Function

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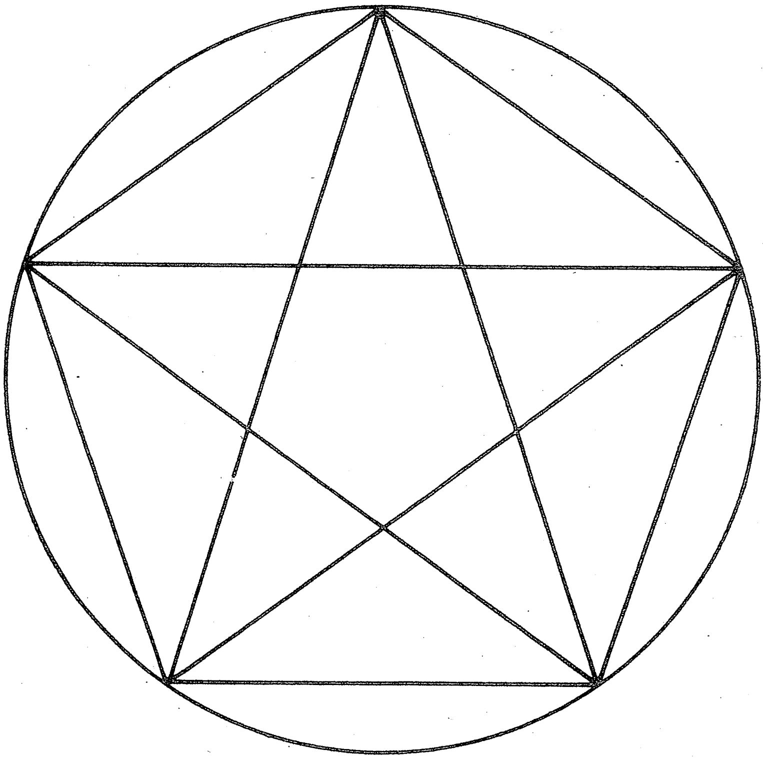

A SCHOOL MATHEMATICS MAGAZINE

FUNCTION is a mathcmatics magazine addressed principally to students in the upper forms of schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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The front cover exhibits a regular pentagon, and its diagonals, inscribed in a circle. The beauty of the regular pentagon, often remarked upon by artists, architects, and mathematicians, partly stems from the fact that the ratio of the length of any diagonal to any side is the golden section number (see *The Front Cover*, and *Fun with Fibonacci*, in this issue). A rectangle, the lengths of whose sides have this ratio, has almost universally been regarded as having the ideal proportions.

We have reproduced in this issue, under the heading TEN YEARS AGO, an article on *Black Holes*, by Colin McIntosh, from *Function* of June 1979. Research on this subject continues to be energetically pursued and Colin's article is as relevant today as it was in 1979.

CONTENTS

THE FRONT COVER

The regular pentagon is one of the figures that exhibits the golden section number $\tau = \frac{\sqrt{5}+1}{2}$, the positive root of the equation $x^2 - x - 1 = 0$. (See *Fun with Fibonacci*, by Richard Whitaker, pp. 73-79, for further situations that involve the golden number.)

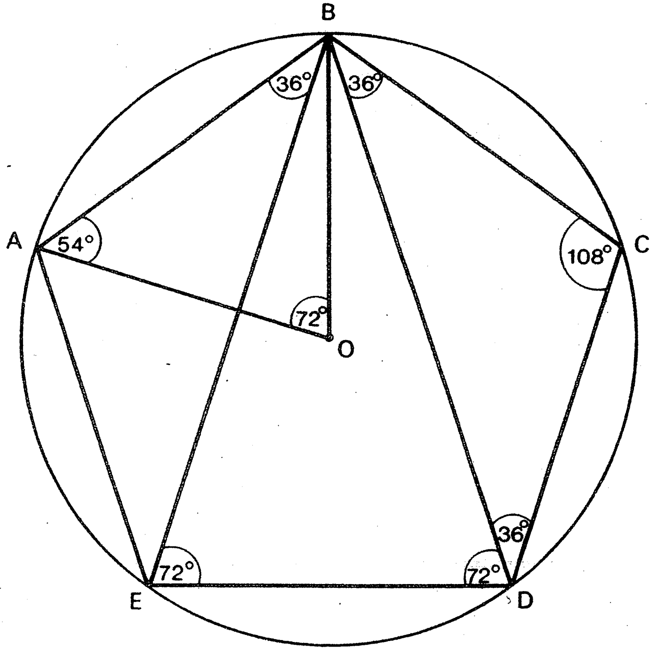

Figure 1

First let us calculate some angles associated with the regular pentagon. Each side of the pentagon subtends an angle of $\frac{2\pi}{5}$ radians $= \frac{360}{5}$ degrees $= 72^\circ$ at its centre O, and consequently subtends 36° at its opposite vertex: in Fig. 1, $\angle ABO = 72^\circ$ and $\angle EBD = 36^\circ$. Thus each of the base angles of isosceles triangle ABO is 54° and the angles of the pentagon are each $(2\times54)^\circ = 108^\circ$.

The isosceles triangle determined by any two adjacent sides thus has base angles of 36° : in Fig. 2 triangle ABC has angles of 36°, 108° and 36°. Let the diagonals AC and BE meet at K (Fig. 2). Then KC is parallel to ED and KE is parallel to CD, so $KCDE$ is a parallelogram. Hence KC is of the same length as ED, a side of the pentagon.

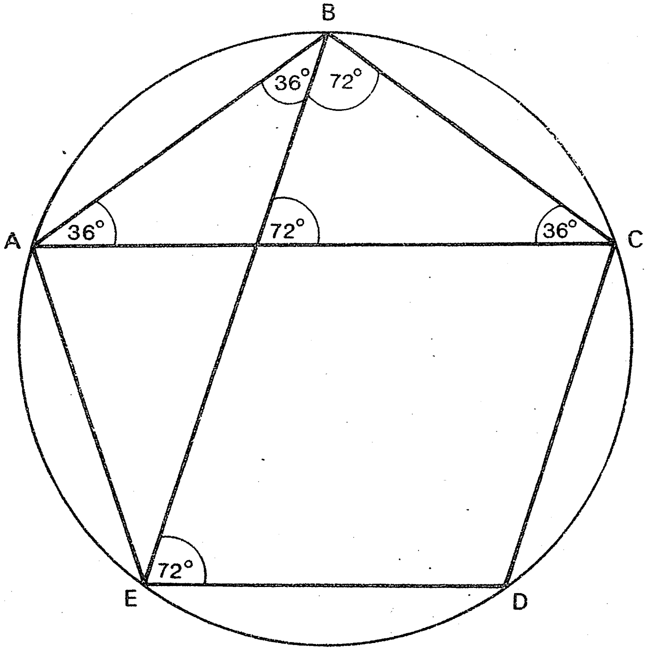

Figure 2.

Suppose we take the length of a side of the pentagon to be 1 and each diagonal to be of length τ. Then $AK = \tau-1 = KB$, and since triangles ABC and AKB are similar we have

$$AK/AB = AB/AC$$

i.e.
$$\frac{\tau-1}{1} = \frac{1}{\tau},$$

i.e.
$$\tau^2 - \tau - 1 = 0,$$

so that $\tau = \frac{\sqrt{5}+1}{2}$, the golden number.

We can also read off from our figures the sines and cosines of various angles. In particular, from Fig. 1, we have

$$\cos 72^\circ = \tfrac{1}{2}ED/BE = \tfrac{1}{2}/\tau = \frac{1}{2\tau}$$

$$= \frac{1}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{4},$$

and

$$\cos 36^\circ = \tfrac{1}{2}BD/CD = \tfrac{1}{2}\tau/1 = \tfrac{1}{2}\tau$$

$$= \frac{\sqrt{5}+1}{4} .$$

How do we construct regular pentagons – using straight edge and compass only? Here are two problems: (a) construct the pentagon when a side is given; (b) construct the pentagon when a circle in which it is to be inscribed is given.

Again, given a segment of length 1, can we construct another segment whose length is τ, the golden number?

Send your favourite methods in to the Editors and await the next issue of *Function*.

* * * * *

Problem 13.3.1, composed by Marta Sved

The importance of being earnest

Tom, Dick, Harry are three brothers,
Hearty, hale and youthful,
And each of them is always lying,
Or – is always truthful.

