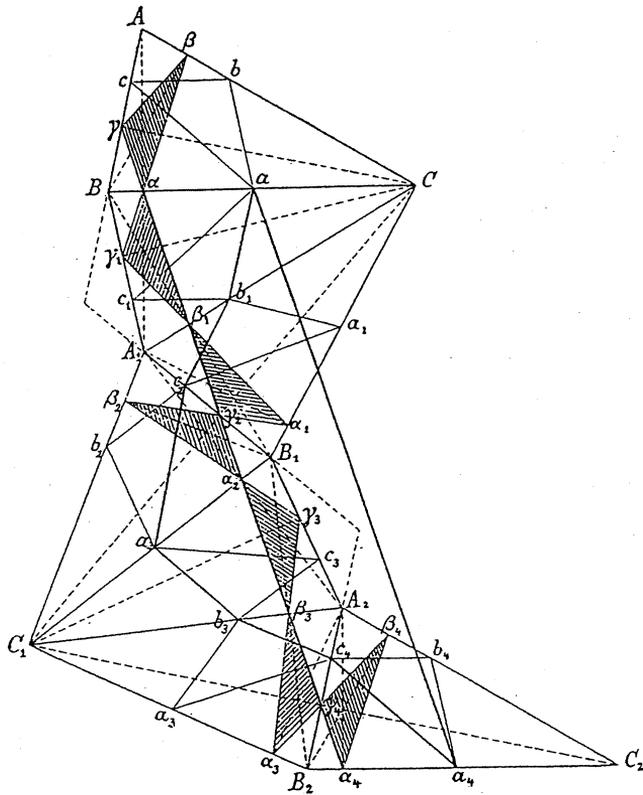


# FUNCTION

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Welcome to the readers of FUNCTION! An outstanding feature of Australia's bicentennial celebrations will be the 29th International Mathematical Olympiad in July. FUNCTION will keep readers informed of the momentous competition.

1988 is opened by FUNCTION with imprecision as its central theme. Mathematics has many ways of dealing with it.

Peter Kloeden's article is a story of fuzzy sets: despite their name they are mathematical objects, and their theory is applied in real life, e.g. decision making for an entire underground subway rail system in Japan - by computers, of course.

More traditional is the imprecision in forecasting the result of tossing dice: G.A. Watterson's article provides a result to be remembered, especially by players of board games like "Escape from Colditz".

A tracking error, another manifestation of imprecision, occurs when needle and record meet on a record player: how to keep track of such an error is recorded in an article by John Barton.

Jandep this time tells us about a way of surpassing Archimedes' sand reckoning: building a miniature word processor capable of incorporating mathematical symbols and simple diagrams.

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# THE FRONT COVER

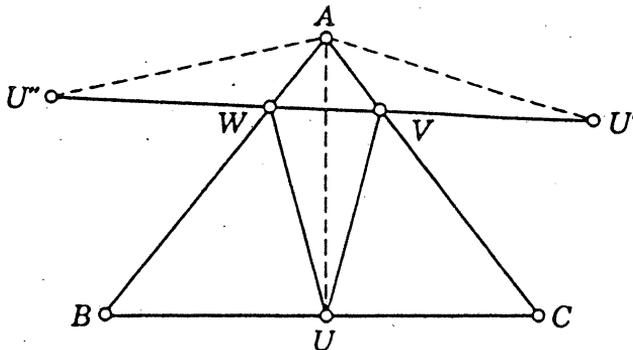
Hans Lausch  
Monash University

Perception and apperception of geometric configurations is a theme that has engaged psychologists, chess players, educators and mathematicians. This issue's front cover presents a diagram originally drawn by Hermann Amandus Schwarz (1843 - 1921).

The drawing delineates a solution to a problem proposed by the 60 year old Marquis Gianfrancesco de Toschi a Fagnano (1715 - 1797), a famous mathematician's son, a master of Euclidian geometry and archdeacon in Senigallia, an ancient Italian city on the Adriatic coast. His treatise *Problemata quaedam ad methodum maximorum et minimorum spectantia* appeared in the German journal *Nova Acta Eruditorum*, the follow-up to *Acta Eruditorum* where a multitude of excellent mathematicians had their articles published.

Fagnano's problem was: "In a given acute-angled triangle  $ABC$ , inscribe a triangle  $\alpha\beta\gamma$  whose perimeter is as small as possible", and Fagnano's answer was: "The triangle of minimal perimeter inscribed in an acute-angled triangle  $ABC$  is the orthic triangle of  $ABC$ ." The term "orthic triangle" (of  $ABC$ ) refers to the triangle whose vertices are the three feet of altitude (of  $ABC$ ).

Fagnano used "pure" geometry as well as calculus in his solution. Knowing the problem and its answer facilitates recovering Schwarz's proof: the drawing, baffling as it may be at first sight on the score of its abounding crisscrossing lines, "speaks for itself" after all.



The editors would like to have your explanation, a legend that reconstructs every single thought Schwarz had when working out his proof. You could e.g. start, as Schwarz did, by saying: "Let  $A_1$  be the reflection of  $A$  in  $BC$  ...".

The mathematician Leopold Fejér (1880 - 1959), while he was a student in Berlin and attended Schwarz's seminar, discovered a very elegant solution to Fagnano's theorem. Inspect Fejér's sketch on page 2. Then tell whether you can recover his proof before reading further.

Consider an arbitrary triangle  $UVW$  with  $U$  on  $BC$ ,  $V$  on  $CA$ ,  $W$  on  $AB$ . Let  $U'$   $U''$  be the images of  $U$  by reflection in  $CA$ ,  $AB$ , respectively.

Then

$$UV + VW + WU = U'V + VW + WU'',$$

which is a path from  $U'$  to  $U''$ , usually a broken line with angles at  $V$  and  $W$ . Such a path from  $U'$  to  $U''$  is minimal when it is straight.

Hence, among all inscribed triangles with a given vertex  $U$  on  $BC$ , the one with smallest perimeter occurs when  $V$  and  $W$  lie on the straight line  $U'U''$ . In this way we obtain a definite triangle  $UVW$  for each choice of  $U$  on  $BC$ .

The problem will be solved when we have chosen  $U$  so as to minimize  $U'U''$ , which is equal to the perimeter. Since  $AU'$  and  $AU''$  are images of  $AU$  by reflection in  $AC$  and  $AB$ , they are congruent and the angle  $U'AU''$  equals twice the angle  $BAC$ .

Thus  $AU'U''$  is an isosceles triangle whose angle at  $A$  is independent of the choice of  $U$ . The base  $U'U''$  is minimal when the equal sides are minimal, that is, when  $AU$  is minimal.

In other words,  $AU$  is the shortest distance from the given point  $A$  to the given line  $BC$ . Since the hypotenuse of a right-angled triangle is longer than either of the other sides, the desired location of  $U$  is such that  $AU$  is perpendicular to  $BC$ .

Thus  $AU$  is the altitude from  $A$ . This choice of  $U$  yields a unique triangle  $UVW$  whose perimeter is smaller than that of any other inscribed triangle.

Since we would equally well have begun with  $B$  or  $C$  instead of  $A$ , we see that  $BV$  and  $CW$  are the altitudes from  $B$  and  $C$ . And Fagnano's theorem has been proved.

Once you have recovered Schwarz's beautiful proof you will be able to appraise the teacher's ideas in his student's approach. And to assess Fejér's clever innovation.

