 REM print result
3000 D$="0123456789"
3010 PRINT MID$(D$,E(1)+1,1);
3020 PRINT ".";
3030 FOR PO = 2 TO PLACE-4
4010 PRINT MID$(D$,E(PO)+1,1);
4020 NEXT PO
4025 PRINT
4030 RETURN
```

* * * * *

Thinking Mathematically

Mathematics is a subject which one can learn best by creating it oneself, by being excited about learning it, by knowing its great achievements of the past and its great promises for the future, by willingness to climb intellectual mountains and by eagerness to formulate mathematically problems of physical, social, biological and management sciences.

J.N. Kapur
(formerly President of the
Indian Mathematical Society)

LETTERS TO THE EDITOR

Graphical Construction of Complex Roots

I was interested to read the article on Tanaka's "Shadows" (*Function*, *Vol.* 15, *Part* 3), the more so as I had a similar idea some years ago. This was published in *The Australian Mathematics Teacher*, *Vol.* 36, *No.* 3 (Oct. 1980).

In that article, I showed that the parabola

$$y = (x - a)^2 + b^2,$$

which has complex roots $a \pm ib$, intersects the line $y = 2b^2$ at the points $a \pm b$. The article then goes on to show that the cubic

$$y = (x - c)\ [(x - a)^2 + b^2]$$

with roots c, $a \pm ib$ intersects the line

$$y = 2b^2(x - c)$$

at the points c, $a \pm b$.

You will notice the similarity between the two methods.

My effort was prompted by something I read in *The Australian Mathematics Teacher* round about 1950 or perhaps before.

Garnet J. Greenbury
Brisbane

The Geometric Mean

In his concluding remarks, K. Evans (*Function*, *Vol.*15, *Part* 4) pointed out that three of the four means he had discussed (the arithmetic and harmonic means and the root mean square) are special cases of the *power mean* (of two positive numbers x_1, x_2) defined by

$$M(n, x_1, x_2) = \left\{ \frac{x_1^n + x_2^n}{2} \right\}^{1/n} \qquad (1)$$

Specifically, the arithmetic and harmonic means and the root mean square correspond to the cases $n = 1$, $n = -1$ and $n = 2$ respectively.

We would like to point out that there is a sense in which the fourth mean considered by K. Evans, namely the geometric mean, can also be regarded as a special case of the power mean, corresponding to $n = 0$. Of course, we cannot just put $n = 0$ in Equation (1), since the expression would be undefined; however, it is the case that

$$\lim_{n \to 0} M(n, x_1, x_2) = \sqrt{x_1 x_2} \tag{2}$$

where the right hand side is the geometric mean of x_1 and x_2.

If we are prepared to take for granted that this limit exists, then Equation (2) can be established as follows. First observe that

$$\left\{ \frac{x_1^n + x_2^n}{2} \right\} = \left[\frac{x_1^{n/2} \quad (x_2^{n/2} x_2^{-n/2} + x_1^{-n/2} + x_2^{n/2})}{2} \right]^{1/n}$$

$$= \sqrt{x_1 x_2} \left[\frac{(x_1/x_2)^{n/2} + (x_1/x_2)^{-n/2}}{2} \right].$$

Therefore it is sufficient to show that

$$\lim \left[\frac{v^{n/2} + v^{-n/2}}{2} \right]^{1/n} = n$$

where $v > 0$.

Let $L = \lim \left[\dfrac{v^{n/2} + v^{-n/2}}{2} \right]^{1/n}$. Now put $m = -n$ to find

$$L = \lim_{m \to 0} \left[\frac{v^{-m/2} + v^{m/2}}{2} \right]^{-1/m} = \left[\lim_{m \to 0} \left[\frac{v^{-m/2} + v^{m/2}}{2} \right]^{1/m} \right]^{-1} = L^{-1} .$$

Hence $L^2 = 1$. Since the limit clearly cannot be negative, we must have $L = 1$.

Peter Grossman, Keith Anker
Monash University, Caulfield

* * * * *

HISTORY OF MATHEMATICS SECTION

EDITOR: M.A.B. DEAKIN

From Two To Three To Four And So On

In the first of these columns this year (*Function*, *Vol.* 15, *Part* 1, pp. 19–21), I looked at the very earliest beginnings of counting. We saw that the first counting systems went: *one*, *two*, *many*. Every culture in the world knows these concepts; every language has words for them. However, not all cultures have made what one influential author (Menninger: author of *Number Words and Number Symbols*) calls "the step to three".