"Most of them are truthful, though",
Claimed their doting mother.
So I went to ask the lads
To tell about each other.

Then Tom declared that Dick denied
That Harry always lied.
"Tom tells a lie, I tell you so",
Brother Harry cried.

I am confused, I must confess,
And now I turn to you.
Can you tell me who was lying
And who is always true?

* * * * *

"Those who have an excessive faith in their ideas are not well fitted to make discoveries."

Claude Bernard, as quoted with approval by Jacques Hadamard in "An essay on the psychology of invention in the mathematical field", Princeton University Press, 1945.

Ten years ago

BLACK HOLES

C.B.G. McIntosh, Monash University

Black holes are objects which exert gravitational influences but cannot be seen because they do not emit light signals. The first suggestion that such objects exist was made by the French mathematician Pierre Simon de Laplace in 1798. Laplace said: "A luminous star, of the same density as the Earth, and whose diameter should be two-hundred and fifty times larger than that of the Sun, would not, in consequence of its attraction, allow any of its rays to arrive at us; it is therefore possible that the largest luminous bodies in the universe may, through this cause, be invisible."

Laplace's prediction was made on the basis of Newton's theory of gravitation; he suggests that the gravitational field of the concentrated object is so strong that light emitted by that object would not have sufficient energy to escape the surface of that object. Thus the object would be invisible to an external observer.

It is Albert Einstein's gravitational theory, the general theory of relativity, dating from 1915, that fully predicts and describes black holes; thus it is in terms of this theory that black holes are discussed in this article.

General relativity describes gravitation in terms of the geometry of 4-dimensional spacetime and Einstein gave a set of equations, the "Einstein field equations" of general relativity, which describes the geometry for the spacetime. These field equations have the form

$$\begin{bmatrix}\text{Function of the geometry}\\ \text{of a given region of}\\ \text{spacetime}\end{bmatrix} = \begin{bmatrix}\text{Function of the matter-}\\ \text{energy content of that}\\ \text{region of spacetime}\end{bmatrix}$$

A solution of these equations in a given region thus gives a description of the geometry of that region for a given type of matter-energy content. One of the first solutions of these equations is one given by Karl Schwarzschild in 1916. This describes the geometry of the region of spacetime in vacuum (i.e. the matter-energy content of the region is zero) surrounding a spherical star. Even though this solution has been known for such a long time, it was not until the 1960's that many of its basic physical properties were understood; indeed, it was not until the 1960's with the discovery of quasars, cosmic background radiation and neutron stars that much work went into understanding general relativity and its implications for astrophysics and other areas of physics.

In 1963, Schwarzschild's solution of Einstein's equations was generalized by Roy Kerr (a New Zealand mathematician) to include rotation; the geometry is no longer spherical but it is symmetric about the axis of rotation. This was generalized in 1965 by Ezra Newman and co-workers to account for the possibility of the star having a net electric charge. It can be shown, under reasonable assumptions, that this solution with three parameters, M, a and e (for mass, angular momentum and electric charge) is the most general solution of Einstein's equations which has the properties of a black hole.

Many questions now arise: In what ways do these solutions represent black holes? How big are black holes? What other properties do they have? How do they form? How can a black hole be detected? Has one been detected?

Mathematically, then, a black hole is a certain type of solution of Einstein's field equations. Physically, a large black hole is formed when an object such as a star has undergone complete gravitational collapse. Small black holes may have been formed in the big bang at the beginning of the Universe. No light can be emitted by a black hole. No matter can be ejected. Anything that falls into the hole loses its identity.

Consider a black hole of mass M but without angular momentum and electric charge (this is just Schwarzschild's solution). Surrounding this black hole there is a spherical surface of radius $r = 2MG/c^2$ (where G is the gravitational constant and c is the speed of light); this radius is known as the Schwarzschild radius. The formula shows that a black hole of the mass of the sun would be about four miles across! The surface at this radius is known as the *event horizon*. The word *horizon* is used because objects such as light rays, radio signals, rocket ships or other stars can cross the surface from the outside to the inside but nothing can cross in the other direction. If you are watching a spaceship go towards the black hole, you will see it disappear from sight; you will never see it again! It goes over the horizon! There is no way in which it could turn round and escape. The people in the spaceship cannot tell you what it is like inside the event horizon because their radio or other signals cannot cross back over the horizon. The spaceship and its occupants would, however, be acted upon by the extremely strong gravitational field from the black hole. They would be torn apart by the force from this field in the direction of their motion and then the constituent parts would eventually be crushed by the extremely large forces near the centre of the black hole. Thus the occupants could not examine the black hole for very long.

Black holes are thus invisible to someone looking through a telescope. Not only are they black, but their small size means that their resulting angular diameter in the sky would be of the order of magnitude of a million millionth of a second of arc.

The only qualities of a black hole that can be measured are its mass, its angular momentum and its electric charge. We cannot even ask, in a meaningful way, from what elements it is made. Two different stars of equal mass, but of differing composition, can form identical black holes.

The physics of a black hole to an observer thus depends on where he or she is. An observer who chooses to follow matter through the horizon will see it crushed to indefinitely high density; but will also be crushed by indefinitely high forces. This crushing will take place at a finite time (as measured by that observer) after the matter (or observer) has crossed the horizon and is inevitable. The observer has no more power to return to a larger r value (outside the black hole r is the radial distance from the centre of the hole) than we have the power to turn back the hands of the clock of life. The tidal gravitational forces experienced by an object or observer resulting from the black hole are proportional to $1/r^3$. (Tides, even on earth, result from forces of this character.) For a black hole of one solar mass, the force at the event horizon (or Schwarzschild's radius $r = 2MG/c^2$) is about a thousand million times the force due to gravitational acceleration on the surface of the Earth. An observer who was near a black hole of this size would be stretched lengthwise in the direction of motion, and crushed sideways by such tidal forces well before he or she reached the event horizon. For a massive black hole (many orders of magnitude greater than the mass of the sun) an observer can cross the event horizon and experience very little force – but can never escape! Inside a black hole of about ten million times the mass of the sun, the observer could last about a day as measured by an atomic clock he or she might be carrying. But then crushing is inevitable. This crushing means that objects and then the atoms that once formed those objects would be ripped apart by the tidal forces.

On the other hand, an observer who stays a long way from the black hole to watch an object falling through the horizon does not actually see it cross the horizon; he or she measures that it would take an infinite time to do so! However, the object does almost suddenly disappear from sight (and from other means of contact) after a finite time as the light from the object is red-shifted enormously and can no longer be seen by the distant observer. (This means that the wavelength of the light emitted becomes progressively longer, and therefore the light appears redder, until the wavelength is so long that the light cannot be seen. This process can take place in extremely short times.)

Black holes more massive than the sun are formed from the gravitational collapse of large stars. When the sun will have used up a lot of its hydrogen in thermonuclear reactions, present theory suggests that it will expand into a red giant and later, after using up more fuel, will contract into a white dwarf of about one hundredth of its present radius. A larger star, say one of twice the solar mass, will probably eventually explode as a supernova and its core will collapse into a neutron star. A neutron star of mass equal to that of a solar mass would have a radius of about one seventy thousandth of the sun's radius!

However, there is no stable equilibrium state for stars of more than about three solar masses. So after gravitational collapse it is expected that such a star will collapse into a black hole. The mass of such black holes may increase by the accretion of other material, by the swallowing up of other stars, or after the collision with another black hole; but it may never decrease.

Small black holes may, however, have been formed in the big bang at the beginning of the Universe. A black hole of about the mass of a fair-sized mountain would have a radius of about 10^{-13}cm! Such black holes cannot be created today, as there is no way that there could be the necessary forces to compress material to form such an object. Stephen Hawking in 1975 showed that such black holes will radiate like a black body due to quantum mechanical processes and that the smallest black holes would have radiated away by now. However, this is too complicated a story to go into here. Large black holes also radiate but at such a slow rate that they virtually seem not to radiate at all.

And detection? Black holes cannot be detected in isolation; but one hope is that one can be found as a partner of a "live" star in a binary system in which the hole and the star rotate round each other. There is good reason to believe that the X-rays from a source known as Cygnus *X*-1 result from material being torn apart just before it plunges into a black hole, an unseen companion of the massive star HDE 226868. The probable black hole has a mass of about four solar masses; big enough for a collapsed star of this mass to have formed a black hole. There is also evidence that there is a black hole at the centre of the galaxy M87.