Hans Freudenthal, mathematician, educator and the inventor of the space language Lincos which he designed for communication with intelligent alien life forms, studied in Berlin soon after Schwarz had passed away, and learned that Fejér had long ago been Schwarz's favourite. In those days, narrates Freudenthal, Fejér's family name had been Weiss [German for "white"], and so it contrasted notably with his teacher's ["schwarz" is German for "black"].

The story continues years later with Fejér about to visit Schwarz, who was Berlin's absent-minded mathematics professor par excellence, after the adoption of his new Hungarian name. Schwarz refused to admit him: "I don't know a Fejér!"

Fagnano's treatise deals with more "extremum" problems, and these are, of course, open to alternative solutions:

Problem 1, proposed by Pierre Fermat (1601 - 1665). In a given triangle ABC, locate a point D whose distances from A, B, C have the smallest possible sum.

Problem 2. In a given triangle ABC locate a point D such that the sum of the squares of its distances from A, B, C is as small as possible.

Problem 3. In a given quadrilateral ABCD locate a point E whose distances from A, B, C, D have the smallest possible sum.

When Schwarz's solution came out in a volume of Jacob Steiner's (1796 - 1863) collected works, it was believed to have emanated from this great Swiss geometer. Steiner was himself an expert in solving extremum problems.

Steiner's methods were, as a rule, pleasing to the eye, and he was not fond of computations: "When it comes to formulae, I am dull." Steiner assaulted successfully the "isoperimetric problem", an extremum problem dating from the early eighteenth century: "To find a closed curve of given length such that the area it encloses is as large as possible."

Many laws in physics can be viewed as "Nature's" response to extremum problems. The mathematical physicist Pierre Louis Moreau de Maupertuis (1698 - 1759), president of the Royal Academy in Berlin, elaborated a "principle of least action" which postulated that Nature minimized "action".

Maupertuis was, however, too vague and excessively general with his claims regarding the applicability of his "principle", which eventually became his obsession. It led to a classic scandal in Berlin, the persecution of the decent mathematician Samuel Koenig, an iniquitous attack on Maupertuis by the witty Voltaire, and Voltaire's banishment from Prussia and his royal friend Frederic II.

It was only William Rowan Hamilton (1805 - 1865) who managed to make the "principle" precise, deriving it rigorously from Newtonian mechanics; the quantity "action" is the product of the quantity "energy" and the quantity "time". "Action" occurs in the celebrated formula  $E = h\nu$  of quantum theory as the constant  $h$ , Max Planck's "quantum of action", where  $E$  stands for "energy" and  $\nu$  for "frequency".

Hamilton's principle, as it is now called - especially by the Germans - played an important role in twentieth-century physics after it had been "translated" into the language of quantum mechanics. Numerous everyday phenomena are explained smoothly by the Hamiltonian principle, e.g. that light is refracted when passing from one medium into a second; one of the principle's consequences is the observation that for moving between any two points in space, light "chooses" a path which permits it to reach its "destination" within the shortest time possible.

The mathematical shape of this extremum problem is: "In a plane a line  $\ell$  is given and two points, A and B, separated by  $\ell$ . Let  $v$  and  $w$  be two positive real numbers. Locate a point C on  $\ell$  such that  $AC/v + CB/w$  is as small as possible." Would you try your skills on this problem?

Imagine a bath tub full of water and in it a coin at point B while you stare from point A upon the water surface; let  $\ell$  be the line on the water surface that is perpendicular to AB, and  $v, w$  be the speeds of light in air and in water respectively. The coin will seem to be located on the line AC (produced) rather than at B.

Stay in your bathroom for an easier extremum problem: look into the bathroom mirror! Can you see that the occurrence of reflexion furnishes an argument favouring the thesis that light "wants to move" as fast as possible?

And another extremum problem is demonstrable in the bathroom, provided you can prepare a good soap solution and you have a piece of wire at hand. Bend the wire into a twisted frame of your choice and fill the frame in with soap film: no other surface spanned onto the frame will have an area smaller than that you have just created.

Hermann Schwarz was noted for "being in love with his soap solution, and he never ceased improving it", according to his obituarist, the mathematician Hamel. To be sure, calculating these "minimal surfaces" is not always simple: Schwarz hit upon complicated mathematical theorems when doing so.

Professor Schwarz was popular with his colleagues and with his students. He confessed to have married the daughter of his teacher Ernst Eduard Kummer (1810 - 1893), as "mathematical talent is well-known to pass from the father to the son-in-law rather than to the son", not without adding that his "real" teacher was not Kummer, "but Weierstrass had no daughter!"

One of Schwarz's grandchildren was the mathematician Roland Percival Sprague (1894 - 1967). FUNCTION has already reported about him, and this issue contains a letter to the editor which addresses an aspect of his work.

## LETTER TO THE EDITOR

I would like to offer the following comments on "The Sprague Sequence", Shyen Wong (Aug 1987, pp 120-123):

Although the approximation  $T_n = \sqrt{2n + \frac{1}{2} \ln n + 1}$  is closer to  $S_n$  than is  $R_n = \sqrt{2n}$ , a better approximation would have been  $\sqrt{2n + \frac{1}{2} \ln n}$ .

However, a considerable improvement can be made.

Retracing the original development of  $R_n$  (vol 10, pt 5),

$$S_n = S_{n-1} + \frac{1}{S_{n-1}}$$

$$S_n^2 = S_{n-1}^2 + 2 + \frac{1}{S_{n-1}^2}$$

$$= S_{n-2}^2 + 4 + \frac{1}{S_{n-1}^2} + \frac{1}{S_{n-2}^2}$$

$$= S_{n-3}^2 + 6 + \frac{1}{S_{n-1}^2} + \frac{1}{S_{n-2}^2} + \frac{1}{S_{n-3}^2}$$

and, in general, 
$$S_n^2 = S_{n-r}^2 + 2r + \frac{1}{S_{n-1}^2} + \frac{1}{S_{n-2}^2} + \dots + \frac{1}{S_{n-r}^2}$$

If  $n = r + 2$  
$$S_n^2 = S_2^2 + 2(n-2) + \frac{1}{S_{n-1}^2} + \frac{1}{S_{n-2}^2} + \dots + \frac{1}{S_3^2} + \frac{1}{S_2^2}$$
,

but 
$$S_3^2 = S_2^2 + 2 + \frac{1}{S_2^2} \quad \text{and} \quad S_2^2 = 4$$

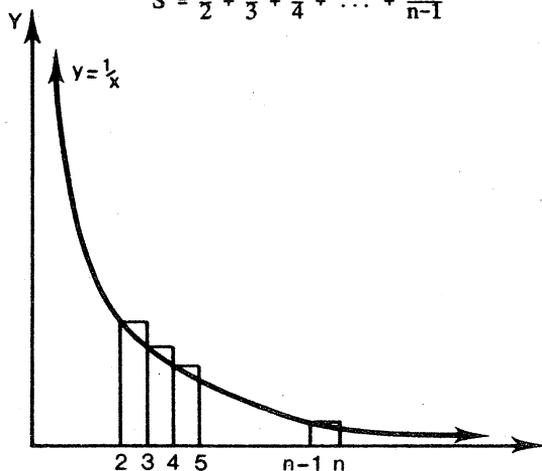
so 
$$S_3^2 \approx 6, \quad S_4^2 \approx 8 \quad \text{and} \quad S_{n-1}^2 \approx 2(n-1)$$

