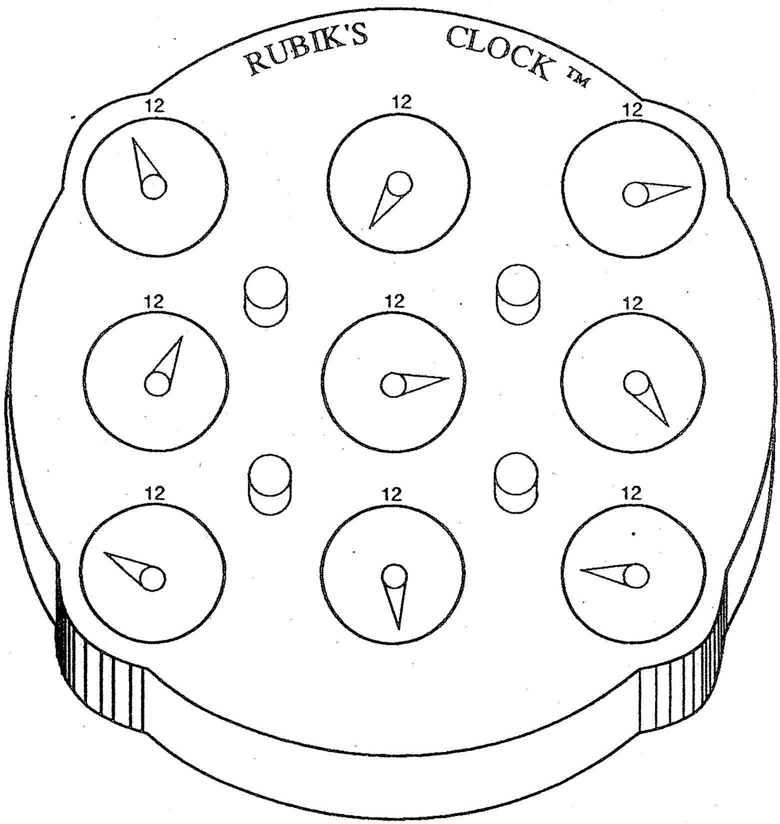
Function

Volume 13 Part 4 August 1989



A SCHOOL MATHEMATICS MAGAZINE

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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
FUNCTION,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

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FUNCTION

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We again, as we did in the last issue, reproduce an article from *Function* of ten years ago. Graph theory, concerned with the kind of graphs consisting of vertices and edges, discussed in this article, is of growing importance in the uses of mathematics involving planning and decision making. Marta Sved's article on fractals introduces a very recent development of mathematics, one that enables mathematics to get to grips with interpreting phenomena that had previously been found too irregular to be amenable to adequate analysis.

The quotation from Truesdell about Leonardo da Vinci's mathematical training is an interesting one. Is your training similar to his?

CONTENTS

FRONT COVER - RUBIK'S CLOCK

Rubik's cube[†] began life in 1974, made its public début in 1978 and by 1980 had spread across the whole world, selling to addicts young and old, from cabinet ministers to five-year olds. Ernö Rubik, a Hungarian architect and teacher of design, followed his cube with Rubik's hinge, marketed in 1985 under the name of "Rubik's magic puzzle: link the rings". In 1988 Rubik's Clock appeared. Neither the hinge nor the clock has developed a band of addicts of a size comparable to those who joined the cube craze. This is almost certainly because the hinge was found easier than the cube. The cube was an Everest puzzle: once beyond the lower slopes the difficulties seemed insuperable, the challenge could not be rejected. The clock was again found easier than the hinge; but it still remains one of the best puzzles around.

Each of the puzzles is concerned with permutations, and each permutation of a system realises a symmetry of that system. The part of mathematics that deals with such symmetries is called *group theory*. And mathematicians were soon writing numerous papers explaining the symmetries of the cube and the hinge in terms of the groups of permutations that the allowed manipulations of the cube (and the hinge) generated.

We explain below how the clock puzzle can be solved. Effectively we are using group theory, but we shall not mention this again in what follows. What we show is that the time on each clock can be set independently of the times shown on the other clocks, with the exception only of the pair of clocks at each corner of Rubik's Clock. The corner clocks are so constructed that any move of the hand on a face of a corner clock is accompanied by the same move of the hand on the clock on the same corner on the reverse side: the hands of the two clocks at a corner are rigidly connected, so that when one moves clockwise the other moves, when viewed from the other side, the same distance anti-clockwise, and *vice versa*.

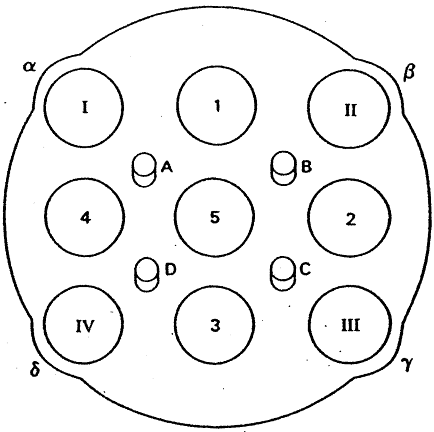
The front cover shows a picture of a face of Rubik's Clock. On the reverse side is an identical set of nine clocks, differing only in their colouring. In the picture the hand of each clock points to 12. Move the control wheels at the four corners and also from time to time press one or more of the four buttons down, or raise them up again, and the clock hands will quickly begin to point to apparently random times, different on each side. *The problem is then to return all the clocks to their original position where all eighteen hands point to 12.*

If someone gives you the Clock with the clock hands pointing all over the place, you will have a harder task trying to solve the problem than you will have if you start with them all pointing to 12, and then experiment by using the control wheels and varying the position of the buttons carefully to find out the effect of each possible operation.

[†] For further information see the international best-seller *Mastering Rubik's Cube* by Don Taylor, 1980, published by Book Marketing Australia P/L, 195 Bridge Rd., Richmond, Victoria, 3121.

So let us suppose that the Clock is in what I shall call its "home" position, i.e. in which all the clock hands point to 12, and proceed to analyse the possible operations that can be performed using the four control wheels and the four buttons.

To facilitate the description it will help to label the buttons, the clocks and the control wheels.

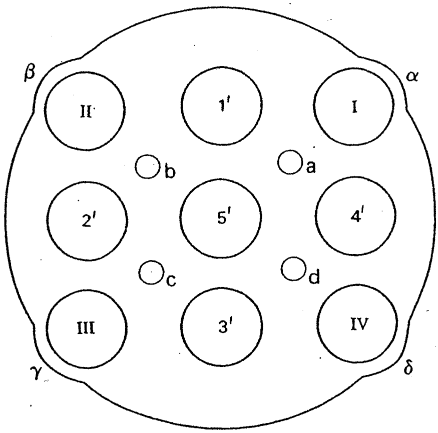


Upper face

Figure 1

As shown in figure 1, label the control wheels α, β, γ, δ, the corner clocks I, II, III, IV, and the other clocks 1, 2, 3, 4, 5, the clock in the centre being clock 5. This is for the upper face. For the reverse side, the lower face of the Rubik's Clock, label the clocks below clocks 1, 2, 3, 4, 5 by $1'$, $2'$, $3'$, $4'$, $5'$, respectively (see figure 2). Examining the action of the control wheels in the various positions of the buttons will quickly show that this gives us enough labels to cover the various clock-hand changes that can occur: each pair of corner clocks rotates its hands in unison, and it is possible to have different times showing on any other pair of clocks.

The buttons have been labelled A, B, C, D when in the positions shown in figure 1, that is, when they have been pressed upwards to project outwards from the upper face to the maximum extent. Lower case letters a, b, c, d, respectively, will be used to label the buttons when in their other possible position, i.e. pressed down so that they project downwards to a maximum extent. Thus the sequence $AbcD$ will indicate that buttons A and D are up and buttons b and c are down. [It is in fact possible to place one or



Lower face

Figure 2

more of the buttons carefully in a half-way position, neither up nor down, (like the Grand Old Duke of York's ten thousand men, in the nursery rhyme). If this is done then the result (on my Rubik's Clock) when a control wheel is operated is to cause the simultaneous occurrence of what would have happened if all the half-way buttons were up and what would have happened if they were all down. Placing the buttons in this neither up nor down position needs delicate positioning and is almost certainly not intended to be part of an allowed operation on the Rubik's Clock. We shall ignore such possibilities in what follows.]