The theory of black holes is thus an extremely interesting one. Some ideas of further properties can be found by reading some or all of the following references. More properties remain to be discovered. We must also wait for news that one or more black holes have been discovered. Theory predicts them; there is a very good chance that they exist - but don't go and visit one!

References for Easy Reading

Roger Penrose. "Black holes". *Scientific American*, May 1972, *226*, 38.

Kip S. Thorne. "The search for black holes". *Scientific American*, December 1974, *231*, 32.

Stephen W. Hawking. "The quantum mechanics of black holes". *Scientific American*, January 1977, *236*, 34.

Also Recommended

Reno Ruffini and John A. Wheeler. "Introducing the black hole". *Physics Today*, January 1971, 30.

Charles W. Misher, Kip S. Thorne and John A. Wheeler. "*Gravitation*". W.H. Freeman and Co., San Francisco, 1973.

Function, June 1979

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FUN WITH FIBONACCI

Richard Whitaker,
Bureau of Meteorology, NSW Regional Office

Leonardo Fibonacci (*circa* 1170-1230) was an Italian mathematician who lived in the famous city of Pisa, and is generally regarded today as one of the outstanding mathematicians of medieval times.

He published several important works, perhaps the best known being *LIBER ABACI* (1202) in which the famous 'Fibonacci sequence' was postulated, in order to explain the breeding pattern of a family of rabbits.†

This sequence of numbers, 1, 1, 2, 3, 5, 8, 13, 21, ..., can be obtained by beginning with 1 and 1, and then adding progressively, so that each term is the sum of the two previous terms.

We write $f_1 = 1,\ f_2 = 1,\ f_3 = 2, \ldots,$ where $f_{n+1} = f_n + f_{n-1}.$ (1)

This deceptively simple sequence does not, at first glance, appear to be much more than a mathematical triviality, but nothing could be further from the truth. In fact, the hidden depths and intricacies surrounding the Fibonacci sequence [or F-sequence] have intrigued and fascinated mathematicians of all persuasions down the centuries.

Appendix 1 is a simple program in BASIC for calculating the Fibonacci sequence. This is suitable for use on such computers as the Commodore 64.

The Fibonacci sequence crops up in unusual places in nature as a type of biological growth pattern; as well as the rabbit population discussed by Fibonacci himself, the F-sequence can be found in the branching habits of trees and the arrangements of leaves on various plants. (There is quite a fascinating literature available which discusses the F-sequence in nature. See e.g. references (1), N.N. Vorobev, *Fibonacci numbers*, Pergamon Press and (2), *Fibonacci Quarterly (Journal of Fibonacci Society)*). Perhaps the most interesting aspect of the F-sequence is revealed when the ratio of successive terms is calculated (larger over smaller) to produce a new sequence:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots, \frac{f_{n+1}}{f_n}, \ldots .$$

Experimentation with a calculator, or by using the program in Appendix 2, will illustrate that this new sequence tends to a limit of approximately 1.6180339 ... as we move up the F-sequence.

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† Fibonacci assumed that his rabbits lived for ever. Initially he had one pair of rabbits who produced their first pair of offspring after 2 months, and thereafter a pair each month. Each pair consisted of a male and female, and followed the same pattern of reproduction.

The way in which this limit is reached is interesting in its own right, as the approach is made by successively "overshooting" then "undershooting" the limit, each time by a progressively smaller amount.

It is a fairly straightforward matter to prove (and a proof is offered below) that this number (we'll call it τ) is equal to $\frac{1+\sqrt{5}}{2}$, which to nine decimal places is 1.618033989.

This number τ is the famous "Golden Section", "Golden Number" or "Divine Proportion", which in the past was accorded almost holy status by various branches of early mathematical philosophies.

It was Euclid who posed the question: If we have a rectangle of long side x and short side 1, can we cut from it a square of side 1 such that the remaining rectangle has the same shape as the original?

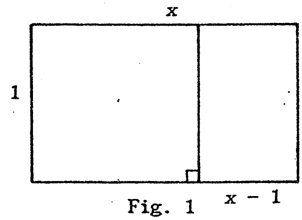

Fig. 1

The answer is yes, provided $\frac{x}{1} = \frac{1}{x-1}$, i.e. $x^2 - x - 1 = 0$, which has as its positive root $x = \tau = \frac{1+\sqrt{5}}{2}$.

Another geometrical problem which leads to τ is as follows:
We have a line AB. We are to find a point C on this line such that $AB/AC = AC/CB$.

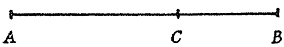

Fig. 2

It is readily shown that for this to be so, $AB/AC = \tau$. This division of the segment AB is said to be a *golden section* of AB.

A simple algebraic consideration shows the connexion between Fibonacci numbers and the golden ratio τ.

As seen above, the equation leading to τ is

$$x^2 - x - 1 = 0.$$

Its two roots τ and $\bar{\tau}$ satisfy

$$\tau + \bar{\tau} = 1$$
$$\tau\bar{\tau} = -1 \qquad (2)$$

[They are $\tau = \frac{1+\sqrt{5}}{2}$ and $\bar{\tau} = \frac{1-\sqrt{5}}{2}$. Notice that

$$1/\tau = \frac{2}{1+\sqrt{5}} = \frac{2(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{\sqrt{5}-1}{2} = -\bar{\tau},$$

in line with $\tau\bar{\tau} = -1$.]

Look now at the sequence defined by

$$u_n = \tau^n - \bar{\tau}^n$$

so that
$$u_1 = \tau - \bar{\tau} = \sqrt{5}$$
and
$$u_2 = \tau^2 - \bar{\tau}^2 = \frac{5+\sqrt{5}}{2} - \frac{5-\sqrt{5}}{2} = \sqrt{5} \qquad (3)$$

Consider the identity

$$u_{n+1} = \tau^{n+1} - \bar{\tau}^{n+1}$$
$$= (\tau+\bar{\tau})(\tau^n-\bar{\tau}^n) - \tau\bar{\tau}(\tau^{n-1}-\bar{\tau}^{n-1}).$$

From this, by using (2), we obtain that

$$u_{n+1} = u_n + u_{n-1}.$$

Comparing this with (1) and (3), it follows that

$$u_1 = \sqrt{5}f_1 \quad \text{and} \quad u_2 = \sqrt{5}f_2,$$

so
$$u_3 = u_1 + u_2 = \sqrt{5}(f_1+f_2) = \sqrt{5}f_3,$$

and generally,

$$f_n = u_n/\sqrt{5}.$$

Thus we have an explicit formula for Fibonacci numbers, in terms of the golden ratio τ.

$$f_n = (\tau^n - \bar{\tau}^n)/\sqrt{5}.$$

From this we can deduce the value of the limit of f_{n+1}/f_n as $n \to \infty$. For

$$\frac{f_{n+1}}{f_n} = \frac{\tau^{n+1}-\bar{\tau}^{n+1}}{\tau^n-\bar{\tau}^n} = \frac{\tau-(\bar{\tau}/\tau)^n}{1-(\bar{\tau}/\tau)^n}$$

and since $-1 < \bar{\tau}/\tau < 0$ (indeed $1/\tau = -\bar{\tau}$, so $\bar{\tau}/\tau = -\bar{\tau}^2$, and $-1 < \bar{\tau} = \frac{1-\sqrt{5}}{2} < 0$), $(\bar{\tau}/\tau)^n \to 0$ as $n \to \infty$, whence

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \tau .$$

In these days of the personal computer, it is always interesting to try and construct an iterative sequence to derive various numbers, and the same temptation exists with τ.

An iterative sequence

$$a_1, a_2, \ldots, a_n, \ldots$$

is a sequence, where each a_n is constructed as a fixed function of the previous term, i.e.

$$a_n = f(a_{n-1}).$$

A simple example is

$$a_n = a_{n-1}^2 \quad \text{where} \quad a_1 = 3.$$

Thus the sequence is

$$3,\ 9,\ 81,\ 6561,\ \ldots$$

which is a fast-growing sequence: "going to infinity". It is more interesting to consider *convergent* iterative sequences, where the terms tend to a *limit* as n goes to infinity, which means that a number A exists such that the difference between A and any of the terms a_n can be made *as small as we wish, provided that n is large enough.*

One well-known iterative sequence which generates τ is

$$\lim_{n\to\infty} \ldots \underbrace{\left(\left(\left(2^{1/2}+1\right)^{1/2}+1\right)^{1/2}+1\right)^{1/2} \ldots}_{n \text{ roots}}$$

This sequence was begun, or "seeded", with the number 2, that is, $a_1 = \sqrt{2}$, $a_2 = \sqrt{\sqrt{2}+1}$, and generally, $a_{n+1} = \sqrt{a_n+1}$; other numbers will do as well, but 2 will be used for the sake of this discussion.