Therefore 
$$S_n^2 \approx 4 + 2n - 4 + \frac{1}{2(n-1)} + \frac{1}{2(n-2)} + \dots + \frac{1}{8} + \frac{1}{6} + \frac{1}{4}$$

$$\approx 2n + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n-1} \right)$$

We now require an approximation for

$$S = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n-1}$$



$S$  is an approximation to the area ( $A$ ) enclosed by  $y = \frac{1}{x}$  and the  $X$ -axis between  $x = 2$  and  $x = n$ .

$$\begin{aligned} A &= \int_2^n \frac{1}{x} dx \\ &= \ln n - \ln 2 \\ &= \ln \frac{n}{2} \end{aligned}$$

$S$  exceeds  $A$  (approximately) by the sum of the areas ( $S_A$ ) of the small "triangles" on top of the curve.

$$\begin{aligned} S_A &= \frac{1}{2} \left( \frac{1}{2} - \frac{1}{3} \right) + \frac{1}{2} \left( \frac{1}{3} - \frac{1}{4} \right) + \dots \quad \dots + \frac{1}{2} \left( \frac{1}{n-1} - \frac{1}{n} \right) \\ &= \frac{1}{2} \left( \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots \quad \dots - \frac{1}{n-1} + \frac{1}{n-1} - \frac{1}{n} \right) \\ &= \frac{1}{2} \left( \frac{1}{2} - \frac{1}{n} \right) \end{aligned}$$

$$\begin{aligned} \text{So } S &\approx \ln \frac{n}{2} + \frac{1}{4} - \frac{1}{2n} \\ &\approx \ln \frac{n}{2} + \frac{1}{4} \end{aligned}$$

$$\text{Therefore } S_n^2 \approx 2n + \frac{1}{2} \left( \ln \frac{n}{2} + \frac{1}{4} \right)$$

$$S_n \approx \sqrt{2n + \frac{1}{2} \ln \frac{n}{2} + \frac{1}{8}} = W_n$$

The table shows values of  $S_n$  together with the three approximations,  $R_n$ ,  $T_n$  and  $W_n$  (corrected to 4 dec. places)

n	$S_n$	$R_n$	$T_n$	$W_n$
10	4.5699	4.4721	4.7065	4.5749
50	10.0839	10.000	10.1467	10.0863
100	14.2137	14.1421	14.2584	14.2155
500	31.6675	31.6228	31.6877	31.6684
1000	44.7569	44.7214	44.7711	44.7575
5000	100.0199	100.000	100.0263	100.0201

Shyen may care to incorporate  $W_n$  in her programs to reproduce the relevant graphs.

# THE MATHEMATICS OF IMPRECISION: FUZZY SETS

**Peter Kloeden**  
**Murdoch University**

To most people mathematics is synonymous with precision and rigour, so the title of this article may seem rather absurd. Indeed the subtitle "fuzzy sets" may even give the impression that this is going to be a mathematical spoof. While some mathematicians still wince at the mention of fuzzy sets, they are certainly no joke. In fact they are now the focus of serious mathematical investigation and are being used extensively in many practical engineering and managerial situations.

Fuzzy sets are the brainchild of Professor Lofti Zadeh, a professor of electrical engineering at the University of California in Berkeley. Typically in many engineering problems, such as the control of a rocket or the regulation of a chemical reaction, precise mathematical models are used to describe the behaviour of the system. These are usually expressed in terms of algebraic or differential equations, which are based on well established scientific laws, such as Newton's laws of mechanics. The engineer solves these equations, usually on a computer, in order to predict and to control the evolution of the system. The advances in aerospace technology over the past fifty years are a clear indication of the stunning success of this methodology.

There are however many situations for which precise mathematical models cannot be determined or, if they can be, are too unwieldy to implement. It was to enable one to predict and to control such processes that led Zadeh to the idea of fuzzy sets in the 1960s. This occurs for instance when an engineer has only a rough idea of the mechanisms involved in a process or has heuristic and often conflicting criteria for making decisions about its operation. Consider for example a transport engineer who is to organize a commuter bus service for a city, which must be cost effective, yet convenient and pleasant. The impression here is due to the indefiniteness and the subjectivity of the task; it has nothing to do with randomness in the probabilistic sense. Rather it is the kind of imprecision involved in deciding, for example, if a woman is beautiful or, in a less emotional context, if a number is much larger than 1; there appears to be no clearcut way of deciding. This contrasts with the impression of not knowing whether a fair coin will come down as a head or a tail when tossed. Excluding absurd situations such as its being sucked up into space by a UFO or the less absurd but rare situation of its landing on its edge, the coin will land either as a head or a tail, and there is an equal probability of it being one or the other; one simply cannot say in advance which of the two specific alternatives will occur.

Before seeing how Zadeh managed, without resorting to magic, to give a precise mathematical meaning to imprecision, that is how he defined fuzzy sets, let us consider the more conventional mathematical sets. These are specified either by listing each element in the set, for example with the set

$$A = \{1, 2, 3, 4\},$$

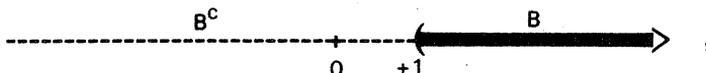
or by giving a mathematical rule that clearly indicates which elements are in the set and which are not, as for example with the set

$$B = \{x: x > 1\}.$$

The two sets  $A$  and  $B$  here are subsets of a larger set  $R$ , the set of all real numbers. Although  $R$  may itself be considered a subset of some other larger set, it often suffices or is convenient to restrict attention, in these cases, to the set  $R$ , which can be represented by a straight line with each number a point at the corresponding distance from the origin in either the positive or negative direction.



We could then indicate a subset of this line by marking in black (or boldface) all those numbers which belong to this subset and in white (or light dashes) all those numbers not in it, that is all those numbers in the complement of the subset. With the subset  $B$  from above we have



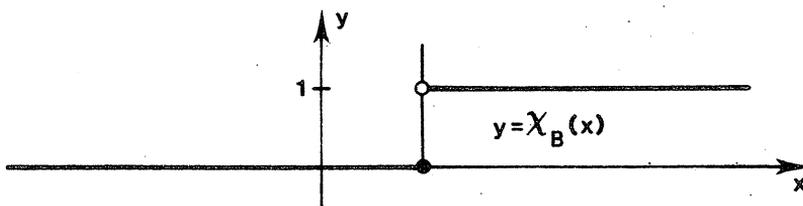
where  $B^c$  denotes the complement of the subset  $B$ , so

$$B^c = R \setminus B = \{x: x \leq 1\}.$$

Instead of using a colour or typographical code to indicate the points belonging or not belonging to a given subset we could just as conveniently characterize the set in terms of the graph of a function, called the characteristic function of the subset, which takes the value 1 at each point belonging to the subset and the value 0 at each point not belonging to it. For a subset  $B$  it is usual to denote the characteristic function by  $x_B$ , so

$$y = x_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}.$$

Hence for the particular subset  $B = \{x: x > 1\}$  we have for the graph of its characteristic function



Note that this graph has a jump or discontinuity in it at  $x = 1$ , a boundary point between the set  $B$  and its complement set  $B^c$ . (The graph may not be easy to draw at all; consider the graph of the characteristic function of the subset  $Q$  of all of the rational numbers in  $R$ .)