The rest of this article is speculative – like the earlier column it predates history, the era for which written records are available. Some parts are more speculative than others and I will try to indicate at each stage what is securely known, what is clear deduction from that, and what is speculation going rather beyond what the available evidence can tell us.

Let us begin with the Australian language *Gumulgal*. Gumulgal is spoken in far North Queensland and it has the following number–words:

1 = urapon 2 = ukasar 3 = ukasar–urapon 4 = ukasar–ukasar.

Two points may be made:

(a) The number 2 (*ukasar*) acquires a special status. Anthropologists refer to Gumulgal as having a *base* of two. This is not quite the same use of the term "base" as that technically employed in Mathematics†, but it has a reasonably close similarity to that use;

(b) Gumulgal may be classified as a "one, two, three, four, many" language.

Put these two points together. It looks very much as if an early "one, two, many" language evolved into today's more developed Gumulgal. As Menninger puts it: the *two*, once a limit of counting, now becomes a base, and the limit of counting is shifted out to *four*. More recent linguistic research among Australian languages (the best basic material for this work) verifies this almost to a certainty.

This leads to a further question. In the earlier article I found evidence, still persisting after all these millenia, of our own ancestors' use of a "one, two, many" system in its faint echoes in today's English. Is there, correspondingly, evidence of "one, two, three, four, many" system?

The answer is – yes, there is; but we have to go outside English to find it. Recall from the earlier article that English is but one of a very large number of related languages, together making up the *Indo-European* family of languages. So is there evidence among the Indo-European languages of there once having been a limit of counting of *four?*

---

†See *Function*, *Vol* 9, *Part* 1, pp. 8-12.

And there is – overwhelming evidence in fact. We saw earlier that our early ancestors, speaking a language now called Proto–Indo–European (PIE) changed the form of the word according to whether it was singular, dual or plural (one, two or many). The form also varied with the *gender* (masculine, feminine or neuter – i.e., approximately, male, female or inanimate) of the thing the word described. To complicate matters further words have changes in their pronunciation and spelling depending on their status (called *case*) in a sentence. Thus, compare:

I asked *her* to call *his* dog
She asked *me* to call *his* dog
He asked *her* to call *my* dog

etc., etc., etc.

Even over and above these complications, we distinguish naming–words (technically called "nouns" or "pronouns"), such as I've been talking about till now, from "adjectives" (describing–words) like "red" or "nice" or "small", etc. All the complications described above apply also to adjectives.

Now here comes the crunch. A describing adjective in PIE had to agree with its described noun in respect of number (one, two, or many), gender and also case. So if we talk of (e.g.) a happy man, the word happy must be singular and masculine to agree with man (as opposed to men, woman or women) and must also agree as to case – i.e. its value in the sentence.

So, after the disgression, back to numbers. Are our number–words nouns or adjectives? That is to say, do we have things called "twos" or two–type things? Let us leave the philosophers to try to sort out the ultimate truth, if any, behind this question. The plain fact is, however, that PIE–speakers had, rightly or wrongly, a clear answer to it.

Their answer went like this

The numbers 1, 2, 3, 4 were adjectives;

The numbers 5, 6, 7, ... were nouns.

How do we know this? Well, start with Sanskrit, the best preserved and most faithful descendant of PIE. In Sanskrit, if masculine things were being counted, the first four numbers went: *ekah, dvi, trayah, chatvarah.* If the objects were feminine, however, the numerals were *ekā, dvi, tisrah, chatasvarah.* Relics of this exist in the Celtic languages (e.g. Welsh has *pedwar* (masculine) / *pedair* (feminine) for "four"). A remnant may be found in French: *un* (masculine) / *une* (feminine) for "one". We had it in English until relatively recently. "Two" is the feminine form; the now archaic "twain" was masculine. These considerations never apply to numbers above four.

Other Indo–European languages distinguish the first four numbers from the others in different ways. In Bengali, the number words for *two, three* and *four* modify when they compound with nouns (as number–words may in that language). In the main, those for *five, six,* etc. do not.

Menninger notes similar breaks in Latin in the naming of children and in aspects of the calendar.

But by far the clearest case comes from the Slavic languages. It even has a technical name, being called the "Slavic squish". ("Squish", believe it or not, is a technical term in linguistic theory.) It takes a number of forms – for more detail, see Menninger or *Language and Number* by J.R. Hurford. Two examples will suffice. In Czech one says "Two and two *are* four" but "Two and three *is* five" The total modifies the previous word. In Russian, they say "One *house*", but "two, three or four *of house*" and "five, six, etc. *of houses*".