To show that the limit of this sequence is τ, consider more generally the sequence

$$\ldots\left(\left(\left(2^{1/2}+a\right)^{1/2}+a\right)^{1/2}+a\right)^{1/2}\ldots$$

We have $a_1 = 2^{1/2}$, $a_2 = (a_1+a)^{1/2}, \ldots,$ $a_{n+1} = (a_n+a)^{1/2}, \ldots$. Suppose that $\lim_{n\to\infty} a_n = A$; then it is intuitively clear that

$$A = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}(a_n+a)^{1/2} = (A+a)^{1/2}.$$

Thus $A^2 = A + a$. In the case considered, in which $a = 1$, we get $A^2 - A - 1 = 0$, and so A, the positive square root of this equation, equals τ.

More generally, we obtain by considering in a similar manner the sequence generated by giving a_1, and for some fixed a, b, k having

$$a_{n+1} = (a+ba_n)^{1/k} \tag{5}$$

that

$$a + bA = A^k. \tag{6}$$

This is rather handy, because it ties in neatly with a special property of τ, namely:

$$1 + \tau = \tau^2;\ 1 + 2\tau = \tau^3;\ 2 + 3\tau = \tau^4;\ 3 + 5\tau = \tau^5;\ 5 + 8\tau = \tau^6 \ \ldots \text{ etc.}$$

Notice the F-sequence running through these equations! You may use induction to prove that $\tau^n = f_{n-1} + f_n\tau$.

Referring back to (1) and (2), we can now construct an infinite number of iterative sequences which generate τ, some examples of which are:

$$\lim_{n\to\infty} \ldots 2\left[2\left[2.2^{1/3}+1\right]^{1/3}+1\right]^{1/3}+1\Big]^{1/3} \ldots = \tau$$

(n operations)

$$\lim_{n\to\infty} \ldots 3\left[3\left[3\left[3.2^{1/4}+2\right]^{1/4}+2\right]^{1/4}+2\right]^{1/4} \ldots = \tau$$

$$\lim_{n\to\infty} \ldots 5\left[5\left[5\left[5.2^{1/5}+3\right]^{1/5}+3\right]^{1/5}+3\right]^{1/5} \ldots = \tau$$

or, if 'b' is the k^{th} term of the F-sequence, 'a' the $(k-1)^{\text{th}}$, then

$$\lim_{n\to\infty} \ldots b\left[b\left[b\left[b.2^{1/k}+a\right]^{1/k}+a\right]^{1/k}+a\right]^{1/k} \ldots = \tau.$$

Appendix 3 is a program for calculating such sequences.

You will find that the larger the values for a, b chosen, the more rapid the convergence to τ. For the student who wishes to ask "how rapid is more rapid?", the following line of argument may be of some assistance.

Consider the sequence

$$\lim_{n\to\infty}\left[g(\tau)\right]^{n}\left[\tau - \ldots b\left(b\left(b.2^{1/k}+a\right)^{1/k}+a\right)^{1/k}\ldots\right]$$

where $g(\tau)$ is some function of τ. Now the second bracket is, of course, *approaching zero at a certain rate*, and if we can choose $g(\tau)$ such that $(g(\tau))^n$ is *increasing at the same rate*, intuitively we can see that a finite, and in general non-zero number will be generated. Without going into detail, it can be shown that $g(\tau)$ must equal $\frac{k\tau^{k-1}}{b}$, and if we replace the "seeding value" of 2 with a variable x, then the following function may be defined.

$$I(x) = \lim_{n\to\infty}\left[\frac{k\tau^{k-1}}{b}\right]^{n}\left[\tau - \ldots b\left(b\left(b.x^{1/k}+a\right)^{1/k}+a\right)^{1/k}\ldots\right]$$

(n operations)

Appendix 4 is a program for calculating an example of a function of this type. [Unless you are using double precision, let the program run for only 7 or 8 iterations, because rounding error takes over soon after. This is because we are multiplying a number that is getting smaller and smaller (the second bracket term) with a number becoming larger and larger (the first bracket).]

If you can find the time to plot a family of $I(x)$ curves for different a, b values you may be pleasantly surprised at the symmetry produced.

And the final observation is that $\left(\frac{k\tau^{k-1}}{b}\right)^{n}$ may be thought of as a rate of convergence of $\ldots b\left(b\left(b.x^{1/k}+a\right)^{1/k}+a\right)^{1/k}\ldots$ to τ.

The Fibonacci sequence is a fine example of the principle that what may appear simple can be far more complicated than we first believe. Or, put another way, you can never judge a book by its cover.

APPENDICES

1.

```
10 N=1
20 A=0
30 B=1
40 C=A+B
50 PRINT N;B
60 A=B
70 B=C
80 N=N+1
90 GOTO 40
RUN
```

2.

```
10 N=1
20 A=0
30 B=1
40 C=A+B
50 PRINT N;C/B
60 A=B
70 B=C
80 N=N+1
90 GOTO 40
RUN
```

3.

```
10 N=1
20 X=2
30 Y=X↑(1/6)
40 PRINT N,Y
50 N=N+1

60 X=8*Y+5
70 GOTO 30
RUN
```

(This calculates the sequence $\ldots 8(8(8(8.2^{1/6}+5)^{1/6}+5)^{1/6})+5)^{1/6})\ldots)$

4.

```
10 N=1
20 X=2
30 A=1.618033989
40 Y=X↑(1/6)
50 PRINT N;(((6/8)*(A↑5))↑N)*
     (A-Y)
60 N=N+1
70 X=8*Y+5
80 GOTO 40
RUN
```

(This will generate the number 4.6269... . By taking different X values, a curve can be plotted)

* * * * *

THE MATHEMATICS OF FAMILY PLANNING INSURANCE

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At a recent UNESCO conference in Dhaka, Bangladesh, Mr Shafat A. Chaudhuri, an actuary with the Delta Life Insurance Company, presented his concept of Family Planning Insurance. One of Function's *editors (M.A.B. Deakin), who was also at the conference, sought and obtained Mr Chaudhuri's permission to publish an edited version in* Function. *Mr Chaudhuri greatly assisted in this process by supplying not only the text of his conference talk, but also a copy of a much longer paper he has written on the subject. He tells us that his scheme is reaching the final stage of preparation of a pilot project. We thank Mr Chaudhuri for allowing us to use this material in* Function *and wish his project the success it most certainly deserves.*

A family planning programme, aimed at curbing the rate of population growth, is currently receiving the Bangladesh government's most urgent attention. But despite all the efforts made and all the money spent, targets are not being met. This is frustrating, but not really unexpected in the circumstances. The problems are many and complex, and no-one would dare to prescribe an easy solution. But two problems can easily be identified, both relating to financial motivation.

First, the government field workers have no financial incentive to work hard, and second, the individual couples (the programme targets) have no inducement to limit family size. If these two deficiencies could be remedied, the success of the programme would surely be greatly enhanced. This paper describes an administrative framework within which this could occur.

Three suggestions are here put forward.

1. The field-worker's income (at least in part) should be linked to his success in achieving birth-preventions.

2. The target couples who succeed in achieving birth-preventions should have their old age secured.

3. The present programme should be restructured as an insurance scheme.

To elaborate these suggestions, consider first insurance on mortality. In this case, a claim arises in the event of mortality, i.e. death. But, on the other hand, there is pension life insurance in which a claim arises with the *non-event* of mortality, and payments are made as long as death does *not* occur. Similarly, there are policies which fall due, i.e. a claim arises, in the case of an event of fertility, i.e. a birth. What is here proposed is the opposite of this: a claim that arises with the *non-event* of fertility. A fertility-based family planning insurance works with probabilities of non-fertility, exactly as a pension works with probabilities of non-mortality. Claims would be paid to field-workers and target couples on the basis of the pattern of births among the insured couples.