Before we can describe the operations possible upon Rubik's Clock, we need a notation to describe their effect on the clocks.

If clock number 1 has its hand moved one hour clockwise we denote this by $(1)^1$, if by one hour anti-clockwise we denote it by $(1)^{-1}$; if it is moved by 2 hours clockwise denote this by $(1)^2$, and if by 2 hours anti-clockwise by $(1)^{-2}$; and so on: if n is a positive integer, then $(1)^n$ denotes a movement of clock number 1 by n hours clockwise, and $(1)^{-n}$ denotes a movement by n hours anti-clockwise. Similarly for clocks numbered 2, 3, 4, 5, 1′, 2′, 3′, 4′, 5′; thus $(3')^{-2}$ denotes a movement by clock number 3′ by 2 hours anti-clockwise.

We treat the corner clocks slightly differently: if n is a positive integer then $(I)^n$ denotes that clock I, looked at from the upper face, has moved n hours clockwise, and that simultaneously (for this necessarily happens) that viewed from the lower face the other side of clock I has moved n hours anti-clockwise.

What happens when a control wheel is turned depends on the position of the buttons. Each button can be either up or down; so there are $2 \times 2 \times 2 \times 2 = 16$ possible positions for the buttons. Thus *Abcd* denotes the button position where the button in the top left corner is up and the other three buttons are down.

We assume in our description of the operations that we have selected one face that we shall consider as the upper face, the other being the lower face, and that this choice is fixed.

Experimenting with the buttons and the control wheels shows that we can perform the following operations upon the clocks. Denote by α, β, γ, δ a turn of control wheel α, β, γ, δ, respectively, through one hour, i.e. 30°, clockwise.

Button position	Control wheel	Effect
Abcd	α	$(\mathrm{I})^1(1)^1(4)^1(5)^1$
Abcd	β or γ or δ	$(\mathrm{II})^1(\mathrm{III})^1(\mathrm{IV})^1(1')^{-1}(2')^{-1}(3')^{-1}$ $(4')^{-1}(5')^{-1}$
ABcd	α or β	$(\mathrm{I})^{(1)}(\mathrm{II})^1(1)^1(2)^1(4)^1(5)^1$
ABcd	γ or δ	$(\mathrm{III})^1(\mathrm{IV})^1(2')^{-1}(3')^{-1}(4')^{-1}(5')^{-1}$
AbCd	α or γ	$(\mathrm{I})^1(\mathrm{III})^1(1)^1(2)^1(3)^1(4)^1(5)^1$
AbCd	β or δ	$(\mathrm{II})^1(\mathrm{IV})^1(1')^{-1}(2')^{-1}(3')^{-1}$ $(4')^{-1}(5')^{-1}$
ABCd	α or β or γ	$(\mathrm{I})^1(\mathrm{II})^1(\mathrm{III})^1(1)^1(2)^1(3)^1(4)^1(5)^1$
ABCd	δ	$(\mathrm{IV})^1(3')^{-1}(4')^{-1}(5')^{-1}$
ABCD	α or β or γ or δ	$(\mathrm{I})^1(\mathrm{II})^1(\mathrm{III})^1(\mathrm{IV})^1(1)^1(2)^1(3)^1$ $(4)^1(5)^1$
abcd	α or β or γ or δ	$(\mathrm{I})^1(\mathrm{II})^1(\mathrm{III})^1(\mathrm{IV})^1(1')^{-1}(2')^{-1}$ $(3')^{-1}(4')^{-1}(5')^{-1}$

If we turn a control wheel through n hours, i.e. through $n \times 30°$, clockwise, then this has the effect of just repeating n times one of the operations listed above. Thus, with button position *ABCd* and wheel δ moved through n hours, i.e. with the operation *ABCd* δ^n, the effect on the clocks is $(\mathrm{IV})^n(3')^{-n}(4')^{-n}(5')^{-n}$.

We have used above 6 button positions. There are 10 more: 3 more where one button only is up; 3 more in which 2 adjacent buttons only are up; one more in which another diagonally opposite pair of buttons are the only buttons up; and 3 more in which three buttons are up. It will be found, by experimenting with the Rubik's Clock, that the effect of control wheel movements for these other button positions is just what we would expect if each of the control wheels behaved exactly as each of the other control wheels, taking into account their relative positions. Thus *aBcd* β has the effect $(II)^1(1)^1(2)^1(5)^1$; etc.

Now notice that if one operation has the effect $(x)^n$ on clock number x and if another has the effect $(x)^m$ on that clock, then following one by the other, in either order, has the effect $(x)^{n+m}$.

So now we have described all possible operations, and can, using the notation we have introduced, write down the effect of a sequence of operations performed on the clock.

We begin by showing that we can change any of the clocks by one hour clockwise, while at the same time leaving all other clocks unchanged in position.

A corner clock

Two operations suffice: for clock I *ABCD* β together with *aBCD* β^{-1}. These two produce the effects

$(I)^1(II)^1(III)^1(IV)^1(1)^1(2)^1(3)^1(4)^1(5)^1$ and

$(II)^{-1}(III)^{-1}(IV)^{-1}(1)^{-1}(2)^{-1}(3)^{-1}(4)^{-1}(5)^{-1}$,

respectively. Combined their effect is $(I)^1$.

The centre clock, upper face: 5

Five operations suffice:

Abcd α : $(I)^1(1)^1(4)^1(5)^1$

abCd γ : $(III)^1(2)^1(3)^1(5)^1$

aBcD α^{-1} : $(I)^{-1}(III)^{-1}(1')^1(2')^1(3')^1(4')^1(5')^1$

ABCD α^{-1} : $(I)^{-1}(II)^{-1}(III)^{-1}(IV)^{-1}(1)^{-1}(2)^{-1}(3)^{-1}$
$(4)^{-1}(5)^{-1}$

abcd α : $(I)^1(II)^1(III)^1(IV)^1(1')^{-1}(2')^{-1}(3')^{-1}(4')^{-1}$
$(5')^{-1}$.

The effect of these five operations is just $(5)^1$; i.e. the central upper face clock is moved clockwise by one hour and all other clocks remain where they were. Notice (check it) that the five operations can be carried out in any order to achieve the result $(5)^1$. To calculate the effect of the five operations we just add up the indices occurring for each of the clocks, interpreting a total of 0 as indicating no change. Thus the sum for clock (5) is $1 + 1 - 1 = 1$; and for clock (III) is $1 - 1 - 1 + 1 = 0$, etc.

A side clock, upper face: 1

Six operations suffice:

abCD γ^{-1} : $(III)^{-1}(IV)^{-1}(2)^{-1}(3)^{-1}(4)^{-1}(5)^{-1}$

AbCd α : $(I)^1(III)^1(1)^1(2)^1(3)^1(4)^1(5)^1$.

The effect of these two is $(I)^1(IV)^{-1}(1)^1$. Now use the operations already given to move clock I one hour back and clock IV one hour forward. The result is $(1)^1$: the clock number has been moved forward by one hour.

So we have now solved the Rubik's Clock problem! Move each of the fourteen clocks I, II, III, IV, 1, 2, 3, 4, 5, 1′, 2′, 3′, 4′, 5′, separately, to their home positions, from whatever initial position they are in.

We have also shown that every possible position of the fourteen clocks can be reached: so there are $12^{14} = 1,283,918,464,548,864$ distinct sets of times the clocks can take up.

Of course the way to solve the Rubik's Clock problem in practice is by a much quicker method than by fixing the time of each clock independently.