All these linguistic peculiarities are seen as constituting strong evidence for the suggestion that *four* was once the limit of counting. It seems reasonable to assume that it became a limit of counting by pressing *two* into service as a base, much as in Gumulgal. However there is no direct evidence for this.

The PIE word for *four* was **kwetwores* (the asterisk indicates that this is a word reconstructed by linguists – there is no direct evidence for it as writing was not invented back then). There have been many attempts to see where this word comes from, but they are highly speculative and do not carry conviction. However, despite this dissappointment, it seems most likely that the limit of counting was reached via a doubling of the previous limit (two) which was pressed into service as a base.

There is some very slight evidence that this process recurred: *four* becoming the base and *eight* the limit of counting. The PIE word for *eight* was **oktou* and this, grammatically, is a dual form. It means "two *oks*", whatever "oks" might have been. The dual, as we saw in the earlier article, has all but died out in modern Indo-European languages. However in Lithuanian it survives with *judu* (masculine) and *judvi* (feminine) meaning "you two". The case of **oktou* looks very similar.

The only trouble is that we have absolutely no idea what, if anything, an "ok" might have been. It doesn't seem to relate in any obvious way to **kwetwores* and other suggestions like "hand not counting thumb" are totally unsupported by any direct evidence whatsoever.

There is another faint suggestion of *eight's* once having been the limit of counting. This is the PIE word **newm* for "nine". This is almost identical with the PIE word for "new" (compare the Latin *novum*) and this would suggest that the original meaning of "nine" was "the new number".

According to the foremost authority on the PIE numerals, Szemerényi, we do not for the most part have any idea where the PIE number-words come from. In only a few cases does he offer suggestions. One of these is the *newm* - *novum* connection. His tentative acceptance of this is all the more interesting because in taking this stance he is in part disagreeing with his own theory, which is that the PIE base *ten* system was preceded by a base *five* system.

The PIE word for "five" was **penkwe* and those who, like Szemerényi, argue for an earlier base five system tend to see this as related to our words "finger", "fist" and the like. This is highly disputed. Some linguists, relying especially on the Slavic languages, accept the connection; others discount it.

I have myself put forward a (very tentative) suggestion as to another origin for **penkwe*. In PIE the suffix *–kwe* could be added to the end of a word to give the same meaning as that achieved in English by putting "and" before the word. Thus **penkwe* should mean "and *pen*" or very possibly "*and pem*" (m's become n's before k). So what is *pen* (or *pem*)?

Well PIE had two words for "one". There was **oykos* (alternatively **oynos*) and there was also **sems*. (**sems* is the origin of our word "simple" meaning literally "one-fold". It also crops up in words like "semi-final" in which it adopts a special meaning, often associated with it, "one of a pair". **sems* becomes *hems* in Greek and so, again as one of a pair, we have "hemisphere" and the like.)

Suppose **sems*, following standard rules of phonetic change (see the earlier article), became **pem*. Then **penkwe* would mean "and one". That is to say "(four) and one".

Rather remarkably, something very like this has happened in our own counting numbers and shows the effect of going beyond ten. We say "eleven" and "twelve" instead of following the usual pattern, which would give "oneteen" and "twoteen" repsectively. "Eleven" derives from *ein-lif*, which means and sounds like "one left (over after ten)"; similarly "twelve" derives from *zwö-lif*, i.e. "two left".

Thus my suggestion is that **penkwe* is similar, but relates to a limit of counting of *four*.

There is, to sum up, evidence that at one stage the limit of counting was *four*. This is quite widely accepted, and it is very plausible that at first this used a base of *two*. It is also very clear that now we use a base of *ten*. What is far from clear is what happened in between. Did *four* become in its turn a base, as I suggest? Or did a new base of *five* arise, as Szermerényi thinks?

We don't really know the answer to this. There is some very sketchy evidence in favour of the base four theory and some (in my view even sketchier) evidence for base five. This latter, of course, leads more readily to base ten. Ten is a logical base to use. Mathematically it is good (see *Function, Vol.* 9 *Part* 1, pp. 8–12), and we do have ten fingers. Only two cultures of advanced numeracy (the Babylonians: *Function Vol.* 15, *Part* 3, pp. 85–91 and the Mayans: *Function, Vol.* 12, *Part* 4, p. 98) have ever used any base other than ten.

