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FUNCTION

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THE
NORMAL
LAW OF ERROR
STANDS OUT IN THE
EXPERIENCE OF MANKIND
AS ONE OF THE BROADEST
GENERALIZATIONS OF NATURAL
PHILOSOPHY ◆ IT SERVES AS THE
GUIDING INSTRUMENT IN RESEARCHES
IN THE PHYSICAL AND SOCIAL SCIENCES AND
IN MEDICINE AGRICULTURE AND ENGINEERING ◆
IT IS AN INDISPENSABLE TOOL FOR THE ANALYSIS AND THE
INTERPRETATION OF THE BASIC DATA OBTAINED BY OBSERVATION AND EXPERIMENT

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It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. *Function* is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Welcome to new readers. We hope you enjoy reading *Function* this year. If there is some subject you are specially interested in and would like to see an article about write to us and we shall try to produce one.

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Front Cover: This word picture is taken from p.143 of the beautiful book by Edward R. Tufte entitled *The visual display of quantitative information.* Tufte describes the picture as a "typographical delight" and attributes it to the statistician W.J.Youden. The book may be purchased (only) by direct order from the Graphics Press, Box 430 Cheshire, Connecticut, USA, 06410, price US$34.

The book is a magnificent account of the art of pictorial presentation of information. It includes a short history with superbly chosen examples. Nearly every page contains two or three memorable pictures, mostly of vividly effective presentation of numerical data but with the addition of some pictures showing how it should not be done.

* * * * *

THE DEVELOPMENT OF ALGEBRA

Lyn Donaldson, Keilor Downs post-primary school

A lot of the mathematics that we study today was discovered many centuries ago. Algebra, 'the science of equations', was first used in 1700 B.C. and developed in three distinct stages. At first the equations were written entirely in words, then some abbreviations were introduced until finally symbols replaced the words. This took place over a long period of time.

Algebra originated in Babylonia at about 1700 B.C. A typical problem from this time is shown below.

1. Length, width. I have multiplied length and width, thus obtaining area: 252. I have added length and width: 32. Required: length and width.
2. Given 32 the sum; 252 the area.
3. Answer 18 length, 14 width.
4. One follows this method: Take half of 32 (this gives 16).
   Multiply this by itself (16x16 = 256).
   256-252 = 4. The square root of 4 is 2.
   16+2 = 18 length. 16-2 = 14 width.
5. Check: I have multiplied 18 length by 14 width.
   18x14 = 252 area.

The above method involves stating the problem, listing the data, giving the answer, showing the method and finally checking the answer. This is not all that different from our methods today but we should note that the Babylonians did not have numbers as simple as ours or simple symbols to represent the operations.

Algebra was further developed by the Greeks (540-300 B.C.). Most of the work from this time was concerned with geometry and two of the greatest mathematicians - Euclid and Pythagoras - made some contributions to algebra. A typical problem would have looked like this:

> If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts, together with twice the rectangle contained by the parts.

What does this mean in our language?

The next step in the development of algebra was made by another famous Greek. His name was Diophantus and he is known as 'the father of algebra'. Diophantus wrote a series of books called the Arithmetica which included problems and their solutions. However, we know very little about him. It is believed that he lived between 250 and 300 A.D. and one clue we have to his life is a problem about him.

> "Here lies Diophantus. The wonder behold – through art algebraic, the stone tells how old: "God gave him his boyhood one-sixth of his life, one-twelfth more as youth while whiskers grew rife; and then yet one-seventh ere marriage begun; in five years there came a bouncing new son. Alas, the dear child of master and sage, met fate at just half his dad's final age. Four years yet his studies gave solace from grief; then leaving scenes earthly he, too, found relief."

How old was Diophantus when he died?

Diophantus was the first to use algebraic abbreviations in problems, which helped make life easier for the mathematicians of the time. He also used special symbols which were previously not thought of. This helped algebra develop because it simplified the problems which previously would have been written in words only. However, the work was still very tedious compared to our methods.

Here is an example of Diophantus' symbols.

Consider the expression

$$x^3 - 5x^2 + 8x - 1.$$

Before Diophantus this would be written in words. Diophantus would have written

$$C^u 1 \text{ NU8 LE Sq5 U1}$$

where C^u = cube of the unknown; NU = the unknown; LE = less (minus); Sq = square of unknown; U = unit

The use of symbols such as these helped with the general development of mathematics because it made life easier. It especially favoured the development of trigonometry and number theory because now there was a way to express higher powers and unknowns. Mathematics still had not developed special symbols for operations that we have now, but over the years mathematicians have worked on these symbols and developed them into the algebra and arithmetic that we now use.

This is a problem from Diophantus' *Arithmetica*

> "Find three numbers so that the product of any two plus the sum of the same two shall be given numbers."

In modern notation this problem reads: given numbers a, b, c find numbers x, y, z such that

$$yz + y + z = a$$
$$zx + z + x = b$$
$$xy + x + y = c$$

[Readers are invited to send solutions. For example try with $a = 11$, $b = 7$, $c = 5$.]