I have a method that will move the clocks from any starting position to the home position, where all hands point to 12, taking at most 20 operations, and usually fewer than this. [In counting these 20 operations, I count each rotation of a control wheel through whatever angle, for given positions of the buttons, as a single operation.]

* * * * *

MONASH UNIVERSITY OPEN DAY MATHEMATICS COMPETITION

The Open Day was held on 6 August, 1989. Some 155 secondary students took part in the competition. The results of the qualifying test, of 8 written questions (in 4 different versions), were:

Score	0	1	2	3	4	5	6	7	8	Total
Students	20	39	40	29	15	6	4	2	0	155

Some questions were at Year 12 level, so it is not surprising that students from lower years might have had trouble. However, two prize winners were actually in Year 11.

The 6 students who achieved scores of 6 or 7 in the qualifying test, and competed in an oral final, were:

Russell Coker	(Yarra Valley,	Year 11)	1st.
Jiun Lai	(Monash High,	Year 12)	2nd.
Abhik Sengupta	(Melbourne High,	Year 11)	3rd.
Steven Siew	(Syndal High,	Year 12)	
Colin Banks	(Glen Waverley High,	Year 12)	
Chris Loeliger	(Murtoa High,	Year 12).	

Congratulations on some very fine performances! Prizes to each of a free subscription to *Function*, 1990. The top three also won prizes of $20, $10, and $5.

Ten years ago

THE TUTTE GRAPH

D.A. Holton, University of Otago

The path to the proof of the Four Colour Map Theorem is paved with a large number of false "proofs". The diagram below (figure 1) shows a graph which arose in the course of one of these attempts.

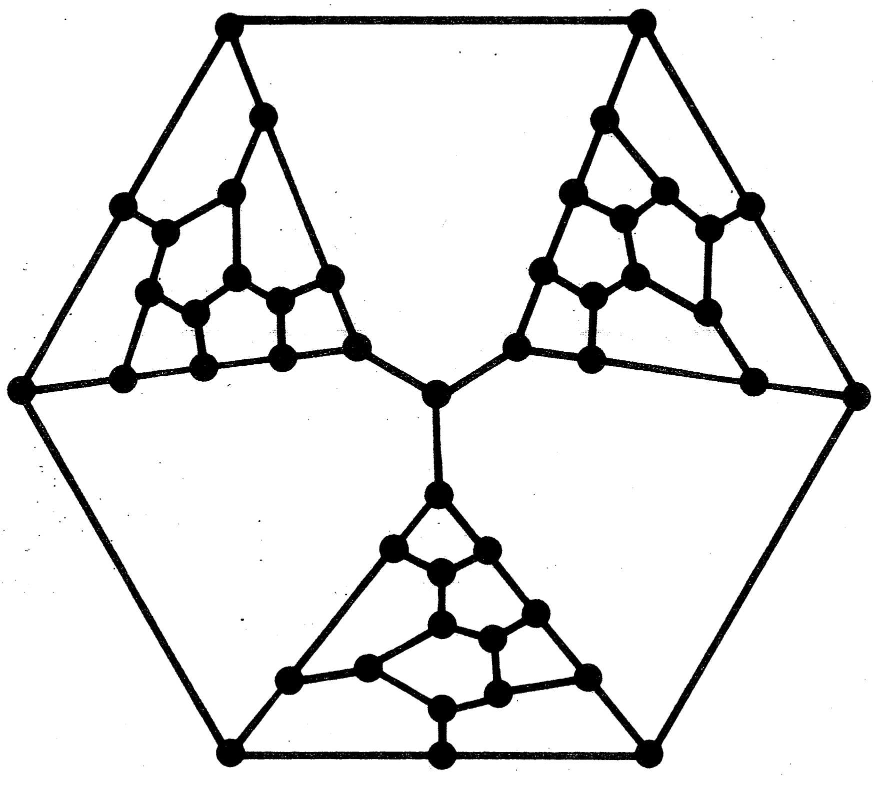


Figure 1

For a background to the problem, see John Stillwell's article in *Function*, Vol. 1, Part 1. The idea, of course, is to prove that the regions of any map can be coloured with four or fewer colours, so that no two adjacent regions have the same colour. This can be converted into a graph theoretical problem by taking a dot to represent the capital of each country and joining two dots by a line if the corresponding countries have a common boundary. Each dot of the resulting graph can then be coloured by one of four or fewer colours, so that no two joined dots have the same colour, if and only if the original map can be coloured in four or fewer colours. For convenience, this graph theoretical formulation was the one used by Appel and Haken in their proof of the Four Colour Theorem.

In 1880, Tait observed that in order to prove this Four Colour Theorem, it would be enough to show that the lines of every cubic, 3-connected planar graph[†] can be coloured in 3 colours so that no two lines with the same colour meet at a given dot.

Tait claimed that, clearly, every cubic 3-connected planar graph had a *hamiltonian cycle*, that is that there is a route along the lines of the graph which starts and ends at one dot and which passes through all the other dots once and only once.

Now if a graph is cubic it is not difficult to prove that it must have an even number of dots. Should the hamiltonian cycle exist, its lines could then be coloured alternately in two colours. The remaining lines could then use the third colour. So if Tait was right about these cubic 3-connected graphs having hamiltonian cycles, the Four Colour Theorem would surely follow.

For a long while people puzzled over Tait's hamiltonian claim. It waited, however, till 1946 before Tutte produced the graph of figure 1 which finally showed that not all cubic 3-connected planar graphs were hamiltonian.

But how do we prove that the Tutte graph is not hamiltonian? One way, of course, would be to illustrate, by exhausting all possibilities, that there is no cycle through every point of the graph. This is enormously tedious.

Is there a characterisation of hamiltonian graphs? If there is, then there is a theorem that says "G is hamiltonian if and only if 'blaa'". All we need to do then is to test Tutte's graph. If it didn't have the property 'blaa' then we would know that it was not hamiltonian.

Unfortunately, despite a great deal of effort, no-one has yet come up with this magic property 'blaa'. If and when they do they

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† A cubic graph is one in which 3 lines come in to every dot and a graph is 3-connected if it takes the removal of at least 3 dots, and their incident lines, to disconnect the graph. A planar graph is one that can be drawn in the plane so that no two lines cross.

will be accorded instant mathematical fame. In the meantime we have a graph at hand. What to do?

Well, fortunately we have some partial 'blaa' results. For instance, Tutte himself has proved that every 4-connected planar graph is hamiltonian. Unfortunately, Tutte's graph is not 4-connected. We can find three vertices whose removal will disconnect the graph. So that test is out.

Another partial 'blaa' comes from two Russians - Kozyrev and Grinberg. Their result says that if G is planar and hamiltonian with n dots, then

$$\sum_{i=2}^{n}(i - 2)(f_i - f'_i) = 0.$$

i.e. $0(f_2-f'_2) + 1(f_3-f'_3) + 2(f_4-f'_4) + \ldots + (n-2)(f_n-f'_n) = 0.$

To explain the f_i and the f'_i look at the planar graph of figure 2, which has the hamiltonian cycle 1, 2, 3, 4, ..., 15, 16, 1.

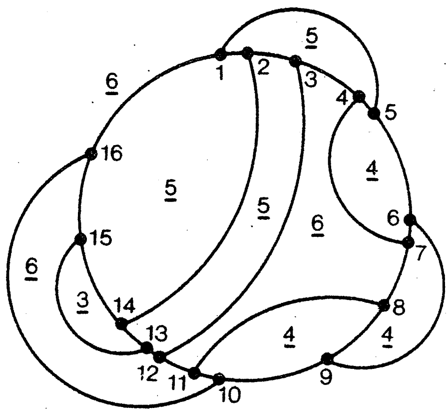

Figure 2

This cycle divides the plane in two. Various lines of the graph divide the inside of the cycle into regions. One region is formed by 1, 2, 14, 15, 16, another by 3, 4, 7, 8, 11, 12. The term f_i is the number of such regions inside the hamiltonian cycle formed by i lines. Hence, for the graph in figure 2 we have $f_2 = 0$, $f_3 = 0$, $f_4 = 2$, $f_5 = 2$ and $f_6 = 1$. The f'_i term tells us the number of regions formed in a similar way on the outside of the cycle. So we see that $f'_3 = 1$, $f'_4 = 1$, $f'_5 = 1$, $f'_6 = 2$.