It is entirely possibly that there were several routes to base ten. Possibly there was a period during which different bases co-existed. This happens in *Kuanua* (the language spoken around Rabaul) where *two* and *five* coexist as bases. Some Indo-European languages show signs of bases other than ten. French has vestiges of a base twenty system. Welsh (*Function, Vol.* 8, *Part* 1, pp. 18–25) shows glimpses of a base nine, and the related Breton language of both six and nine as bases.

The more we find out, in other words, the less we seem to know.

* * * * *

Modular Arithmetic?

Lucette [Perea] was forty-six, half a dozen years younger than [her husband] Jean. ... . Paul Sentenac ... was four years older than Perea and younger than Lucette by ten ... .

Leonard Gribble, *Notorious Crimes*, p.87.

* * * * *

PROBLEMS AND SOLUTIONS

The specialist editor of this section is away overseas and so we use the column to give solutions to some long outstanding problems. Eds.

SOLUTIONS

Problem 12.1.1 (proposed by F.C. Klebaner).

It is a well known fact that if $-1 < q < 1$ then $1 + q + q^2 + \dots q^n$ tends to $\frac{1}{1-q}$ as $n \longrightarrow \infty$.

Let $\{a_n\}$ be any sequence of numbers that converges to q. Show that the limit of the sum does not change if we replace q by a_n, q^2 by $a_n a_{n-1}$ and so on; q^n we replace by $a_n a_{n-1} a_1$.

Namely show that

$1 + a_n + a_n a_{n-1} + \dots + a_n a_{n-1} \dots a_1$ tends to $\frac{1}{1-q}$ as $n \longrightarrow \infty$.

Solution. This problem requires two things to be shown. First it must be established that

$$1 + a_n + a_n a_{n-1} + \dots + a_n a_{n-1} \dots a_1$$

tends to a limit at all as $n \longrightarrow \infty$. This is actually the hard part and neither Dr. Klebaner nor we have been able to produce a proof that is both complete and elementary. However, if we abbreviate this expression by the notation S_n, then we may also write

$$S_{n-1} = 1 + a_{n-1} + a_{n-1} a_{n-2} + \dots + a_{n-1} a_{n-2} \dots a_1 .$$

It now follows that

$$S_n = 1 + a_n S_{n-1} . \qquad (*)$$

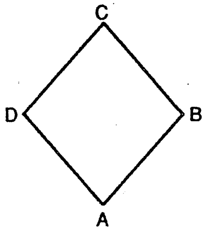
This equation does allow a fairly ready proof that the limit exists (via a technique known as the d'Alembert ratio test – we omit the details) if $-1 < q < 1$. Once it is known that the limit exists, we have $S_n \longrightarrow S$ (say) as $n \longrightarrow \infty$. Now, as $n \longrightarrow \infty$, $a_n \longrightarrow q$. Thus Equation (*) may be approximated by

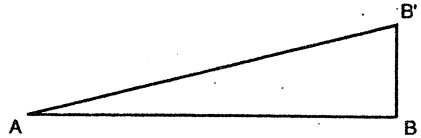
$$S = 1 + q S \qquad (\dagger)$$

and this approximation can become arbitrarily good merely be taking n large enough. The solution of Equation (†) is $S = 1 / (1 - q)$ as required.

Problem 12.3.1 (proposed by Michael A.B. Deakin).

Today (28.3.'88) I visited Calcutta's Birla Museum of Science and Technology and saw a device which consisted of a lozenge-shaped base on each leg of which stood a triangle,





all four triangles being congruent. On this base, a double cone rolled with its axis parallel to BD. It appeared to roll "uphill" as it came to equilibrium with the axis vertically above BD. What condition(s) must hold for this motion to occur?

Solution. Figure 1 gives a 3-D drawing of the base, and Figure 2 shows a view from above.

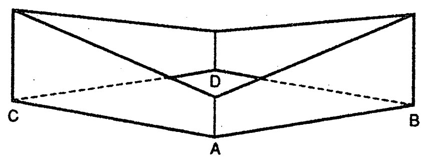


Figure 1

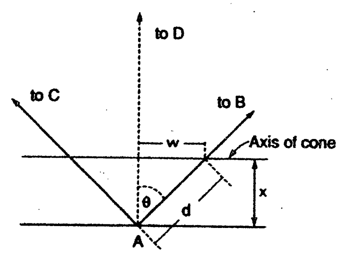


Figure 2

Use notation as shown in Figure 2. Then

$$d = x \sec \theta \qquad\qquad w = x \tan \theta.$$

When the cone has its axis as shown in Figure 2, that axis will be at a height y above the plane $ABCD$. This is made up from two components: the height y_1 of the frame at that point and the thickness y_2 of the cone. See Figures 3, 4 for diagrams that allow us to calculate y_1, y_2. Use the notation of those figures.