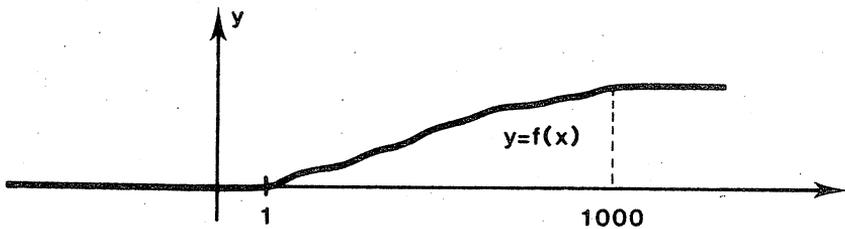
The characteristic function of a subset is so called because it characterizes that subset; the value that it takes at each point of the real line indicates the degree of membership of the point in the subset. Here there are just two alternatives, either the point is in the subset (and has membership degree 1) or it is not in the subset (and its membership degree is 0). Abstracting, as is a feature of modern mathematics, we can thus think of the subset of real numbers as a characteristic function. This may seem artificial and useless, but is the key to Zadeh's concept of a fuzzy set; he very simply allowed points to have an intermediate membership grade between 0 and 1, as well as 0 or 1, with these intermediate grades representing "partial" membership. Thus according to Zadeh a fuzzy set of real numbers is characterized by, and may thus be thought of as being equivalent to, a function

$$f: R \rightarrow [0,1],$$

with the value  $f(x)$  of the function at a point  $x \in R$  indicating the degree of membership of the point  $x$  in this fuzzy set. Consider for example a fuzzy set of real numbers *much greater* than 1, that is the set

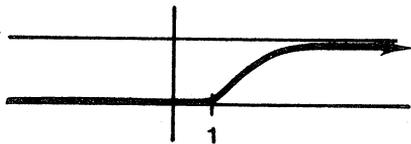
$$\{x: x \gg 1\}$$

This set does not make sense as a conventional subset of  $R$  as the meaning of  $\gg$  is imprecise. It does however make sense as a fuzzy set. In fact there are many such fuzzy sets, depending on how we specify the membership function. Certainly we would expect that  $f(x) = 0$  for  $x \leq 1$  for all of them and then for  $f(x)$  to increase slowly or at least not decrease, towards 1 as  $x$  gets larger and larger.



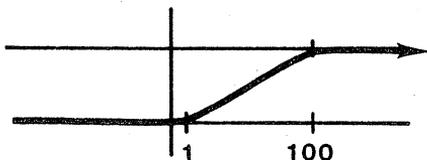
Two specific examples are

$$(i) \quad y = f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 1 - e^{(-x+1)} & \text{if } x > 1 \end{cases}$$



and

$$(ii) \quad y = f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ \frac{1}{99}(x - 1) & \text{if } 1 \leq x \leq 100 \\ 1 & \text{if } x \geq 100 \end{cases}$$

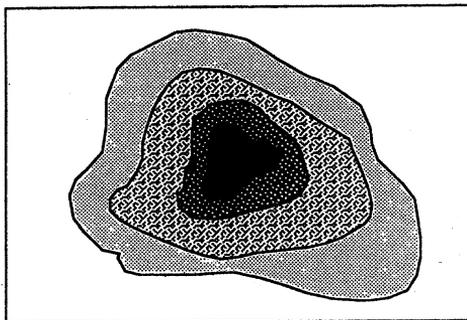


In choosing a specific membership grade function, that is a specific fuzzy set, we are in effect specifying what we mean by a number's being much greater than 1. In many cases it is not necessary to specify a particular fuzzy set; often general, qualitative, facts can be deduced from the general qualitative properties that such a function must satisfy. In other cases, if numerical or quantitative results are required, then the engineer or manager must specify a particular membership function; this is usually done on the basis of past experience and observations. (This is in fact what must be done when nonfuzzy or "crisp" mathematical or probabilistic models are used; it is often called "parametrizing" the model. In doing this it is important to know that small changes in the model parameters do not drastically change the behaviour of the model system, that is the model is a robust one. For this reason a general qualitative investigation is important before numerical calculations are undertaken).

Fuzzy sets need not be defined only on the set of real numbers  $R$ . In general, a fuzzy set can be defined on any set of elements  $S$  and consists of a membership function

$$f: S \rightarrow [0,1]$$

which assigns a degree of membership  $f(s)$  to each element  $s \in S$ . In our example above  $S = R$ , although it could be the plane  $R^2$  or three-dimensional space  $R^3$ , or even something nonmathematical like a collection of Year 12 students. To see where the name "fuzzy" comes from, let  $S = R^2$  and consider a colour coding scheme for the membership grade, with black for 1, white for 0, and progressively lighter shades of grey as the membership grade decreases from 1 to 0. A picture in the plane could typically show a black core, representing full membership, surrounded by a fuzzy grey boundary region consisting of points of intermediate membership values, and finally a pure white region consisting of points of zero membership grade.



As you may already have suspected, fuzzy sets are very useful in pattern recognition, for example where a photographic signal from a satellite has to be converted into a visual picture. They are also used extensively in robotics and in management decision making, where the usual dichotomous TRUE/FALSE logic is limited and various intermediate grades of truth or falseness are useful. (This is what mathematicians call "fuzzy logic", quite different from the meaning implied in everyday speech!). For instance in cost accounting, all unstarted jobs are given value 0, incompleted jobs value 1/2 and completed jobs value 1. Possibly one of the most impressive uses of fuzzy sets in decision making is in the northern Japanese city of Sendai, where the entire underground subway rail system is run by computers using fuzzy decision-making algorithms.

Unfortunately, it would take too much time and advanced mathematics to go into any detail on the above applications of fuzzy sets. To end, I shall list a few elementary set theoretic operations using fuzzy sets. For simplicity these will all be defined on the same base space  $R$ . The interested reader may wish to verify them in terms of the usual crisp sets, using the characteristic functions of these sets as their membership grade functions.

(i) *union*             $h = f \vee g$  :         $h(x) = \max \{f(x), g(x)\}$

(ii) *intersection*    $h = f \wedge g$  :         $h(x) = \min \{f(x), g(x)\}$

(iii) *complement*     $f^c$  :                     $f^c(x) = 1 - f(x)$

Incidentally, the *empty fuzzy set*  $e$  has zero membership grade for all points:

$$e(x) = 0 \text{ for all } x.$$

We illustrate the operation "union":

Let  $A$  and  $B$  be two subsets of  $S$ ,  $f$  the characteristic function of  $A$  and  $g$  the characteristic function of  $B$ . We have to verify that  $h(x) = \max\{f(x), g(x)\}$  is the characteristic function of the union  $A \cup B$ . Let  $c$  belong to  $A \cup B$ , then  $c$  belongs to  $A$  or  $c$  belongs to  $B$ . It follows that either  $f(c) = 1$  or  $g(c) = 1$  (or both), so  $h(c) = \max\{f(c), g(c)\} = 1$ . If  $c$  does not belong to  $A \cup B$ , then  $c$  belongs neither to  $A$  nor to  $B$ , consequently  $f(c) = g(c) = 0$ ; in this case  $h(c) = \max\{f(c), g(c)\} = 0$ . Therefore  $h(x)$  is the characteristic function of the union  $A \cup B$ .