Substituting these values in the Kozyrev-Grinberg expression

$$\sum_{i=2}^{n} (i - 2)(f_i - f'_i)$$

we get

$$(3-2)(0-1) + (4-2)(2-1) + (5-2)(2-1) + (6-2)(1-2)$$

which is indeed zero.

Now Tutte proved his graph was not hamiltonian, but a nicer proof was given by an amateur mathematician, Watts, in 1972. This proof uses the Kozyrev-Grinberg result. It is a little too long to include here but the basic idea is to assume that Tutte's graph has a hamiltonian cycle and use the Kozyrev-Grinberg equation to obtain a contradiction. The complete proof can be found in the book "Mathematical Gems", by R. Honsberger (published by the American Mathematical Society, 1974). In the same book the Kozyrev-Grinberg theorem is also proved.

For more results in the area of hamiltonian graphs you should read L. Lesniak-Foster: "Some recent results in hamiltonian graphs", *Journal of Graph Theory*, Vol. 1, 1977, 27-36.

* * * * *

Comment, from D.A. Holton, ten years later

A few things have happened in the area of hamiltonian graphs and planarity since "The Tutte Graph" appeared in *Function*. However, we appear to be no closer to a "nice" proof of the Four Colour Map Theorem.

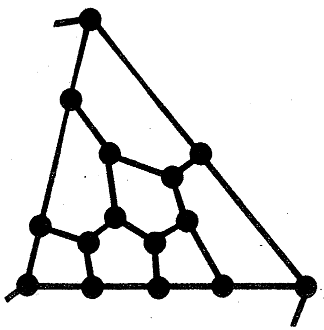

Figure 3

One comment on planar hamiltonian graphs may be of interest. Look at the part of the Tutte graph shown in Figure 3. You will (eventually) see that if you want to enter at one of the vertices 1, 2, 3, pass through all the unnumbered vertices once each and exit at one of the other vertices 1, 2, 3, then you will have had to have entered or exited through 2. There's no way of coming in through 1 and out through 3, or *vice versa*.

This should help you to find another proof of the fact that the Tutte graph is non-hamiltonian.

If you place the part graph of Figure 3 as shown in the graphs of Figure 4, you'll find that you have six non-hamiltonian graphs on 38 vertices. The placement is done by blowing up each of the vertices in Figure 4, denoted by a large black dot, into the graph of Figure 3, and letting the edges labelled 1, 2, 3, at each such vertex, terminate, in the blown-up diagram, at the vertices labelled 1, 2, 3, respectively, in Figure 3. It is now known that these are the only non-hamiltonian 3-connected cubic planar graphs on 38 vertices and that all 3-connected cubic planar graphs on 36 or fewer vertices are hamiltonian.

It is now unlikely that there exists a simple 'blaa' property for hamiltonian graphs. Though, of course, in 10 years' time I may be proved wrong.

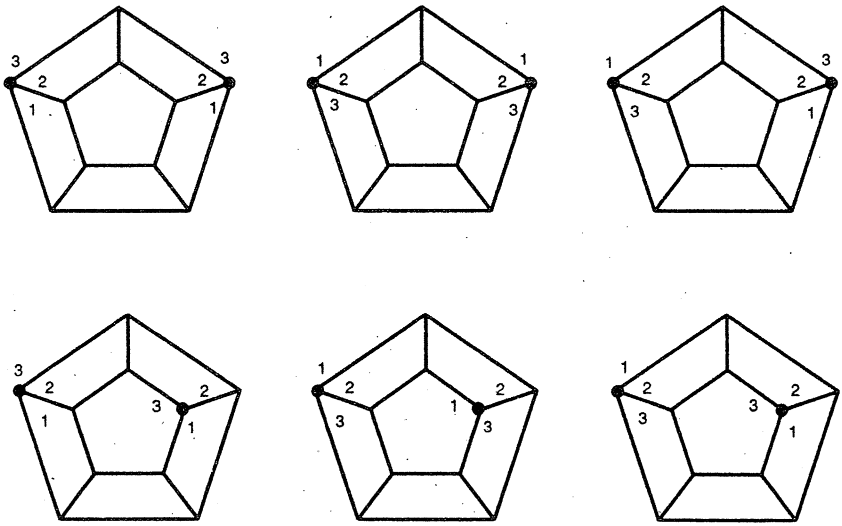

Figure 4

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ON FRACTALS

Marta Sved, University of Adelaide

Classical geometry as handed down to us from mathematicians of ancient Greece, and pursued through many generations until our own days, deals with things like lines and planes, circles and spheres, triangles and polygons, also with more exciting objects, like cylinders and cones, giving ellipses, hyperbolas, parabolas as their plane sections. Regular polygons and regular polyhedra hold their own fascination.

Though geometry offers a rich and endless variety for our studies, its objects are idealisations of things which occur in our every-day experience. Some of the objects occurring in nature show still remarkable, though more complex geometrical structures, like we see in waves, eddies of water, ripples of sand, snow flakes, shells, clouds, only naming a few, while it is hard to find regularity in rugged shore-lines, faces of rocks, or other things which appear, at least on first sight, random, irregular, fragmented.

In the 1970's, it was the French mathematician Benoit Mandelbrot, who was the first to create a connection between complex structures occurring in nature, and structures emerging from earlier mathematical studies. Considering the problem of finding *accurately* the length of a shore-line, he came to the conclusion that the problem has no satisfactory answer as long as we apply to it our ordinary notions of measuring, for the shore-line may *appear* straight from a certain height, but getting nearer, we observe some inlets and promontories, and getting still nearer, we find that these have their own wrinkles and folds, and continuing the approach to the individual grains of sand, and resorting to a microscope, the process of measuring the length never ends.

Figure 1, picturing the "Koch Island", shows an idealised model of the situation. The figure shows stages of construction, beginning with an equilateral triangle, representing "straight" shores of the island. Next we divide each side into three parts and construct on the middle parts equiangular triangles as "promontories". In the next step each of the individual smaller triangles is given the same treatment, continuing (theoretically) to infinity.

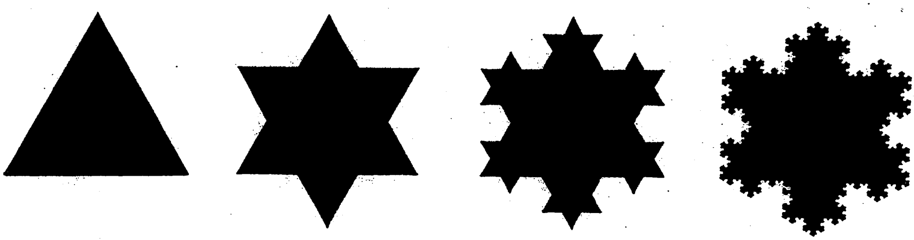

Figure 1

Mandelbrot coined the word "fractal" to characterise such "fragmented" objects. In Euclidean geometry we measured length (say) in *millimetres* (mm), areas in *square millimetres* (mm^2), volumes in *cubic millimetres* (mm^3), and talk about lengths having dimension 1, areas dimension 2, and volumes dimension 3. When we compare a line-segment with our unit line segment, we say that its length is ℓ, if we can place exactly ℓ units to cover the segment. When on the other hand we want to compare a square of side-length ℓ, with the unit square, then we must place ℓ^2 unit squares to cover our square, and for a cube of edge ℓ we need ℓ^3 unit-cubes to fill up the cube. So the dimension expresses an index: 1 in the case of a length, 2 in the case of an area, 3 in the case of a volume. In each case we had simple objects and compared them with *similar* unit-objects. Now Mandelbrot observed that although very small stretches of a shore-line (or you may think of the pictures of the Koch Island) are *similar* to larger stretches, comparing their lengths with some given measuring unit does not lead to a sensible answer. He drew the bold conclusion that the *dimension* of such objects is not an integer, like that of a straight interval, or a square, or a cube, but must be given a different definition. He found that if such configurations are assigned appropriate dimensions, which are *not integers*, then measuring them becomes meaningful. In his earlier writings he gave the name *fractal* to configurations of which the dimension was not an integer.