In essence, what is proposed is a contract system. The government would opt to have its family planning programme carried out *via* insurance contracts. Such contracts would have two tiers. The upper tier would involve the government and the insurance company. The government would pay a premium to the insurance company and set a target of birth-reduction to be achieved for this premium. In the event of the company's failing to achieve the target, part of the premium would have to be refunded, the amount in question depending on the extent of the shortfall.

The lower tier of the contract would be between the company on the one hand and the field-workers and target couples on the other. The company would pay to the field-workers and the target couples a remuneration directly related to the birth-reductions achieved. In measuring "birth-reduction", note would be taken of both the number of births and "birth-deferment". Thus, a worker in the scheme would earn an income directly related to success, as measured by the lowering of births among the client couples involved. Similarly with each couple.

Important to the whole concept is the measurement of birth-prevention and birth-deferment. This latter is significant because it is found to have a significant effect on the overall population growth rate.

It is not, in fact, a difficult actuarial exercise to calculate for each couple a notional number of "birth-preventions" based on the number of previous births, the age of the mother, the length of the marriage and so on. This is what the family planning programme ultimately entails.

Such calculations involve mathematics, which, although readily understood by those with adequate training, is unfortunately too complicated to be followed by most of those concerned with implementing the programme.

Thus, although it would be ideal to have the scheme based on actual births prevented (a BP-based scheme we call this), it is unrealistic to expect to be able to calculate this, and we have had to opt for a simpler scheme involving what we term *claim points*, giving rise to a CP-based scheme in the technical terminology.

Claim points are awarded to each target couple on the basis of their success in postponing or preventing births as compared with the population average. The aim is to derive a simple but reasonably fair and actuarially sound formula which allows the ready calculation of the number of claim-points, i.e. the value of CP.

There are many possible ways in which this might be done. Here is one of them.

Multiply:

Months after marriage but before the birth of the first child	by 6
Months subsequent to the birth of the first child but before the birth of the second	by 5
Months subsequent to the birth of the second child but before the birth of the third	by 3

and then add.

This formula would give CP in the case of a couple with 0, 1 or 2 children. If three or more children are born to a couple, then a separate table of deductions would apply.

On this basis, tables may be constructed from which it is possible to read off the number of claim points from the history of fertility for each target couple. The level of remuneration for each worker and each couple would then be fixed by putting a monetary value on each claim point.

Whatever method of determining claims is employed (whether it use BP or CP, or CP as measured by some other formula), a mathematical model will be required to put the scheme on a sound theoretical basis – just as is the case with the different forms of life insurance.[†]

Consider that case as background. If

$\ell(x)$ is the number of living people x years old,
$q(x)$ is the probability that a person aged x years dies before attaining the age of $(x+1)$ years,
$d(x)$ is the number of deaths of persons aged x years,

[†] For more on life insurance, see *Function, Vol. 5, Part 2.*

then

$$\ell(x)q(x) = d(x). \tag{1}$$

Furthermore,

$$\ell(x+1) = \ell(x) - d(x). \tag{2}$$

The probability that a pension will be payable to a person, now aged x years, t years in the future is $\ell(x+t)/\ell(x)$. To pay for that pension we invest the premiums at a rate $R\%$ (say) of interest. For each dollar we need in the future we invest v^t dollars now, where

$$v = 1/(1+R/100). \tag{3}$$

Life insurance companies gather data on $q(x)$ and use these together with equations (1), (2), (3) to determine the premiums payable for such pensions. Different companies approach the matter in somewhat different ways, and ultimately a certain judgement must be exercised. However, the main outlines of the approach are clear.

In the case of family planning, the mathematical model is necessarily more complicated. While each person dies only once, a woman can and usually does give birth to several children and we need to consider not merely how many of these there are but also how they are spaced.

Analogous to the function $q(x)$ in the life insurance case discussed above is a more complicated function $q(y,i)$; defined as being the probability that a woman who has been married for y years will give birth to her ith child before her $(y+1)$th anniversary.

The theory then develops in analogy with the simpler case, via functions $d(y,i)$, $\ell(y,i)$ corresponding to $d(x)$, $\ell(x)$.

Field surveys allow us to estimate $q(y,i)$ and the challenge is to produce models applicable to family planning insurance in the way that the more traditional models apply to pensions. My own company is active in this area and hopes to run its first pilot study this year.

* * * * *

Miscalculation wings Stealth

Any brilliant product of technology is only as good as the worst piece of science that went into creating it. An error made by two aerodynamicists 43 years ago may have resulted in the most brilliant product of modern aeronautical engineering, the United States Air Force's new B-2 Stealth bomber, being fundamentally flawed in its performance.

The 12 May edition of the journal *"Science"* reports that Joseph Foa, emeritus professor of engineering at George Washington University, has accused those associated with the Stealth bomber project of a concerted effort to cover up the facts about what he claims to be the inferior range of so-called flying wing aircraft, as represented today by the Stealth bomber.

The world's first "flying wing" bomber, Northrop's propellor-driven YB-49, was cancelled in 1949 after only 15 airframes had been built, ostensibly for budget reasons. Air force officials testified later that the real reason was that its range was inadequate, despite calculations by two Northrop aerodynamicists that the flying wing was the optimum configuration for range and fuel economy. It was this assertion, emphasised in their report, that convinced the air force to experiment with flying-wing aircraft.

But in 1947, a young aerodynamicist working at the Cornell Aeronautical Laboratory in Buffalo, New York, was performing theoretical work when he came upon a remarkable contradiction. The researcher was Joseph Foa, and his calculations indicated that not only was the flying wing shape inherently inferior to a wing-and-fuselage design, aerodynamically it was the worst of all possible configurations.

He found that the original Northrop researchers, William Sears and Irving Ashkenas, had performed all their calculations properly, but had accidentally reversed their answers in a calculation [of] the maximum and minimum values for the ratio of total aircraft volume to wing volume. They concluded – wrongly – that for an aircraft with almost its total volume in its wing, the range should be maximised, when the proper conclusion was that it would be at a minimum.

He wrote to Sears and Ashkenas in 1947 to point out the error, and Dr Sears replied, saying that while the error was embarrassing, nobody had taken any serious action as a result. He dismissed the idea that the error should be retracted, saying it was unlikely the air force would be excited by the flying wing proposal.

In 1984, the Stealth bomber underwent a major redesign costing $1 billion, which changed its wing structure to decrease its weight. The air force said it was to improve the bomber's terrain-hugging capabilities so it could fly beneath radar – almost a contradiction of the original design proposal, which called for a high-flying bomber that could elude detection by being almost invisible to radar.

The redesigned Stealth has still not flown, although it was rolled out for public display last November. When it does, initial interest will probably centre on its range, not its radar signature.

Taken from "LEONARDO: TECHNOLOGY"
of *The Age*, 22 May 1989

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Fermat's last theorem

"I have proved that the relation

$$x^m + y^m = z^m$$

is impossible in integral numbers (x,y,z different from 0, m greater than 2); but the margin does not leave me room enough to inscribe the proof."

Found, after Pierre de Fermat's death, written (in Latin) in the margin of his copy of Diophantus' works.

BOWLING AVERAGES

J.C. Burns, Australian Defence Force Academy

In a talk at the Canberra Mathematical Association, Neville de Mestre discussed the following question relating to bowling averages:

> Adams and Brown have each taken 28 wickets for 60 runs before the last match of the season in which Adams takes 0 wickets for 24 runs, while Brown takes 5 wickets for 40 runs. Which ends the season with the better average?†

In the last game Brown performs much better than Adams, so it comes as a considerable surprise that Adams with 28 wickets for 84 runs and an average of 3 runs per wicket comes out ahead of Brown who has 33 wickets for 100 and an average of just over 3.

It is an instructive exercise to analyse this problem in more generality. The mathematics involved is of course entirely elementary, although there is a useful emphasis on inequalities, an area which deserves more attention than it often receives. The challenge offered by the problem lies rather in planning the investigation efficiently and expressing the results simply. The discussion that follows may provide an opportunity for students (perhaps in groups) to gain experience in these important aspects of problem solving.

We suppose that as they enter the last match of the season, Adams and Brown have each taken x wickets for X runs and that in the final match, Adams takes a wickets for A runs and Brown takes b wickets for B runs. The season's averages for the two players are thus $(X + A)/(x + a)$ and $(X + B)/(x + b)$ respectively.