\* \* \* \* \*

## TWO METAPHYSICIANS - TWO VIEWS

My son, so long as you are engaged in studying the Mathematical Sciences and Logic, you belong to those who go round about the palace in search of the gate. Thus our Sages figuratively use the phrase: "Ben-zoma is still outside." When you understand Physics, you have entered the antechamber; and when, after the study of Natural Philosophy, you master Metaphysics, you have entered the innermost court, and are with the king in the same palace. - Maimonides (1135 - 1204; *The Guide for the Perplexed*)

\* \* \* \* \*

Philosophy, that lean'd on Heav'n before,  
Shrinks to her second cause, and is no more.  
Physic of Metaphysic begs defence,  
And Metaphysic calls for aid on Sense!  
See Mystery to Mathematics fly! -  
Pope (1688 - 1744; *The Dunciad*)

\* \* \* \* \*

## THE AVERAGE ACHIEVED IN TOSSING TWO DICE

G.A. Watterson  
Monash University

Over last Christmas, I played an enjoyable game called "Escape from Colditz". In the game, you toss two dice. If you get a pair, you get another toss. If you get another pair, then you get another toss again. And so on. My son asked, "What is the average accumulated total?" Here is my attempt to find out.

When you first toss the two dice, the probability that you get a pair of "1's" is  $1/36$ , because assuming each dice has probability  $1/6$  of turning up "1", and assuming the dice are independent in their behaviour, then

$$\begin{aligned} \text{Pr (both dice fall "1")} &= \text{Pr (first dice falls "1")} \times \text{Pr (second dice} \\ &\quad \text{falls "1")} \\ &= 1/6 \times 1/6 = 1/36. \end{aligned}$$

As there are 6 different ways of getting a pair with two dice (i.e. both "1's" or both "2's" or ... or both "6's"), then the probability of getting any pair is  $6 \times \frac{1}{36} = 1/6$ , and the probability of *not* getting a pair is the complementary amount  $1 - 1/6 = 5/6$ .

Suppose you *do* get a pair. Then the average total of the two dice would be 7, because the results  $1+1 = 2$ ,  $2+2 = 4$ ,  $3+3 = 6$ ,  $4+4 = 8$ ,  $5+5 = 10$   $6+6 = 12$  do average at 7:

$$\text{Average(pair case)} = \frac{2 + 4 + 6 + 8 + 10 + 12}{6} = 7.$$

On the other hand, suppose you *do not* get a pair. Then the average total would be the average of 30 possible results such as  $1+2 = 3$ ,  $2+1 = 3$ ,  $1+3 = 4$ ,  $3+1 = 4$ ,  $5+6 = 11$ ,  $6+5 = 11$ . Again the average is 7.

$$\text{Average (no pair case)} = \frac{3 + 3 + 4 + 4 + \quad + 11 + 11}{30} = 7.$$

(See if you can check this out.)

Suppose you did not have another toss whenever you obtain a pair. Then one way that you could work out the average total would be by the formula

$$\begin{aligned}
 \text{Average} &= \text{Pr (pair)} \times \text{average (pair case)} \\
 &+ \text{Pr (no pair)} \times \text{average (no-pair case)} \\
 &= \frac{1}{6} \times 7 + \frac{5}{6} \times 7 \\
 &= 7.
 \end{aligned} \tag{1}$$

Of course there are many other ways of working out this answer, too.

But if you *are* allowed repeated tosses whenever a pair is obtained, let us write  $\mu$  for the average accumulated total. If we start by tossing a pair, we average 7 for that toss and, as we are allowed to continue tossing, an extra average amount  $\mu$  is added to this to allow for the results of further tosses.

$$\text{Average (pair case)} = 7 + \mu.$$

On the other hand, if we first do *not* toss a pair, then the average total is 7 without any extra allowance for further tosses. Thus instead of (1), we get

$$\begin{aligned}
 \mu &= \text{Pr (pair)} \times \text{average (pair case)} \\
 &+ \text{Pr (no pair)} \times \text{average (no-pair case)} \\
 &= \frac{1}{6} \times [7 + \mu] \\
 &+ \frac{5}{6} \times 7, \\
 &= 7 + \frac{1}{6} \times \mu.
 \end{aligned}$$

This equation can be re-arranged to say that

$$\frac{5}{6} \mu = 7,$$

that is,

$$\mu = \frac{42}{5} = 8.4.$$

You average 8.4 for the accumulated total, when you toss two dice and are allowed further tosses whenever you get a pair.

# DESIGN FOR A PICK-UP

John Barton  
N. Carlton

A record player of conventional type, Fig. 1, has a pick-up arm PKN pivoted at P with a needle at N tracking the record groove. Usually the pick-up cartridge KN is off-set at an angle  $\beta$ , as shown, to the line PN, so that when the needle N sits in the groove, the vertical plane containing the longitudinal axis of the cartridge KN is as near as possible to being tangential to the track of the groove at the point of contact.

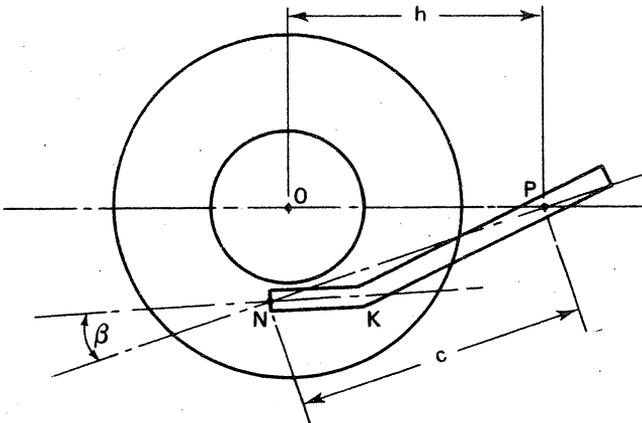


Fig. 1

Our design is to show how to calculate the position of P and the angle  $\beta$  to get the best approximation to this tangential condition for all the grooves across the record. It will become clear that, with the conventional type of pick-up arm, we cannot fulfil the condition exactly for all grooves. There will always be some "tracking error" for most of the grooves, that is, a non-zero angle between the vertical plane KN and the tangent to the track of the groove at the point of contact. The larger this angle the greater is the stress established at the contact surfaces of needle and record during the running of the record, and consequently the greater the wear on the equipment. This is one reason why we try to keep the angle as small as possible.

## THE TRACKING ERROR

This can be taken as the variation, across the record, of the angle ONP, Fig. 2, which we denote by  $\phi$ . For if  $\phi$  were constant across the record, we could then choose the angle  $\beta$  equal to  $\frac{1}{2}\pi - \phi$  so that, at all times, KN is perpendicular to the radius ON, in which case KN is very nearly tangential to the record groove at all times.

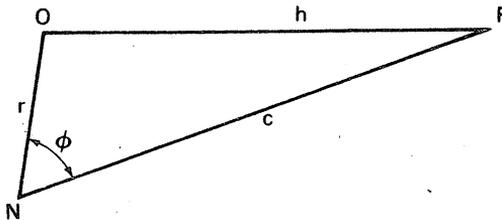


Fig. 2

Bear in mind that a groove is not exactly circular, but is an arithmetical spiral whose polar equation is  $r = k(\theta_0 - \theta)$ ,<sup>†</sup> where  $k$  is a very small constant and  $\theta_0$  a large one. The "pitch" of the spiral,  $2\pi k$ , is the change in the radius  $r$  for one revolution and is so small relative to  $r$  that the spiral can be regarded as having constant radius, that is, as being for practical purposes a circle. For a modern record, spinning at  $33\frac{1}{3}$  rpm, the pitch is of the order of 1/10 mm.