Without going any deeper into the mathematics of fractal dimensions, let us look at some examples of fractal configurations, which come from mathematical theories. Modern computer graphics applied to some mathematical procedures produce amazingly beautiful pictures such as shown in figures 2 and 3.

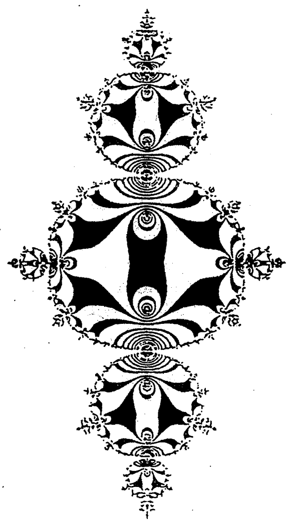

Figure 2

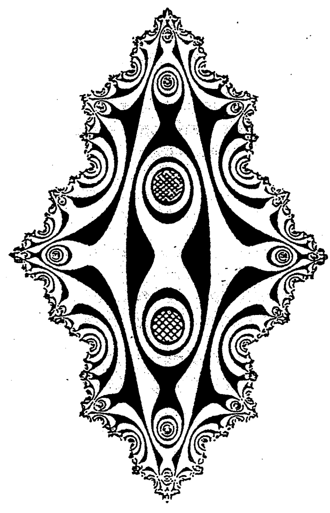

Figure 3

The book:

The Beauty of Fractals, by H.O. Peitgen & P.H. Richter (Springer-Verlag)

offers a beautiful collection of such configurations appearing on computer screens. Many of these are in colour. Without understanding the mathematics behind it, you can still get much enjoyment looking at these reproductions of pictures which could grace any exhibition of abstract art. We recommend that you get hold of this book from some library or a friend who has bought it, and look at the pictures.

Figure 4 shows a simpler picture. You can amuse yourself by producing an approximate cardboard copy. The figure is called the Sierpinski gasket, named after the Polish mathematician who produced it before the birth of fractal theory. The figure is one which arises after an infinite sequence of repeated operations, of which you may try a few.

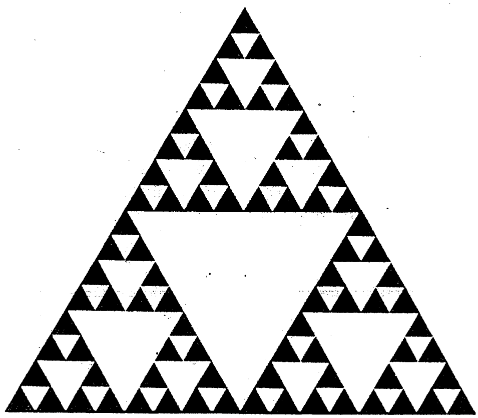

Figure 4

Begin with a large equilateral triangle as shown on figure 5. Divide each side into two equal parts, joining the points of bisection. Now cut out the shaded middle part, cheating a little at the corners, so that the figure should not fall apart. Repeat the procedure on the remaining three smaller triangles. Keep repeating carefully as long as you can. You obtain a fragile object, if not yet a fractal.

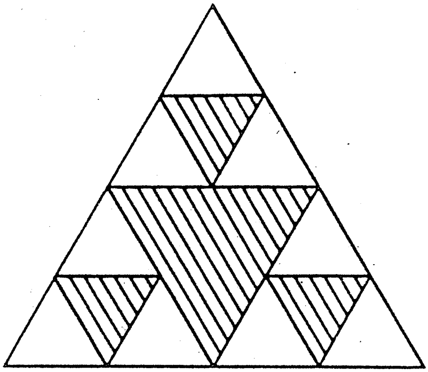

Figure 5

Now comes the surprise, from a field of mathematics that you would not suspect that it has anything to do with all this. Write down the first 16 lines of the Pascal triangle of binomial coefficients, which you could copy from mathematical tables, or produce quickly yourself using the formula

$$\binom{n+1}{k+1} = \binom{n}{k} \quad \binom{n}{k+1}.$$

You obtain

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

Oops, I am running out of space, and find it hard to print out all 16 lines on this page.

But it does not really matter, for what I want to do next is to put a | instead of each odd number, and a dot instead of each even one.

I obtain now

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                  | | . . | | . . | | . . | |
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                | | | | | | | | | | | | | | | |
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You may have noticed that I have produced now all 16 lines without really referring to my original Pascal triangle, because instead of the Pascal addition formula I use the simple rule:

even + even = even
even + odd = odd
odd + odd = even.

Now look at this figure and your Sierpinski gasket!

You may find a little more about even and odd binomial coefficients if you get hold of an older issue of *Function* (Vol. 11, 2, April 1987) and look at the article

"What are the odds that $\binom{n}{r}$ is even?"

However, you could get another fractal figure out of your original Pascal triangle, if you divide each of the coefficients by 3, and write down the remainder of the division in the place of the original coefficient. Now try to produce this new pattern to 27 rows.

Finally, for a little mathematical exercise, try to use your figure to determine the following:

If n and k are any positive integers, under what condition is

$\binom{3n}{k}$ a multiple of 3?

* * * * *

MY DENTIST'S CLOCK

G.A. Watterson, Monash University

I was enduring the filling of some teeth the other day, when I looked across at the clock on the dentist's surgery wall. What I saw surprised me. Had my eyes or brain been damaged by the injections in my gums? The clock looked decidedly wonky.

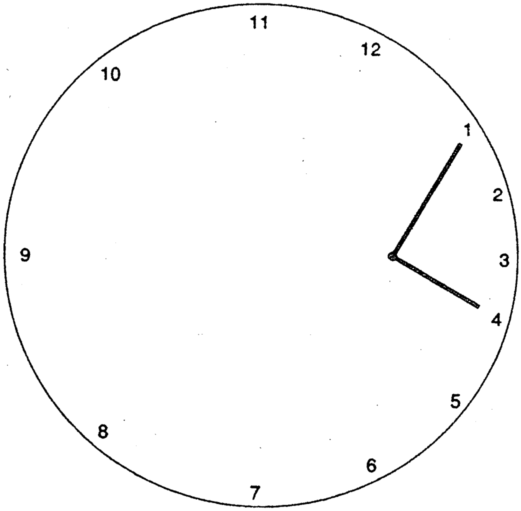


Figure 1

I kept looking. The clock still looked wonky. The placement of the pivot of the hands, away from the centre of the circular clock face, meant that the hour numbers on the face had been displaced from their normal positions. The numbers were, of course, still 30° apart as the hour hand turned about its pivot. But that meant that they were unevenly spaced around the circumference of the face. The morning hours *looked* as if they would be slow to get through, while the afternoon hours *looked* as if they would go quickly!

Suppose that we take the centre of the clock face as origin, O, and the radius of the clock face as 1 (our unit of measurement). Suppose that we introduce the axes Ox and Oy in the usual way, with $A = (a,0)$ as the pivot position for the hands on the x axis.

Consider the hour number, N, one of the numbers $1,2,3,\ldots,12$.