It is sufficient to consider the conditions under which a particular player, say Adams, has the better average for the season. This will be so if and only if

$$\frac{X+A}{x+a} < \frac{X+B}{x+b}.$$

(To consider the case in which Brown has the better average, we simply interchange A, a with B, b.)

A couple of special cases can be cleared up at once. If both bowlers take the same number of wickets in the final game so $a = b$, then Adams has the better average if he concedes fewer runs than Brown, i.e. $A < B$; and if they both concede the same number of runs in the last match so $A = B$, then Adams has the better average if he takes more wickets, i.e. $a > b$.

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† The average is the total number of runs divided by the total number of wickets taken. The lower, the better.

The interesting cases therefore are when $a \neq b$ and $A \neq B$. Decisions are needed on how best to consider the range of cases in such a way that all possibilities have been included. We can begin by supposing that Adams is the player who takes more wickets in the last match, i.e. $a > b$ and seek the conditions under which he has the better average for the season. Do we need to consider the opposite case $a < b$? No: this can be dealt with by interchanging Adams and Brown in the analysis that follows. We could equally well start with the assumption that Adams concedes fewer runs in the last match, i.e. $A < B$; we shall see that a comparison of the two approaches is not without interest.

Adams, the player who takes the greater number of wickets in the last match, has the better average for the season if and only if

$$\frac{X+A}{x+a} < \frac{X+B}{x+b},$$

i.e. $$X(a-b) - x(A-B) > bA - aB = ab(A/a-B/b).$$

Since $a > b$, this means that Adams has the better average if and only if

$$X > \{x(A-B) + ab(A/a-B/b)\}/(a-b). \qquad (1)$$

We now consider the conditions under which this inequality (1) is satisfied.

(a) $a > b;\ A < B$

If Adams concedes fewer runs in the last match than Brown so $A < B$ (as well as $a > b$), his average in that match is better than Brown's, i.e. $A/a < B/b$. Thus, the right-hand side of (1) is negative. The expected result is therefore obtained that if Adams takes more wickets for fewer runs in the last match than Brown, then he ends the season with the better average.

Let us write C as a shorthand for $ab(B/b-A/a)/(A-B)$, which equals $(aB-bA)/(A-B)$.

(b) $a > b;\ A > B,\ A/a < B/b;\ x \leq C$

If, on the other hand, Adams concedes more runs in the last match than Brown so $A > B$, he may still have a better average than Brown in that game, i.e. $A/a < B/b$. The sign of the right-hand side of (1) then depends on x. If $x(A-B) \leq ab(B/b-A/a)$ i.e. if $x \leq C$ the right-hand side of (1) is negative or zero and Adams will have the better average, whatever the value of X. For example, if $a = 5$, $b = 3$, so $a > b$; $A = 70$, $B = 60$ so $A > B$ and $A/a = 14$, $B/b = 20$, giving $A/a < B/b$, then $C = 9$. Hence, provided $x \leq 9$, Adams will have the better average for any X.

Let us write $F(x)$ for the right-hand side of (1).

(c) $a > b;\ A > B;\ A/a < B/b;\ x > C$ and $X > F(x)$.

When $x > C$, however, the right-hand side of (1), $F(x)$, is positive and (1) is satisfied, i.e. Adams has the better average, if and only if $X > F(x)$. Thus, in the example used in (b) above, if $x = 25$, then $F(x) = 80$ and Adams has the better average for the season if $X > 80$:

when $X = 90$, say, the averages for Adams and Brown are 5.33 and 5.36 respectively; when $X = 80$, each average is 4; and when $X = 70$, say, the respective averages are 4.67 and 4.64.

(d) $a > b$; $A > B$; $A/a \geq B/b$; $X > F(x)$

Finally, we have the case in which Adams not only concedes more runs than Brown in the last match but also has an average in that match which is no better than Brown's. In these circumstances, $F(x)$, the right-hand side of (1) is positive for all x but even so (1) will be satisfied for any given x, provided $X > F(x)$ when Adams will then have the better average for the season.

For example, if $a = 3$, $b = 2$, so $a > b$; $A = 60$, $B = 20$ so $A > B$ and $A/a = 20$, $B/b = 10$ giving $A/a > B/b$; and $x = 10$, then $F(x) = 460$. Hence, provided $X > 460$, Adams will have the better average for the season: when $X = 490$, say, the averages for Adams and Brown are 42.31 and 42.5 respectively; when $x = 460$, each average is 40; and when $X = 400$, say, the respective averages are 35.38 and 35.

The analysis has revealed a surprisingly complex array of circumstances in which the player who takes more wickets in the final game has the better average for the season; this will be so if

(a) he concedes fewer runs in the last game;

(b) he concedes more runs in the last game but has the better average in that game and the number of wickets taken (by each player) before that game is not larger than a number C determined by the performances of both players in the last game;

(c) he concedes more runs in the final game but has the better average in that game and, although the number of runs taken before the last game is greater than C, the number of runs conceded (again by each player) before that game is greater than a number $F(x)$ determined by the performances of both players in the final game and the number of wickets $x > C$ taken before that game;

(d) he concedes more runs in the final game and does not have the better average in that game but the number of runs conceded (by each player) before the final game is greater than a number $F(x)$ determined as in (c).

The results can be summarised more briefly and perhaps more strikingly, but less precisely, as follows: the player who takes more wickets in the final game will certainly have the better average for the season if he concedes fewer runs in that game; but if he concedes more runs in the final game, having the better average in it will not necessarily give him a better average for the season; neither, however, will having the poorer average in the final game necessarily prevent his having the better average for the season.

All of this analysis began with the decision to look at what happens when $a > b$. As was remarked at the time, we could equally well have used the condition $A < B$ as the starting point and an exactly similar analysis would have followed. Once the two analyses are complete, it can readily be verified that there is a dualism between them in that one can be obtained from the other by interchanging symbols (and their verbal equivalents) according to the following pattern:

$$a,b;\ x,X \Leftrightarrow B,A;\ X,x.$$

This transformation has the effect of interchanging runs conceded and wickets taken both during the season and in the final match. At the same time, the rôles of Adams and Brown in the last game are interchanged. It is noteworthy, however, that this double interchange leaves the relationships between averages unaltered: the relations $A/a >, =, < B/b$ become $b/B >, =, < a/A$ which are the same as the originals; and the same applies to the relations $(X+A)/(x+a) >, =, < (X+B)/(x+b)$. One is reminded of duality in three-dimensional projective geometry: when points and planes are interchanged, lines are left unchanged, e.g. "three points determine a plane" has as its dual "three planes determine a point", but the dual of "two points determine a line" is "two planes determine a line".

Here are some examples of the effect of the transformation involved in our problem: $a < b$ becomes $B < A$ (i.e. $A > B$); $x \le C = (aB-bA)/(A-B)$ becomes $X \le D = (Ba-Ab)/(b-a)$;

$$X > \{x(A-B) + ab(A/a-B/b)\}/(a-b) = F(x) \quad \text{becomes}$$

$$x > \{X(b-a) + AB(b/B-a/A)\}/(B-A) = f(X)$$

i.e. $$x > \{X(b-a) + ab(A/a-B/b)\}/(B-A) = f(X);\ \text{and so on.}$$

Finally, the following brief summary of the analysis starting with the assumption $A < B$ can be compared with that given above for the analysis starting with $a > b$:

The player who concedes fewer runs in the final game will certainly have the better average for the season if he takes more wickets in that game; but if he takes fewer wickets in that game having the better average in it will not necessarily give him a better average for the season; neither, however, will having the poorer average in the final game necessarily prevent his having the better average for the season.

* * * * *

CALCULATING AGE BY FORMULA

Garnet J. Greenbury, Brisbane

The formula is $70x + 21y + 15z - 105n$.

Suppose a friend is 36 years old. When this age is divided by 3, 5 and 7 in that order, the remainders are 0, 1 and 1. These remainders are x, y and z in that order. From these remainders you can calculate the age.

Using the above formula his age is

$$\begin{aligned} &70(0) + 21(1) + 15(1) - 105(0) \\ &= 21 + 15 \\ &= 36 \end{aligned}$$

where the multiplier of 105 is chosen to give the predicted age within a reasonable range.