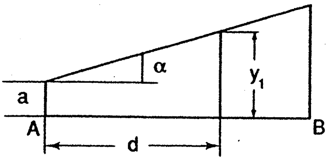


Figure 3

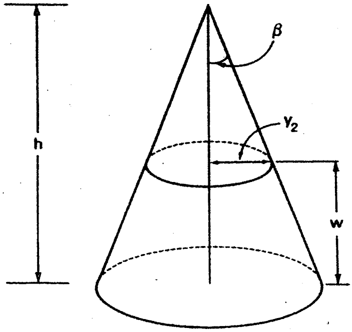


Figure 4

From Figure 3,

$$y_1 = a + d \tan \alpha$$

and from Figure 4,

$$y_2 = (h - w) \tan \beta.$$

Now, putting all the equations together, we get

$$y = y_1 + y_2 = (h - x \tan \theta) \tan \beta + a + x \sec \theta \tan \alpha$$

i.e.

$$y = (h \tan \beta + a) + x (\sec \theta \tan \alpha - \tan \theta \tan \beta).$$

Now the axis of the cone must really roll *downhill*, and so the coefficient of x must be negative. That is to say:

$$\sin \theta \tan \beta > \tan \alpha.$$

This is the condition under which the device can operate.

Although the cone rolls, as it must, *downhill*, the optical illusion that it rolls *uphill* is very strong. Devices not unlike this are sold from time to time in Melbourne toyshops.

Problem 12.3.2 (proposed by Michael A.B. Deakin).

If n is a non-negative integer, then $n!$ is a defined integer, given by $0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, etc. Let $n?$ satisfy $n?! = n!?$, where $n?$ is to take positive integral values (or zero). What values can $n?$ in fact take?

Solution.

$$\text{Since } 0?! = 0!?, \quad 0?! = 1? \tag{1}$$

$$\text{and since } 1?! = 1!?, \quad 1?! = 1? \tag{2}$$

Thus $1?$ is a number satisfying the equation

$$n! = n. \tag{3}$$

There are only two such numbers: 1 and 2. Thus either $1? = 1$ or $1? = 2$. If $1? = 1$, then by Equation 1 we have $0?! = 1$ and so $0? = 0$ or 1. If on the other hand $1? = 2$, then $0?! = 2$, and so $0? = 2$.

$$\text{Now } 2?! = 2!?, \text{ so } 2?! = 2?\ . \tag{4}$$

In other words $2?$ satisfies Equation (3) and so $2? = 1$ or 2.

Thus for the first three values of n we have the possibilities:

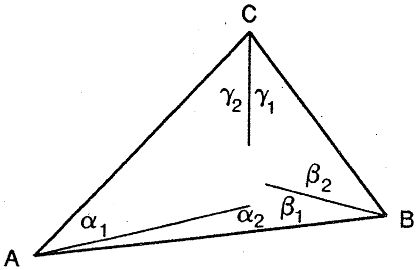
0	1?	2?
0 or 1	1	1 or 2
2	2	1 or 2

a total of 6 separate solutions so far.

For higher values of m, the possiblities are infinite. 3 is not a factorial number and so $3?$ may be assigned arbitrarily. Set $3? = m$. Similarly put $4? = p$, $5? = q$. The next number, 6, is a factorial number and so we have, because $6 = 3!$, $6? = 3!? = 3?! = m!$. Similarly $24? = p!$ and $120? = q!$. Thus we may continue, assigning values arbitrarily, except in the case of the factorial numbers: 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, etc. and, as we have seen, there are also restrictions on the value of $0?$

Problem 12.4.1 (proposed by A.W. Sudbury).

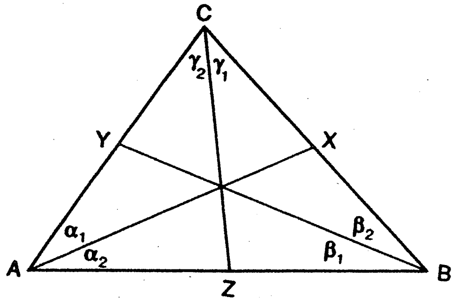
From each of the three vertices A, B, C of a triangle, a ray is drawn in the direction of the interior of ABC. Let α_1, α_2, β_1, β_2, γ_1, γ_2 be the angles as indicated in the diagram below.