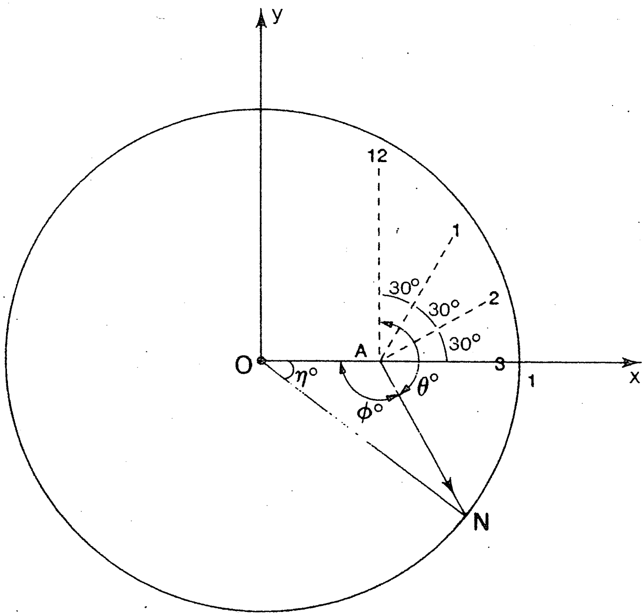


Figure 2

In figure 2 the angle $\theta°$, between the vertical (12-o'clock) position and the hour number N, has to be

$$\theta° = N \times 30°.$$

The angle $O\hat{A}N$, $\phi°$ say, is given by

$$\phi = 360 - 90 - \theta.$$

Consider the triangle ΔOAN, in figure 3.

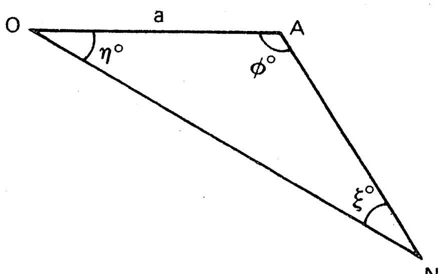


Figure 3

The "sine rule" says that for the three angles, ϕ°, ξ°, and η° as in figure 3, we have

$$\frac{a}{\sin\xi} = \frac{1}{\sin\phi} = \frac{AN}{\sin\eta}.$$

We wish to find η, but as AN is not yet known, we must make use of the first equation to yield

$$\begin{aligned} \sin\xi = a \sin\phi &= a \sin(360-90-\theta) \\ &= -a \sin(90+\theta) \\ &= -a \cos(N\times30), \end{aligned}$$

so that

$$\xi^\circ = \sin^{-1}(-a \cos(N\times30)).$$

Further, because

$$\eta + \xi + \delta = 180,$$

then

$$\begin{aligned} \eta^\circ &= 180^\circ - \phi^\circ - \xi^\circ \\ &= -90^\circ + N \times 30^\circ - \sin^{-1}(-a \cos (N\times30)). \end{aligned}$$

As an example, if we take the pivot A to be half way along the clock face radius, then $a = \frac{1}{2}$, and to the nearest degree, η° is given in the following table for the various hours, N.

N	1	2	3	4	5	6
η	-34	-16	0	16	34	60

N	7	8	9	10	11	12
η	94	135	180	224	266	300

A table like this allows us to see that the hours 3 and 9 are, of course, in their usual positions on the clock face, but that hours on either side of 3 differ from 3 itself by about 16° (more accurately, by 15.52°), using O as centre. On the other side of the face, the hours either side of 9 differ from 9 by about 44° (more accurately, by 44.48°).

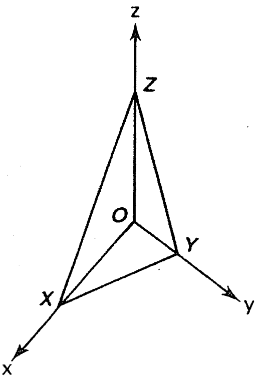
If only the afternoons *did* speed up like the clock suggests!

* * * * *

3-D PYTHAGORAS

Aidan Sudbury, Monash University

Nearly everyone has heard of Pythagoras' theorem: the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides. In fact it is so famous that there are some truly dreadful puns based on it – the squaw on the hippopotamus ... but this article is in far too good taste to mention things like that. However, what is not so well-known is that there is a simple generalisation of the theorem to 3-dimensions. Consider the following figure:



Here Ox, Oy and Oz are three co-ordinate axes in 3-dimensions forming a corner at O just like the corner of a room. XYZ is a triangle, so to speak, resting in the corner. The generalisation of Pythagoras' theorem is:

$$(\text{Area } \Delta\ XYZ)^2 = (\text{Area } \Delta\ OYZ)^2 + (\text{Area } \Delta\ OZX)^2 + (\text{Area } \Delta\ OXY)^2.$$

This is quite easy to prove if you know the formula for the area of a triangle that was discovered by Hero of Alexandria in the 1st century, A.D. If a triangle has sides of length a, b, c then its area is

$$\Delta = \sqrt{(s(s-a)(s-b)(s-c))}$$

where $s = (a+b+c)/2$ is the semi-perimeter. (For a proof of Hero's formula see Editor's comment at end.) Now you should be able to show that

$$s(s-a)(s-b)(s-c) = \frac{1}{16}(2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4). \qquad (*)$$

Then, if you put $OX = x$, $OY = y$, $OZ = z$, so that the sides of XYZ are $\sqrt{y^2+z^2}$, $\sqrt{z^2+x^2}$ and $\sqrt{x^2+y^2}$ (by 2-D Pythagoras), then you can obtain after some simplification

$$(\text{Area } \Delta\ XYZ)^2 = \frac{1}{16}(4y^2z^2 + 4z^2x^2 + 4x^2y^2)$$

$$= \left(\frac{yz}{2}\right)^2 + \left(\frac{zx}{2}\right)^2 + \left(\frac{xy}{2}\right)^2$$

$$= (\text{Area } \Delta\ OYZ)^2 + (\text{Area } \Delta\ OZX)^2 + (\text{Area } \Delta\ OXY)^2.$$

It is possible to generalise Pythagoras' theorem to spaces of any dimension, n, but it is very hard to visualise what it means for $n > 3$.

Editor's comment: Hero's formula follows from the formula $\Delta = \frac{1}{2}bc \sin A$, by eliminating $\sin A$ using the cosine formula $a^2 = b^2 + c^2 - 2bc \cos A$. Thus

$$\Delta^2 = \frac{1}{4}b^2c^2 \sin^2 A = \frac{1}{4}b^2c^2(1-\cos^2 A)$$

$$= \frac{1}{4}b^2c^2\left(1 - \left(\frac{b^2+c^2-a^2}{2bc}\right)^2\right)$$

$$= \frac{1}{16}(4b^2c^2-(b^2+c^2-a^2)^2)$$

which easily reduces to the right-hand side of equation (*), above. On the other hand, you can derive Hero's formula directly as follows:

$$\Delta^2 = \frac{1}{16}((2bc)^2-(b^2+c^2-a^2)^2)$$

$$= \frac{1}{16}(2bc+b^2+c^2-a^2)(2bc-(b^2+c^2-a^2))$$

$$= \frac{1}{16}((b+c)^2-a^2)(a^2-(b-c)^2)$$

$$= \frac{1}{16}(b+c+a)(b+c-a)(a-(b-c))(a+(b-c))$$

$$= \frac{1}{16}(a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

$$= \frac{1}{16}2s(2s-2a)(2s-2b)(2s-2c)$$

i.e. $$\Delta^2 = s(s-a)(s-b)(s-c),$$

which is Hero's formula.

* * * * *

QUATERNIONS AND THE PRODUCTS OF TWO VECTORS

J.A. Deakin
Goulburn Valley College of TAFE

"Quaternions" were originally introduced into mathematics as a generalization of the concept of a complex number by the Irish mathematician Sir William Hamilton. We define a quaternion Q to be the 'hypercomplex' number

$$Q = a + bi + cj + dk$$

where a, b, c, d are real numbers, and i, j, k are quantities having the properties

$$\left.\begin{array}{l} i^2 = j^2 = k^2 = -1 \\ ij = k,\ jk = i,\ ki = j \\ ji = -k,\ kj = -i,\ ik = -j \end{array}\right\} \quad (A)$$

Geometrically we could interpret the quaternion Q as being a "4-dimensional" vector, with real component a and quaternion component $bi + cj + dk$. The norm, or length, of Q is

$$|Q| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

and the conjugate of Q is the quaternion

$$\overline{Q} = a - bi - cj - dk.$$

Also $Q\overline{Q} = \overline{Q}Q = a^2 + b^2 + c^2 + d^2$.