Suppose the age is 37. The remainders are 1, 2 and 2, and substituting, the age is

$$\begin{aligned} &70(1) + 21(2) + 15(2) - 105(1) \text{ (note multiple 1)} \\ &= 70 + 42 + 30 - 105 \\ &= 142 - 105 \\ &= 37. \end{aligned}$$

Suppose the age is 38. The remainders are 2, 3 and 3 and the age is

$$\begin{aligned} &70(2) + 21(3) + 15(3) - 105(2) \\ &= 38. \end{aligned}$$

Can a reader explain mathematically why this method works? Are there any exceptions?

* * * * *

PROBLEMS AND SOLUTIONS

Problem 12.5.1 Find all positive integer pairs of solutions x, y of

$$\frac{x^2+y^2}{xy+1} = k^2,$$

where k is a positive integer.

There has been a number of letters from readers on the solution to this problem. Before announcing the problem, we had the solution of Dr Strzelecki, the leader of the 1988 Australian Olympiad team. We first offer the solution of Martin O'Hely, a member of the Australian Olympiad team, and now a first-year undergraduate at Monash University.

Let $x = a_1$ and $y = a_2$ be a solution, and assume, without loss of generality, that $a_1 \geq a_2$.

Then

$$\frac{a_1^2+a_2^2}{a_1a_1+1} = k^2$$

which is equivalent to

$$a_1^2 - k^2a_2a_1 + a_2^2 - k^2 = 0. \tag{1}$$

Consider the quadratic in x

$$x^2 - k^2a_2x + a_2^2 - k^2 = 0.$$

One solution of it is a_1. The other one is (say) a_3. By the theory of the quadratic,

$$a_1 + a_3 = k^2a_2$$

or

$$a_3 = k^2a_2 - a_1.$$

Clearly a_3 is an integer, since a_1, a_2 and k^2 are. If it were true that $a_3 \geq a_2$, then

$$a_2^2 - k^2 = a_1a_3 \tag{2}$$

by the theory of the quadratic, and

$$a_1a_3 \geq a_2^2 \tag{3}$$

since $a_1 \geq a_2$ and $a_3 \geq a_2$ and these are positive integers (by assumption). But (2) and (3) together mean

$$a_2^2 - k^2 \geq a_2^2,$$

which is impossible as k^2 is positive. Thus it must be the case that

$$a_3 < a_2.$$

Notice that, if $a_3 > 0$, $x = a_2$, $y = a_3$ is also a solution to (1) with $a_2 > a_3$; so it is possible to construct a_4, $a_5, \ldots$ (where $a_{i+1} = k^2a_i - a_{i-1}$, which satisfy

$$\frac{a_i^2+a_{i+1}^2}{a_ia_{i+1}+1} = k^2 \tag{4}$$

and

$$a_{i+1} < a_i$$

whenever $a_{i+1} > 0$.

Since this $a_1, a_2, \ldots,$ is a monotonically decreasing sequence until (at last!) a non-positive term B is reached, the fact that all these a_i are integral means that *such a* B *exists*.

If $B = a_j$, let $a_{j-1} = A$. Then $A > 0$ and $B \leq 0$. This means $AB \leq 0$.

Now if $AB \leq -2$, $AB + 1 \leq -1$ so $\frac{A^2+B^2}{AB+1} \leq 0$ (as $A^2 + B^2 \geq 0$). But $\frac{A^2+B^2}{AB+1} = k^2 > 0$. So $AB > -2$. If $AB = -1$, $\frac{A^2+B^2}{AB+1}$ is undefined, and certainly not equal to k^2 as (4) requires. Hence $AB \neq -1$.

This leaves (as A and B are integers) only the possibility for AB of

$$AB = 0$$

and so

$$B = 0 \quad (\text{since } A > 0).$$

Consequently $k^2 = \frac{A^2+B^2}{AB+1} = A^2$, i.e. $A = a_{j-1} = k$, and so the entire sequence $a_1, a_2, \ldots, a_j$ is obtained from $a_{i-1} = k^2 a_i - a_{i+1}$ (this is a transposition of (*)).

Consequently it must be the case, for any solution $x = a$, $y = b$, that a and b are consecutive terms of the sequence b_i, $i = 1,2,\ldots,$ defined by

$$b_1 = k,$$
$$b_2 = k^3,$$
$$b_{i+1} = k^2 b_i - b_{i-1} \quad (i = 2,3,\ldots).$$

The above solution of O'Hely shows how successive solutions are obtained, but does not give a formula for the solutions. We now show that the constructed sequence is of length 1 when $k = 1$, is infinite when $k > 1$, and then give Dr Strzelecki's formula for $k > 1$.

Let us deal with the slightly troublesome case, when $k = 1$. Then starting, as in the previous solution, with a solution $x = a_1$, $y = a_2$, with $a_1 \geq a_2$,

$$a_1 + a_3 = 1^2 a_2 = a_2,$$

and hence, since $a_1 \geq a_2$, $a_3 \leq 0$. So the downward sequence $a_1 \geq a_2 > a_3 \ldots > a$, of positive integers, stops at a_2. Moreover, $a_1 = a_2 = 1$. For this is clearly a solution and we quickly check that there is no solution $x = n > 1$ with y an integer.

If $k > 1$, we never have a solution with $x = y$, for this would give a positive integer solution to

$$2x^2 = x^2k^2 + k^2$$

i.e. to $x^2 = \dfrac{k^2}{2-k^2} < 0$, if $k > 1$.

Hence, applying the argument of O'Hely, we have an infinite sequence $b_1 < b_2 < \ldots$ such that all solutions, with $x \leq y$, are given by $x = b_n$, $y = b_{n+1}$, $n = 1,2,\ldots$.

As just established, any pair of solutions $x = a$ and $y = b$, with $0 < a \leq b$, are, when $k > 1$, successive terms of the sequence $b_1 < b_2 < \ldots < b_n < \ldots$, with $b_1 = k$, $b_2 = k^3$ and

$$b_{i+1} - k^2b_i + b_{i-1} = 0 \qquad (5)$$

for $i = 2,3,\ldots$. Note that $x = 0$, $y = k$ is also a solution of (1); and, setting $b_0 = 0$, (5) also holds for $i = 1$.

The general solution of the recurrence relation (5) is of the form

$$b_n = \alpha z_1^n + \beta z_2^n$$

where z_1 and z_2 are the roots of the equation

$$z^2 - k^2z + 1 = 0.$$

Thus
$$z_{1,2} = \frac{k^2 \pm \sqrt{k^4-4}}{2}.$$

Hence

$$b_n = \alpha\left(\frac{k^2+\sqrt{k^4-4}}{2}\right)^n + \beta\left(\frac{k^2-\sqrt{k^4-4}}{2}\right)^n,$$

for $n = 0,1,\ldots$. Set $n = 0$ and $n = 1$ and we get

$$b_0 = 0 = \alpha + \beta$$

$$b_1 = k = \alpha\frac{k^2+\sqrt{k^4-4}}{2} + \beta\frac{k^2-\sqrt{k^4-4}}{2} .$$

Solving these equations for α and β gives

$$\alpha = \frac{k}{\sqrt{k^4-4}}, \quad \beta = \frac{-k}{\sqrt{k^4-4}} .$$

Hence,

$$b_n = \frac{k}{\sqrt{k^4-4}}\left[\left(\frac{k^2+\sqrt{k^4-4}}{2}\right)^n - \left(\frac{k^2-\sqrt{k^4-4}}{2}\right)^n\right], \qquad (6)$$

and hence, any positive integer solution of (1), with $x \le y$, and $k > 1$, is of the form $x = b_n$, $y = b_{n+1}$, $n = 1,2,\dots$, where the b_n is given by (6).

Problem 11.3.1. If a and b are positive and $a + b = 1$, show that

$$(a + 1/a)^2 + (b + 1/b)^2 \ge \frac{25}{2}.$$

A solution to this problem, using the calculus, appeared on p. 128 in Part 4 of *Function*, Volume 11. A second solution, just using the algebraic manipulation of inequalities, appeared on pp. 31f. in Part 1 of *Function*, Volume 12. Another elegant algebraic solution was published in Part 2, Volume 12, pp. 37f. This was followed by two further algebraic solutions in Volume 12, Part 4, pp. 123-4. Here is a sixth algebraic solution sent to us by Andy Liu (University of Alberta, Edmonton, Canada). Used in Liu's solution is the arithmetic-mean geometric-mean inequality (*A-G* inequality): for any non-negative numbers u and v

$$\sqrt{uv} \le \frac{u+v}{2}.$$

Here $\sqrt{uv}$ is the *geometric mean* of u and v,
while $\frac{u+v}{2}$ is the *arithmetic mean* of u and v.