Find a trigonometric equation, involving only the angles α_1, α_2, β_1, β_2, γ_1, γ_2, which holds if and only if the three rays intersect in one point. Demonstrate directly from their definitions that the orthocentre, in-centre, circumcentre and centroid satisfy your equation.

Solution (by the editors).

Look at the amended diagram at the right. First we use the sine rule in the triangle *ABC*.



We have

$$\frac{AB}{sin(\gamma_1+\gamma_2)} = \frac{BC}{sin(\alpha_1+\alpha_2)} = \frac{CA}{sin(\beta_1+\beta_2)} = d \text{ (say).}$$

[d is, in fact, the diameter of the circle passing through A, B and C.]

Thus

$$AB = d\ sin(\gamma_1+\gamma_2),$$

etc.

Now use the sine rule again in the triangle *AXB*.

$$\frac{BX}{sin\ \alpha_2} = \frac{AB}{sin\ \angle AXB} = \frac{d\ sin(\gamma_1+\gamma_2)}{sin\ (\alpha_2+\beta_1+\beta_2)}$$

since $sin\ \angle AXB = sin\ (\pi-\alpha_2-\beta_1-\beta_2) = sin\ (\alpha_2+\beta_1+\beta_2)$

Thus

$$BX = \frac{d\ sin(\gamma_1+\gamma_2)\ sin\ \alpha_2}{sin(\alpha_2+\beta_1+\beta_2)}.$$

Similarly

$$CX = \frac{d\ sin(\beta_1+\beta_2)\ sin\ \alpha_1}{sin(\alpha_1+\gamma_1+\gamma_2)}.$$

Now $sin\ (\alpha_1 + \gamma_1 + \gamma_2) = sin\ (\alpha_2 + \beta_1 + \beta_2)$ because $\alpha_1 + \gamma_1 + \gamma_2 + \alpha_2 + \beta_1 + \beta_2 = \pi$.

Therefore

$$\frac{BX}{CX} = \frac{\sin(\gamma_1+\gamma_2)\ \sin\ \alpha_2}{\sin(\beta_1+\beta_2)\ \sin\ \alpha_1}\ . \tag{*}$$

We now proceed by using Ceva's Theorem (see *Function, Vol.* 12, *Part* 5, pp. 147-152), which says that if (and only if) AX, BY, CZ pass through a common point, then

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1.$$

Use Equation (*) and similar equations for the other ratios in this formula and simplify. The result is

$$\sin\ \alpha_1\ \sin\ \beta_1\ \sin\ \gamma_1 = \sin\ \alpha_2\ \sin\ \beta_2\ \sin\ \gamma_2\ .$$

This is the required formula.

It is clear that this holds for the in-centre ($\alpha_1 = \alpha_2$, etc.). We leave it to the reader to see that it holds in the other cases mentioned. It is best in doing this to work from Ceva's theorem directly.

Problem 13.2.2 (taken from the *Argus* in the 1930's from its Education Column).

A man who had no watch was about to leave for a friend's home when he noticed that his clock had stopped. He went to the home of his friend, and after listening to a wireless programme for a couple of hours, returned home and set his clock. How could he do this with any degree of accuracy without knowing beforehand the length of the trip from his friend's place?

Solution. Clearly we have to supply some further information over and above what the wording actually specifies. There are several possible "solutions" to this problem, all depending on our supplying some further (but plausible) information about the case. This one (not the one the editors first thought of) is that supplied by the *Argus*.

Before he left, the man wound his clock but he did not adjust it. He did however take a note of the time recorded by it. Thus, when he returned, he knew the total time he'd been away. This was two hours plus twice the travel time. From this data he would be able to deduce the travel time. Assuming that he also took note of the time when he left his friend's place, he could then add the travel time to that and so set his clock.

Just one new problem this time.

Problem 15.5.1

In an Australian Rules match, the Galahs beat the Goannas. One fan noticed that the Galahs scored as many goals as the Goannas scored behinds and vice versa. He also noticed that the total points score of the Galahs (read from right to left) equalled that of the Goannas (but read from left to right).

What were the scores registered by the teams?

* * * * *

INDEX TO VOLUME 15

Registered for posting as a periodical – "Category B"
ISSN 0313 – 6825

* * * * *

Published by Monash University Mathematics Department