If we consider the 'pure' quaternions, i.e. quaternions with no real components

$$Q_1 = a_1 i + b_1 j + c_1 k$$

$$Q_2 = a_2 i + b_2 j + c_2 k$$

and find the product, we obtain

$$Q_1Q_2 = (a_1 i + b_1 j + c_1 k)(a_2 i + b_2 j + c_2 k)$$

$$= a_1a_2i^2 + a_1b_2ij + a_1c_2ik$$

$$+ b_1a_2ji + b_1b_2j^2 + b_1c_2jk$$

$$+ c_1a_2ki + c_1b_2kj + c_1c_2k^2$$

$$= -(a_1a_2 + b_1b_2 + c_1c_2)$$

$$+ (b_1c_2 - b_2c_1)i - (a_1c_2 - a_2c_1)j + (a_1b_2 - a_2b_1)k$$

$$= -(a_1a_2 + b_1b_2 + c_1c_2) + \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

Let $\underset{\sim}{u}, \underset{\sim}{v}$ be the vectors

$$\underset{\sim}{u} = a_1\underset{\sim}{i} + b_1\underset{\sim}{j} + c_1\underset{\sim}{k}$$

$$\underset{\sim}{v} = a_2\underset{\sim}{i} + b_2\underset{\sim}{j} + c_2\underset{\sim}{k}.$$

Then the scalar product

$$\underset{\sim}{u}\circ\underset{\sim}{v} = a_1a_2 + b_1b_2 + c_1c_2$$

is *minus* the real part of the quaternion product Q_1Q_2 and the vector product

$$\underset{\sim}{u} \times \underset{\sim}{v} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

is the quaternion part of the quaternion product Q_1Q_2.

Hamilton was impressed with the possibilities for the application of quaternions to physical problems, and indeed was so enthusiastic that one Sunday when walking with his wife (16 October 1843), he carved the fundamental formulae (A) of his quaternion algebra in the stone of a bridge in Dublin, which can still* be seen today[1].

Nevertheless, the use of quaternions as such was relatively short-lived, and it was the genius of the American mathematician/physicist/chemist, J. Willard Gibbs, who identified the real and quaternion parts of the product of two 'pure' quaternions with what we now call the scalar and vector product of two vectors.

* Ed.: I am told that it is no longer visible; weather and time have taken their toll.

The algebra of vectors was developed in the form we use today by Gibbs, and it is of interest that the new vector algebra and its application to elementary geometry and mechanics was popularized by the Australian mathematician, Weatherburn, then Professor of Mathematics at the University of Western Australia, in his two standard textbooks[(2),(3)] – one case of an important topic of the upper secondary school curriculum to which an Australian mathematician made real contributions.

References

(1) E.T. Bell, *Men of Mathematics*, New York, Simon & Schuster, 1986, p.360.

(2) C.E. Weatherburn, *Elementary Vector Analysis*, London, G. Bell & Sons (various editions).

(3) C.E. Weatherburn, *Advanced Vector Analysis*, London, G. Bell & Sons (various editions).

* * * * *

Why Mathematics Works

Everyone knows that if you want to do physics or engineering, you had better be good at mathematics. More and more people are finding out that if you want to work in certain areas of economics or biology, you had better brush up on your mathematics. Mathematics has penetrated sociology, psychology, medicine, and linguistics. Under the name of cliometry, it has been infiltrating the field of history, much to the shock of old-timers. Why is this so? What gives mathematics its power? What makes it work?

One very popular answer has been that God is a Mathematician. If, like Laplace, you don't think that deity is a necessary hypothesis, you can put it this way: the universe expresses itself naturally in the language of mathematics. The force of gravity diminishes as the second power of the distance; the planets go around the sun in ellipses; light travels in a straight line, or so it was thought before Einstein. Mathematics, in this view, has evolved precisely as a symbolic counterpart of the universe. It is no wonder, then, that mathematics works; that is exactly its reason for existence. The universe has imposed mathematics upon humanity.

Philip J. Davis and Reuben Hersh: *The Mathematical Experience*,
p. 68, Birkhäuser, 1981, Boston.

A SPY STORY

A. Liu, University of Alberta, Canada

A message was to be sent to agent Double-O Zero. Just as a courier was about to be dispatched, another agent staggered into headquarters. He had been fatally wounded, but managed to reveal that there were two enemy agents in the courier pool. Unfortunately he died before naming them.

"This has serious repercussions", lamented the Chief, "but at the moment, we must get this message to Double-O Zero. What can we do?"

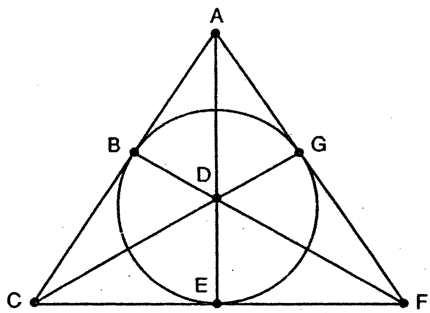
"We can send three copies of the message by three couriers", offered an aide. "This way, Double-O Zero is bound to get one."

"That will not do", said the Chief. "It is essential that the message does not fall into enemy hands."

"We can break up the message into several parts", another aide suggested, "in such a way that it is unintelligible unless all parts are available. We make three copies of each part, and send several couriers. Each courier carries several parts, and no duplicate copies."

"That sounds all right", said the Chief, "but we must still make sure that no two couriers carry all the parts between them. Can someone come up with a practical scheme?"

The aides conferred for a while. Then one of them recalled a diagram she had seen in a mathematics book. She drew a copy for the assembly.



"This diagram has seven points and seven lines", she pointed out.

"I assume", remarked another aide, "that you consider the circle passing through *B*, *E* and *G* as a line. I beg your pardon. I use the term 'line' very loosely here. Anyway, each line passes through three points and each point lies on three lines."

"What is the relevance of all this to our current problem?" the Chief wanted to know.

"We can break up the message into seven parts, each represented by one of the lines. We can dispatch seven couriers, each represented by a point. A courier carries a copy of a part of the message if and only if the point representing the courier lies on the line representing the point."

"Very good", the Chief nodded with approval. "Since each part is carried by three couriers, Double-O Zero will get all the parts. Since each courier carries three parts, no two couriers carry all the parts between them. Implement this scheme immediately!"

Exercises

1. (a) If seven couriers are to be dispatched, show that we can break up the message into only five parts.

   (b) Prove that it cannot be done with four parts.

2. (a) If the message is to be broken up into seven parts, show that we can dispatch only six couriers.

   (b) Prove that it cannot be done with five couriers.

3. (a) What is the minimum number of parts into which the message must be broken up?

   (b) What is the minimum number of couriers required in this case?

4. (a) What is the minimum number of couriers that must be dispatched?

   (b) What is the minimum number of parts required in this case?

* * * * *

Astrology

The role of astrology in the development of mathematics, physics, technology, and medicine has been both misrepresented and downplayed; contemporary scholarship has been restoring proper perspective to this activity. We are dealing here with a prescience and a failed science. It can be called a false or a pseudoscience only insofar as it is practiced with conscious deception.

Philip J. Davis and Reuben Hersh: *The Mathematical Experience*, p. 101, Birkhäuser, 1981, Boston.

Fifteenth Century Mathematics Teaching - A Comment on Leonardo da Vinci's Training in Mathematics

Boys [like Leonardo da Vinci] from the intermediate stream – the "semilearned", we might call it – attended "abacus schools" from the age of seven to twelve, more or less. Their instruction in arithmetic was in spirit much like teaching of mathematics today in American university classes for physicists, chemists, engineers, biologists, economists, pre-medical students, and computer scientists: recipes to be got by heart, innumerable routine examples differing from each other in principle not a whit, just mindless drill, as monotonous as an assembly line, with no justification even of the roughest sort, not a trace of proof or logical criticism, no hint that to think might serve any purpose. As the word "mathematics" derives from a Greek verb meaning "to ascertain, understand, comprehend", these schools' approach to mathematics is the embodiment of the antimathematical.