Referring to Fig. 2, we now have, by the cosine rule,

$$\cos \phi = \frac{r^2 + c^2 - h^2}{2rc} = g(r) \text{ , say,}$$

for  $a \leq r \leq b$ , where  $a, b$  are the respective radii of the innermost and outermost record grooves,  $c$  is arm length PN,  $h$  is the distance between the turntable axis and the pivot axis of the arm.

<sup>†</sup>If  $O$  is the centre of the record and  $t$  a fixed ray coplanar with the record and emanating from  $O$ , then a point  $P$  on the record is said to have polar coordinates  $(r, \theta)$  if  $r$  is the distance between  $O$  and  $P$  and  $\theta$  is the angle between  $t$  and  $OP$ , measured counterclockwise from  $t$ .

We wish to make the variation of  $g$  over the domain  $(a, b)$  as small as possible, so that the corresponding variation of  $\phi$  is likewise small. Writing

$$g(r) = \frac{r}{2c} + \frac{c^2 - h^2}{2cr} \quad (1)$$

we can easily sketch graphs of  $g(r)$  against  $r$  for various choices of  $c$  and  $h$ . Such a family of graphs is shown in Fig. 3.

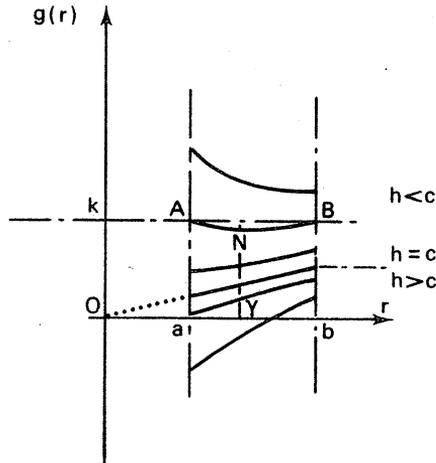


Fig. 3

If  $h = c$ ,  $g(r)$  varies linearly across the record, that is, for  $a \leq r \leq b$ .

For  $h > c$ ,  $g(r)$  increases steadily with  $r$ , the total variation  $g(b) - g(a)$  increasing as  $h$  increases. This is clearly a state of affairs to be avoided, as compared with the case  $h = c$ .

For  $h < c$ , the graph suggests that it may be possible so to choose  $h$ , given  $c$ , that the variation of  $g(r)$  for  $a \leq r \leq b$  can be minimised.

To avoid tedious calculations, we shall assume that the curve ANB, for which  $g(a) = g(b)$ , is the best one available. For this, we have

$$\frac{a}{2c} + \frac{c^2 - h^2}{2ca} = \frac{b}{2c} + \frac{c^2 - h^2}{2bc},$$

$$ab = c^2 - h^2.$$

This allows us to calculate  $h$  from the known design values of  $a$ ,  $b$ ,  $c$ . As an example, for a typical record and pick-up, we have

$$a = 70 \text{ mm}, \quad b = 145 \text{ mm}, \quad c = 225 \text{ mm},$$

so that  $h = 201 \text{ mm}$  approximately.

That is, the pick-up is to be so mounted that the needle point overlaps the turn-table centre by 24 mm.

For this case  $g(r)$  has its minimum value at  $N$ , where  $g'(r) = 0$ . Differentiating (1), we have

$$g'(r) = \frac{1}{2c} - \frac{c^2 - h^2}{2cr^2} = \frac{1}{2c} - \frac{ab}{2cr^2},$$

so that  $g'(r) = 0$  for  $r = (ab)^{\frac{1}{2}} = \gamma$ , say, the geometric mean of  $a$ ,  $b$ .

At this point  $g(r) = (ab)^{\frac{1}{2}}/c = \gamma/c$ , and, at the extremes of the radius,  $g(a) = g(b) = (a + b)/(2c) = \alpha/c$ , where  $\alpha = \frac{1}{2}(a + b)$  is the arithmetic mean of  $a$ ,  $b$ .

Thus the maximum variation of  $g(r)$  is  $(\alpha - \gamma)/c$ .

A reasonable choice for the "best" value of  $g(r)$ , that is, of  $\cos \phi = \cos(\frac{1}{2}\pi - \beta) = \sin \beta$ , to keep the variation of  $g(r)$  across the record small, is the value  $k$  as shown on the graph, where

$$k = \frac{1}{2}\{g(a) + g((ab)^{\frac{1}{2}})\} \\ = (\alpha + \gamma)/(2c).$$

This allows us to calculate the design value for the angle  $\beta$ . Referring to the above typical values we have

$$\alpha = \frac{1}{2}(70 + 145) = 107.5$$

$$\gamma = (70 \times 145)^{\frac{1}{2}} = 100.75$$

$$k = \sin \beta = 208.25/450 = 0.4628$$

$$\beta = 27\frac{1}{2}^\circ \text{ approximately.}$$

Note that, with this choice of design, there are two radii (grooves) on the record where the tracking error is zero. Between these radii the tracking error has one sign, outside them it has the opposite sign. It is a simple matter to calculate from (1) the values of these radii, by substituting for  $g(r)$  the value  $(\alpha + \gamma)/(2c)$ . This gives

$$\alpha + \gamma = r + ab/r$$

and, using the above numbers, we find the values of  $r$  to be 87.8 mm, and 130.4 mm approximately.

A little further arithmetic, not given here, will give the tracking error, that is, the variations in  $\frac{1}{2}\pi - \phi$  as between the zero value at these radii and the non-zero values at the extreme radii  $a, b$  on the one hand and the radius  $\gamma$  on the other.

This article is based on an actual pick-up made by the writer, about 1938, from part of a bicycle pedal, some pieces of brass channel and one or two other odds and ends. As well as the cartridge, of course. The device is still working well.

\* \* \* \* \*

## ACTUARIAL PROBLEMS OF 1693 WERE:

First, In that the *Number* of the People was wanting. Secondly, That the *Ages* of the People dying was not to be had. And Lastly, That both *London* and *Dublin* by reason of the great and casual Accession of *Strangers* who die therein, (as appeared in both, by the great Excess of the *Funerals* above the *Births*) rendered them incapable of being Standards for this purpose; which requires, if it were possible, that the People we treat of should not at all be changed, but die where they were born, without any Adventitious Increase from Abroad, or Decay by Migration elsewhere.

Halley (1656 - 1742; An Estimate of the Degrees of the Mortality of Mankind)

# BUILD YOUR OWN TECHNICAL WORD PROCESSOR OR BYO TWP

Jandep

In a previous article<sup>1</sup> we described how to combine EDLIN and DOSEDIT<sup>2</sup> to make a miniature word processor for the IBM PC and compatibles. DOSEDIT allows you to save in memory a stack of DOS commands which can then be inserted into a batch file by means of EDLIN. The 'DOS commands' need not be real ones at all so you can use the same technique to write a letter if you like. Now we would like to extend this notion a little further so that we have a miniature *technical* word processor, i.e. a word processor capable of incorporating mathematical symbols, foreign (or rather non-English) words and some simple diagrams such as histograms. Anyone studying mathematics or science should be interested in such a word processor - they are sometimes called *mathematical* or *scientific* word processors. Archimedes may have written his mathematics in the sand but we guarantee that a floppy disk is more portable and travels better in the mail.