Andy Liu makes the following sequence of observations which combine to provide the solution.

1) If $0 < x \le y < 1$, then

$$\left(x + \frac{1}{x}\right) - \left(y + \frac{1}{y}\right) = \frac{(y-x)(1-xy)}{xy} \ge 0,$$

i.e. $x + \frac{1}{x} \ge y + \frac{1}{y}$.

2) By the *A-G* inequality,

$$\sqrt{ab} \le \frac{a+b}{2} = \frac{1}{2},$$

i.e.
$$ab \le \frac{1}{4}.$$

3) By 1) and 2),

$$ab + \frac{1}{ab} \ge \frac{1}{4} + 4 = \frac{17}{4}.$$

4) By the *A-G* inequality

$$\frac{a}{b} + \frac{b}{a} \ge 2\sqrt{\frac{a}{b}\,\frac{b}{a}} = 2.$$

5) By 3) and 4) and the A-G inequality

$$\begin{aligned}(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2 &\geq 2(a + \frac{1}{a})(b + \frac{1}{b}) \\ &= 2(ab + \frac{1}{ab} + \frac{a}{b} + \frac{b}{a}) \\ &\geq 2(\frac{17}{4} + 2) = \frac{25}{2}.\end{aligned}$$

6) It is easy to verify that equality holds if and only if $a = b$.

A seventh solution was provided by J.A. Deakin in which he used the so-called method of Lagrange multipliers to show that the minimum value of $(a + \frac{1}{a})^2 + (b + \frac{1}{b})^2$, subject to the condition $a + b - 1 = 0$, occurs when $a = b = \frac{1}{2}$.

Mr Deakin observes that his method can be immediately used to demonstrate that, if $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 + a_2 + \ldots + a_n = 1$, then

$$(a_1 + \frac{1}{a_1})^2 + (a_2 + \frac{1}{a_2})^2 + \ldots + (a_n + \frac{1}{a_2})^2 \geq \frac{(n^2+1)^2}{n},$$

with equality only if $a_1 = a_2 = \ldots = a_n = \frac{1}{n}$.

We also received the following letter:

"Dear Sirs,

I seem to recall seeing four solutions, all sweet, of the inequality $(a+a^{-1}) + (b+b^{-1})^2 \geq 25/2$ for $a > 0$, $b > 0$, $a+b = 1$ in recent numbers, but none using calculus. Calculus is under attack from the computer monstrosity which technologists have unleashed, and *FUNCTION* has a tendency to lean towards the static, algebraic, logical, set-theoretic sort of mathematical thinking as contrasted with the dynamic, calculus, variational sort, that it might provide a little balance if you were to publish a calculus solution of the above inequality. Anybody could supply it; the main thing is, probably, to keep it neat. I try to do this below, but, before embarking on the details I remark that it is not unrelated to the old results that the sum of two numbers, whose product is constant, is least when they are equal, and its companion, that the product of two numbers, whose sum is constant, is greatest when they are equal.

Let us then take the function, of x,

$$u = (x+x^{-1})^2 + (y+y^{-1})^2, \text{ where } x + y = 1.$$

The domain is $\{x : 0 < x < 1\}$.
We have

$$u = x^2 + x^{-2} + y^2 + y^{-2} + 4,$$

so that, using the chain rule for differentiation,

$$\begin{aligned} u' &= 2(x-x^{-3}) + 2(y-y^{-3})(-1) \\ &= 2(x-y) - 2(x^{-3}-y^{-3}) \\ &= 2(x-y)\left\{1 + \frac{x^2+xy+y^2}{x^3y^3}\right\} \end{aligned}$$

which has the sign of $x - y = 2x - 1$.

Hence $u' < 0$ for $0 < x < \frac{1}{2}$ and

$u' > 0$ for $\frac{1}{2} < x < 1$.

Hence u is (absolute) minimum for $x = \frac{1}{2}$ and then

$$u = (\tfrac{1}{2} + 2)^2 + (\tfrac{1}{2} + 2)^2 = 25/2,$$

and the result is established.

J.C. Barton"

* * * * *

Problem 12.4.2. The game NORTH-EAST is played on the rectangular array of points in the plane with integral coordinates (n,m) where $0 \le n \le N$, $0 \le m \le M$. Player A selects a point (p,q) and removes all those points for which $n \ge p$, $m \ge q$. Player B then selects a point (r,s) and removes all those points still left for which $n \ge r$, $m \ge s$, etc. The loser is the player who takes $(0,0)$. The problem is to show that A has a winning strategy.

Solution (by J.C. Barton).

We consider various cases.

(i) $N = 0$, $M = 0$. There is one point $(0,0)$ only.
A loses. We reject this case.

(ii) $N = 0$, $M > 0$. There is a single line q points $(0,m)$.
A selects the point $(0,1)$ leaving the single point $(0,0)$.
A wins.
The sub-case $N > 0$, $M = 0$ is dealt with similarly.

(iii) $N = 1$, $M \ge 1$. A selects $(1,1)$ leaving the three points $(0,1)$, $(1,0)$, $(0,0)$. Whichever of $(1,0)$ or $(0,1)$ B choses, A selects the other. A wins.
Similarly if $N \ge 1$, $M = 1$.

(iv) If $N > 1$, $M > 1$, A selects $(1,1)$ as in (iii).

Problem 12.4.3. (Proposed by a puckish Lewis Carroll in *The Monthly Packet* beginning in April, 1880). Place twenty-four pigs in four sties so that, as you go round and round, you may always find the number in each sty nearer to ten than the number in the last.

Solution (by J.C. Barton)

6, 8, 10, 0 will work if we agree that "nothing is nearer to ten than 10".

Problem 13.3.1 (suggested by Esther Szekeres)

Given a circle centre O, radius R, and a point P outside of it. Construct a straight line passing through P that meets the circle in points A and B such that $PB = 2PA$.

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PERDIX

IBM Olympiad Training School, 1989

The following questions formed the trial Olympiad examination set in April this year to contenders for places in the Australian International Mathematical Olympiad team. Try your hand at them. Let me have your solutions and comments.

Question 1

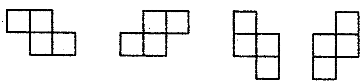
Find, with proof, all solutions of the equation

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1$$

in positive integers x, y, z.

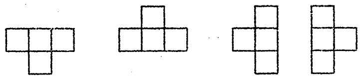
Question 2

(i) Decide whether or not the squares of an 8×8 chessboard can be numbered by the numbers 1, 2, 3, ..., 64 in such a way that the sum of the numbers in each of its sections of one of the shapes



is divisible by 4.

(ii) Repeat with the shapes



Question 3

Let $f(n)$ be a function defined on the set of all positive integers and having its values in the same set. Suppose $f(f(n) + f(m)) = m + n$ for all positive integers n, m. Find all possible values of $f(1989)$.

Question 4

Let O be the circumcentre of the triangle ABC, and let X and Y be points on AC and AB respectively such that BX intersects CY in O. Suppose $\angle BAC = \angle AYX = \angle XYC$; determine the size of this angle.

Question 5

Let K denote the set $\{a,b,c,d,e\}$. F is a collection of 16 different subsets of K and it is known that any three members of F have at least one element in common. Show that there are exactly 5 possibilities for F.

Question 6

Let a be the greatest positive root of the equation $x^3 - 3x^2 + 1 = 0$. Show that $[a^{1788}]$ and $[a^{1988}]$ are both divisible by 17. ($[x]$ denotes the integer part of x.)

Question 7

Let $ABCD$ be a tetrahedron having each sum of opposite sides equal to 1. Prove that

$$r_A + r_B + r_C + r_D \le \sqrt{3}/3$$

where r_A, r_B, r_C, r_D are the inradii of the faces, equality holding only if $ABCD$ is regular.

Question 8

Prove: for every integer $n \ge 2$ and for any n functions $f_1, f_2, \ldots, f_n$ that are defined for $0 \le x \le 1$, there exists real numbers $a_1, a_2, \ldots, a_n$ such that

(i) $0 \le a_i \le 1$ for $i = 1, \ldots, n$;

(ii) $|a_1 \cdot a_2 \cdot \ldots \cdot a_n - \sum_{i=1}^{n} f_i(a_i)| \ge \frac{1}{2} - \frac{1}{2n}$.