From "The Page-Barbour Lectures for 1985", by Clifford Truesdell: *Experimental Science without Experiments: Leonardo da Vinci*, page 5, published by the University of Virginia Press, 1987.

* * * * *

PROBLEMS AND SOLUTIONS

Problem 13.4.1 Show that, on a 4 by 4 chessboard, a knight cannot start at any square and then visit once only, in turn, each other square of the board.

Problem 13.4.2 If you watch a game of snooker on video, using the Fast Forward facility, not only is the action greatly speeded up, but the balls come to rest with astonishing rapidity. Why?

PERDIX

The 30th International Mathematical Olympiad took place from July 13th to July 24th in Braunschweig (Brunswick), the birthplace of Gauss, West Germany. Fifty countries competed, slightly fewer than in 1988. China came first with 237 points, Rumania second with 223 points, USSR third with 217 points, East Germany fourth with 216 points, USA fifth with 207 points. The maximum possible points is 252 (at most six in each team, and 42 points the highest possible score for each contestant). Last year, 1988, China, Rumania and the USSR also held the first three places, but in the order USSR, first, with Rumania and China equal second. Australia was 22nd this year (17th in 1988), with 119 points.

Congratulations to all members of the Australian team. The team was slightly changed from that announced in Part 2 (p. 64) of this year's *Function*: Danny Calegari withdrew at the last moment and his place was taken by the reserve Christopher Eckett. Mark Kisin and Alan Offer both obtained silver medals and Kevin Davey and Brian Weatherson bronze medals. Our individual congratulations to Mark, Alan, Kevin and Brian for an excellent result.

Here are the competition questions. What do you think of them? Send me your solutions.

XXX. INTERNATIONALE MATHEMATIK-OLYMPIADE

13.-24. Juli 1989

Bundesrepublik Deutschland
Braunschweig·Niedersachsen

1989
XXX
C. F. Gauss 1777-1855
INTERNATIONALE
MATHEMATIK
OLYMPIADE

English version

FIRST DAY

Braunschweig, July 18th 1989

1. Prove that the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets A_i $(i = 1, 2, \ldots, 117)$ such that

   (i) each A_i contains 17 elements;

   (ii) the sum of all the elements in each A_i is the same.

2. In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly.

   Prove that

(i) the area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$;

(ii) the area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

3. Let n and k be positive integers and let S be a set of n points in the plane such that

(i) no three points of S are collinear, and

(ii) for every point P of S there are at least k points of S equidistant from P.

Prove that

$$k < 1/2 + \sqrt{2n}.$$

Time: 4.5 hours
Each problem is worth 7 points.

SECOND DAY

Braunschweig, July 19th 1989

4. Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$.
There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$.
Show that

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}} .$$

5. Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

6. A permutation $(x_1, x_2, \ldots, x_{2n})$ of the set $\{1, 2, \ldots, 2n\}$, where n is a positive integer, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$.
Show that, for each n, there are more permutations with property P than without.

Time: 4.5 hours
Each problem is worth 7 points.

* * * * *

I received the following letter from Mr John Barton.

North Carlton
16 July 1989

Dear Perdix,

Herewith one trigonometric solution of Q. 4, p. 96 (vol. 13, No. 3). There are several others, of similar style, in which Ceva's theorem is not explicitly used. The frustrating thing, however, is that a 'pure euclidean' demonstration eludes me. In principle, I suppose, every trigonometric solution has its corresponding pure geometrical parallel solution, but this trigonometrical solution, and the other similar ones, are hardly to be described as simple.

Question 4, *FUNCTION*, Vol. 13, Part 3 (June 1989), p. 96

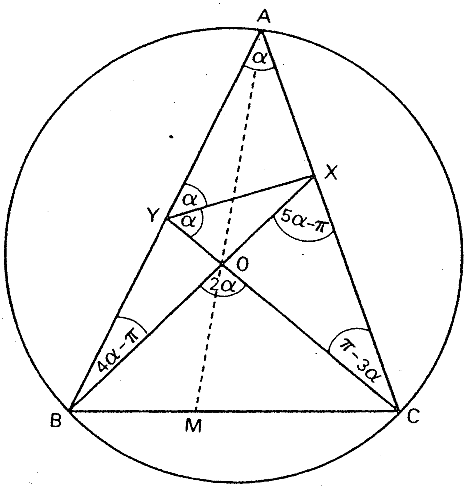
Draw the line AOM, as shown. It is easy to find the angles, as marked, in terms of $\angle BAC$, denoted by α. If $BC = a$, the radius R of the circum-circle is $a/(2 \sin \alpha)$. We have, by Ceva's theorem,

$$\frac{AY}{YB} \cdot \frac{BM}{MC} \cdot \frac{CX}{XA} = 1.$$

$$\frac{2XY \cos \alpha}{a/(2\sin \alpha)} \cdot \frac{BM}{MC} \cdot \frac{CX}{XY} = 1,$$

since triangles YBO and AYX are isosceles, so that $YB = OB = R$ and $XA = XY$. Hence

$$\frac{4\sin \alpha \cos \alpha}{a} \cdot \frac{BM}{MC} \cdot CX = 1.$$



By the sine rule for $\Delta\ COX$,

$$\frac{cx}{\sin(\pi-2\alpha)} = \frac{a/(2 \sin \alpha)}{\sin(5\alpha-\pi)}, \quad \text{so that}$$

$$CX = -a \sin 2\alpha/(2 \sin \alpha \sin 5\alpha).$$

Hence $\dfrac{BM}{MC} = -\dfrac{2 \sin \alpha \sin 5\alpha}{\sin 2\alpha} \dfrac{1}{4 \sin \alpha \cos \alpha} = -\dfrac{\sin 5\alpha}{2 \sin 2\alpha \cos \alpha}$.

But $\dfrac{BM}{MC} = \dfrac{\sin \angle BOM}{\sin \angle MOC} = \dfrac{\sin 2(4\alpha-\pi)}{\sin 2(\pi-3\alpha)} = \dfrac{\sin 8\alpha}{-\sin 6\alpha}$. Hence

$$\sin 5\alpha \sin 6\alpha = 2 \sin 2\alpha \cos \alpha \sin 8\alpha$$

$$\cos \alpha - \cos 11\alpha = 2 \cos \alpha(\cos 6\alpha - \cos 10\alpha)$$

$$= \cos 7\alpha + \cos 5\alpha - \cos 11\alpha - \cos 9\alpha,$$

so that

$$\cos\alpha - \cos 7\alpha = \cos 5\alpha - \cos 9\alpha$$

$$\sin 4\alpha \sin 3\alpha = \sin 7\alpha \sin 2\alpha,$$

whence, cancelling $\sin 2\alpha$,

$$2\cos 2\alpha \sin 3\alpha = \sin 7\alpha$$

$$\sin 5\alpha + \sin\alpha = \sin 5\alpha(1-2\sin^2\alpha) + \cos 5\alpha \sin 2\alpha.$$

Cancel $\sin 5\alpha$, and divide through by $\sin\alpha$ to get

$$1 = 2(\cos 5\alpha \cos\alpha - \sin 5\alpha \sin\alpha)$$

$$= 2\cos 6\alpha.$$

Hence $6\alpha = \pm\frac{\pi}{3} + m.2\pi$ (m an integer).

The first few positive solutions are 10°, 50°, 70°, 110°, 130°. If $\alpha = 10^\circ$ and we take O between Y and C, then $AO > CO$ which contradicts data. If $\alpha = 70^\circ$, X is not between A and C, which contradicts data. Hence $\alpha = 50^\circ$.

There surely must be some relatively simple construction to allow a "pure euclidean" solution? Note that the points B, Y, O, M are concyclic, but how can one show this without first solving the problem?

** Can anyone solve Mr Barton's problem: can you find a solution not
** involving the trigonometric complications he has used?

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