Your IBM PC/PS or compatible has quite a number of mathematical symbols and Greek letters tucked away out of sight and the first thing to do is to see what is available. To see these characters on your screen hold down the Alt (Alternate) key and type in the appropriate number *on the right-hand numerical keyboard*. (Typing the numbers with the central top number keys will not do the trick.)

To start with, try the following numbers in combination with the Alt key to see on your screen (when Alt is released) the symbols indicated in the table below:

Number	224	227	228	229	230	235	236	239
Symbol	$\alpha$	$\pi$	$\Sigma$	$\sigma$	$\mu$	$\delta$	$\infty$	$\parallel$

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<sup>1</sup> See *Function* Vol 11, Part 5, 1987.

<sup>2</sup> EDLIN is the line editor provided with DOS, DOSEDIT is a famous public domain program.

To see the full range of mathematical symbols you should try out *all* of the numbers between 224 and 253. There are a few others at 124, 164 - 167 and 171 - 175, for example, which might also be of interest. To get an integral sign experiment with 244, 245, and 124. Unfortunately, there seems to be no logical arrangement about the association between number and symbol.

Sometimes a mathematician or scientist needs to use some non-English letters and a similar trick will produce some of these on the screen. The relevant numbers are mainly those between 128 and 163. Here are a few examples:

Number	128	130	143	168
Letter	ç	é	À	ı

(Linguists should also be interested in this aspect of the extended character code.) The characters supplied are those used in modern European languages; there are no Russian, Gothic, Hebrew, or Chinese characters.

There are also some characters which could be useful for producing big brackets, histograms and other simple diagrams or 'boxes' for enclosing text; for these the numbers range between 169 and 223 mostly. Here are a few of these items:

Number	187	206	218	179	193
Character	⌈	⌋	⌈		⌋

You may be able to find a table showing all the IBM character codes in your textbook on BASIC or in your printer manual, or you could construct your own on screen with the following True BASIC program:

```

For n = 128 to 254
    Print n; chr$(n),
Next n
end

```

To print the table produced by this program on the screen requires the True BASIC command RUN; if you wish it printed to your printer use the True BASIC command ECHO followed by RUN. This may be the quickest way to find out whether your printer can copy the symbols you see on your screen.

Having found out how to display the hidden characters on the screen we can now use EDLIN and DOSEDIT in combination as before to produce a mathematical document. Let DOSEDIT take over by typing in

DOSEDIT

and pressing Enter.

Then enter the following text line by line:

The formula for the  
arithmetic mean is

$$\mu = \Sigma X/n$$

and for the variance is

$$\sigma^2 = \Sigma x^2/n$$

where  $x = X - \mu$

Ignore all the grumbling from DOS about 'bad' commands; each line will be saved on the stack by DOSEDIT even if DOS gets furious. Incidentally, the exponent in the second last line is at 253.

Now start to make your document file by entering

EDLIN math.doc

and

I

to go into input mode. Use the cursor keys to pull down from the stack the lines stored there and finish with

Ctrl Z

(hold down Ctrl and press Z) and then type

EXIT

followed by Enter.

Your mathematical document will be saved on disk. Notice that as we are discussing only text or document files here we use the extension .doc for the filenames. By using the TYPE command you can subsequently read your document on the screen. The DOS command would be, in this case,

TYPE math.doc

since the filename was math.doc.

If you want your mathematical document to be printed you may have to adjust your printer to graphics mode; on an Epson LX86, for example, this means setting the dip switch 1-2 up (for on) but see your printer manual for the dip switch settings. Otherwise, if your LX86 is set for standard printing you will get various italic letters being printed in place of the graphics characters you see on your screened document. To print your document (assuming you have a suitable printer) use: COPY math.doc PRN or one of the alternative DOS commands for printing. Some IBM PC printers do not

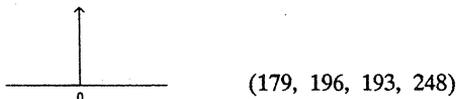
display characters as they appear on the screen; you must experiment or consult your printer manual.

Try some further experiments with the characters - for example, prepare a file (call it fame.doc perhaps) with the following content:

François Viète	(135,138)
L'Hôpital	(147)
Gödel	(148)
Dieudonné	(130)
Pólya	(162)

The numbers on the right are there to help you find the right foreign language letter - there is no need to put them in the document. The names are all those of famous mathematicians.

Or you might like to try a simple graph, for example, try to make a graph like this (the numbers are character codes):



Our miniature technical word processor is quite limited in its capabilities and certainly does not have all the features one would expect to find in a fully developed word processor. From a mathematical point of view one of its major defects is that it does not allow for the inclusion of *both* italics and mathematical symbols in the text - you may have either, according to the setting of your printer, but not both. Most mathematicians would regard this as a fatal flaw and a thoughtful glance at any well-printed mathematical text should tell you why.

Some word processors exploit the IBM extended set of characters we have been examining in order to make available a limited handling of mathematical and foreign language text. A notable example is PC Write in its latest version (2.71). As PC Write is a well-known and widely used shareware product you should be able to get a free copy from a friend or User Group. It has many powerful features far beyond the capabilities of our miniature technical word processor and *provided your mathematical requirements are not very great* you might be very happy with the facilities provided by PC Write. The manual obtainable on registering is very extensive and might provide sufficient instruction for someone just beginning to use a PC and thinking about the need for a word processor. The method of entering the mathematical characters is, however, not much of an improvement over the method used by our miniature technical word processor and the range of mathematical and foreign language fonts is just as limited. If you intend

to be a professional mathematician, scientist or engineer you should consider going straight to a fully-fledged technical word processor.

There are now many technical word processors and the area is one which is rapidly developing. One place you can find up-to-date information about technical word processors and their capabilities for typing mathematics is in the *Notices of the American Mathematical Society* where there is an occasional series of articles edited by Richard S. Palais devoted to this subject. The articles have appeared over the last two or three years and are still continuing. Two recent articles giving a list of the capabilities, or lack of them, of about twenty technical word processors on the market and, in the later article, offering reviews of each are in the February and April 1987 issues of the *Notices* pp 262 - 281 (February) and pp 462 - 491 (April). Other sources of information about this variety of software are unfortunately not very well informed at this time. The AMS is fortunate to have the active co-operation of the Boston PC User Group.

The only technical or scientific word processor we know of for which there exists a public domain version is Chiwriter<sup>3</sup> - often rather misleadingly described as a 'multifont' word processor in recognition of its many fonts but ignoring its special relevance to scientific and mathematical text. Because it is marketed through the public domain user network you should be able to get it from a scientific PC user friend or a science-minded User Group. The public domain version comes ready equipped with an on-disk tutorial which is an excellent electronic first lesson in word-processing in general as well as a training for the use of Chiwriter in particular. The tutorial does not, however, demonstrate all the Chiwriter facilities - for this you really need the manual obtainable on registration.

PC Write has been advertised in the popular PC magazines for some years. Very recently we noticed, for the first time, a similar advertisement for Chiwriter. Amongst all the high-powered advertisements appearing in such magazines these small ads are not easy to spot so you need to look carefully if you want to know about current prices for registration, supplier's address and other matters for these two word-processors. Alternatively, the public domain version of each is accompanied by an explanatory document providing such details.

Time to own up! This paper was composed using Chiwriter which is also used to print *Function*. All examples, however, were given a trial run using only EDLIN and DOSEDIT. Equipment used included an IBM PC clone with CGA and an Epson LX86 for draft versions and a Hewlett Packard Series II Laser Jet for the final version.

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<sup>3</sup> Anyone having information about any other should please let us know.

## PROBLEM SECTION

### 12.1.1

It is a well known fact that if  $0 < q < 1$   
 then  $1 + q + q^2 + \dots + q^n$  tends to  $\frac{1}{1-q}$  as  $n \rightarrow \infty$ .

Let  $\{a_n\}$  be any sequence of numbers that converges to  $q$ . Show that the limit of the sum does not change if we replace  $q$  by  $a_n$ ,  $q^2$  by  $a_n a_{n-1}$  and so on,  $q^n$  we replace by  $a_n a_{n-1} \dots a_1$

Namely, show that

$1 + a_n + a_n a_{n-1} + \dots + a_n a_{n-1} \dots a_1$  tends to  $\frac{1}{1-q}$  as  $n \rightarrow \infty$ .

### 12.1.2

Find the angle  $x$  as shown in Fig. 12.1.2.

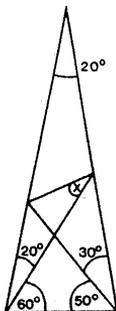


Fig. 12.1.2

Both problems were passed to FUNCTION by Dr.F.C. Klebaner of Monash University.

## TWO NORWEGIAN UNIVERSITIES - TWO VIEWS

I have often set myself problems beyond my powers. I have nevertheless arrived at a large number of general results which throw a strong light on the nature of those quantities whose elucidation is the object of mathematics.

Niels Henrik Abel,  
Kristiania University (Norway).

\* \* \* \* \*

We have not succeeded in answering all our problems. The answers we have found only serve to raise a whole set of new questions. In some way we feel we are as confused as ever, but we believe we are confused on a higher level and about more important things.

Posted outside the mathematics reading room,  
Tromsø University (Norway)

\* \* \* \* \*

## PERDIX

Australia is host to the International Mathematical Olympiad (IMO) in the bicentennial year, 1988. Despite the distance of Australia from many of the countries participating a record number of teams is expected to compete this year. The IMO takes place in Canberra in mid-July, one of the official bicentennial events for 1988.

Teams are due to arrive in Sydney on July 9 and after a couple of days rest and a welcoming function, they travel to Canberra on Monday July 11. The first IMO paper, consisting of three questions with  $4\frac{1}{2}$  hours allowed for their solution, is tackled on July 15. On the next day the second paper of three questions, also taking  $4\frac{1}{2}$  hours, is tackled. Medal awards are made and results are announced on Wednesday July 20. Perdix will be there to ensure that all goes smoothly.

Each team that participates is accompanied by a team leader and by a deputy team leader, and may offer a maximum of 6 questions to be considered as possible questions for selection for the Olympiad competition. If 50 teams compete then this may result in 300 questions being offered from which the final 6 are eventually selected for the two IMO papers. The job of selection is a formidable one.

The team leaders are separated from their teams shortly after arrival and it is their task to select these six questions. The team leaders will not be speaking 50 different languages, but most of them will be fluent only in their native language, and the job of ensuring clear communication will be as difficult as that of judging between the merits of the 300 questions.

This task has to be completed sufficiently before the day of the first competition paper, taking place only six days after the arrival of the teams in Australia, for there to be time for the papers to be printed, and for each paper to be translated into the appropriate language for each team. The translations have to be carefully discussed to ensure that they are logically correct and are written clearly and idiomatically. This is a vast task for which a large number of Australian mathematicians have been collected who combine a knowledge of the languages concerned with their understanding of the mathematics.

Equally difficult is the marking process once the competition is over. Each team's papers are first marked by its leaders. Then it is necessary to try and ensure that the same criteria are being applied to all candidates. Interpreters are again required to make the discussions (and disputes) possible.

It might perhaps surprise some to hear that it is difficult to agree on a mark for the solution to a mathematical problem. If the solution is fully correct, and this is not questioned, then of course the mark to be given is not too difficult to decide. But even then there is a problem. Some correct solutions are long-winded. Some seem to have been produced without true understanding. Others are elegant. Occasionally a solution involves insight that none of the leaders had expected: a totally new and beautiful solution has been discovered. Are those solutions all worth full marks?

When the solution is not complete, then the mark to be offered is much harder to agree on. Team leaders argue for their own teams, which ensures justice is done. Has a convincing case been made that what a candidate wrote was on the right track? These and similar questions are argued until the small hours of the morning: the time is so short (fortunately!), the results have to be announced four days later!

\* \* \* \* \*

Let us now turn to a problem I set last year, Problem 11.3.1 (page 96, Volume 11, pt 3). The problem is :

If  $a$  and  $b$  are positive and  $a + b = 1$ , show that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

A solution involving the Mean Value Theorem, of the calculus, produced by Dr.J.G. Kupka, was printed in the next issue (p. 128).

The Mean Value Theorem is not a result studied in most high schools, and when I set the problem I was hoping for a solution not involving the calculus. One has not been forthcoming. So I now offer one.

Showing that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 \geq \frac{25}{2} \text{ is equivalent to}$$

showing that

$$\left(a + \frac{1}{a}\right)^2 - \frac{25}{4} \geq \frac{25}{4} - \left(b + \frac{1}{b}\right)^2$$

i.e.

$$a^2 - \frac{17}{4} + \frac{1}{a^2} \geq \frac{17}{4} - b^2 - \frac{1}{b^2}$$

i.e.

$$\left(\frac{4}{a} - a\right) \left(\frac{1}{4a} - a\right) \geq \left(\frac{4}{b} - b\right) \left(b - \frac{1}{4b}\right). \quad (1)$$

If  $a = b = \frac{1}{2}$ , then we get equality.

Otherwise, without loss of generality, we can assume  $0 < a < b < 1$ .

Then

$$\frac{4}{a} - a > \frac{4}{b} - b, \text{ (because } \frac{4}{a} > \frac{4}{b} \text{ and } -a > -b) \quad (2)$$

hence

$$\frac{4}{a} - \frac{4}{b} > a - b$$

Also,

$$\frac{1}{4a} - a > b - \frac{1}{4b}, \quad (3)$$

because it can be rearranged to assert that

$$\frac{1}{4}\left(\frac{1}{a} + \frac{1}{b}\right) > a + b$$

i.e.

$$1 > 4ab, \text{ since } a + b = 1$$

i.e.

$$1 > 4a(1 - a)$$

i.e.

$$(2a - 1)^2 > 0, \text{ which is so since } a \neq \frac{1}{2}.$$

Hence the result (1) follows from (2) and (3), the factors in (1) being the terms considered in (2) and (3).

\* \* \* \* \*

## PERDIX - A WOMAN?

Robert von Ranke-Graves wrote from Deyá, Mallorca, Spain: "Perdix (<partridge>) was the name borne by the sister of Daidalos, the smith". The writer is known to have composed a treatise with the title "The White Goddess".

(Ed.)

\* \* \* \* \*