THE DOMESDAY* METHOD

DAVID JOHNSON, UNIVERSITY OF NOTTINGHAM

Of the many methods for computing the day of the week upon which a given date falls, my favourite is that devised by Professor J.H. Conway, F.R.S., of Gonville and Caius College, Cambridge. Apart from his contributions to Group Theory, Topology ("An enumeration of knots and links and some of their algebraic properties", pp. 329-358 in Computational Problems in Abstract Algebra, Pergamon, Oxford, 1970 : despite the title, a joy to read), and Logic ("All Numbers Great and Small!"), Professor Conway is justly famous as the inventor of many mathematical games, puzzles and diversions, such as the "Game of Life", upon which several million dollars' worth of computing time has been expended. The method described below is not the least of his brainchildren.

There are two reasons why evaluating the function d : calendar date → day of the week is non-trivial. First, the number of days in a year is never divisible by 7, and second, in the words of W.S.Gilbert,

> Although, for such a beastly month as February,
> Twenty-eight days are as a rule considered plenty,
> It has been decreed that, one year in every four,
> Its days shall be numbered as nine-and-twenty.

On the other hand, the function d takes only seven values, and the last day of February is fairly near the beginning of the year.

The first step capitalizes on this fact, and goes straight to the heart of the problem by declaring that the day of the week that is the *last* day of February in any year be the Domesday* for that year. For example, February 28 falls on a Saturday in 1987, and so the Domesday for this year is Saturday. In 1988, February 29 will fall on a Monday, and so Monday is the Domesday for 1988.

We proceed to find (at least) one Domesday in every other month, using the following simple rules:

a) for *even* months, April = 4, June = 6, ..., December = 12, a Domesday is the day whose date is the number of the month; for example, June 6 is always a Domesday: check, for example, that June 6 and February 28 are both Saturdays in 1987;

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* Domesday is the Middle English spelling of Doomsday. [Ed.]

b) for odd months after February, it is the day that has date equal to the number of the month $\pm$ 4, with + for long months (31 days) like May, and − for short months (30 days) like November;

c) January 31 is a Domesday, except in leap years, when it is a day later, so in this case, pick your favour from 4, 11, 18, 25 to find which day is a Domesday.

Month	Number	Days	Domesday
January	1	31	(See (c) above)
February	2	28 or 29	last
March	3	31	7
April	4	30	4
May	5	31	9
June	6	30	6
July	7	31	11
August	8	31	8
September	9	30	5
October	10	31	10
November	11	30	7
December	12	31	12

With this information, it is now a matter of simple arithmetic (modulo 7) to arrive at the day of the week for any day in any month.

For example, suppose someone you know was born upon December 22 in 1963. Given that the Domesday for that year was Thursday, you know that so is December 26 (= 12 + 14) (Boxing Day is always a Domesday, as is American Independence Day and possibly some special dates of your own) and by simply counting back four days, you deduce that they are a Sunday's child.

There remains the problem of establishing the Domesday for a given year, and this is neatly solved as follows. First note that a given century also has a Domesday, namely that of the first (really the zeroth!) year of that century. The Domesday for the twentieth century is Wednesday. Note that (since $365 = 7 \times 52 + 1$), the Domesday moves forward one for each normal year, plus an extra one for leap years, and thus by fifteen, i.e. to the next day of the week, every 12 years. So to obtain the Domesday for nineteen hundred and k, $0 \le k < 100$, move on N days from Wednesday, where N is obtained as follows: divide k by 12 :

$$k = 12q + r \quad , \quad 0 \le r < 12 \quad ,$$

so that $q = [k/12]$, the "integer-part" of $k/12$, whereupon

$$N = q + r + [r/4].$$

Thus, for 1988,

$$q = \lfloor 88/12 \rfloor = 7 \ , \ r = 4 \ , \ \lfloor r/4 \rfloor = 1 \ ,$$

$$\text{and} \quad N = 7 + 4 + 1 = 12 = 7 \times 2 - 2$$

and the Domesday is two days before Wednesday, i.e. Monday, as we saw earlier.

In case you should wish to carry this out for dates in the last century, I should mention that the Victorians' Domesday was a Friday, since the year 1900 was not a leap year (they miss it out every hundred years); but the Domesday for the 21st century will be Tuesday (take $k = 100$ in the previous paragraph) since the year 2000 *is* a leap year (they put it back every four hundred years). There is also a version of the method that works for the Julian calendar, which was replaced by the Gregorian calendar in 1752 (in Britain at least - in Russia, this did not take place until 1923).

Now the method may look complicated, but facility comes with remarkably little practice (best done with an equally quick-witted friend!), and you will find you can perform the whole calculation in a few seconds, and have what is among other things an impressive party trick. In fact, by asking for year first, then month, and finally day and using some glib patter, you can give the appearance of producing the answer instantaneously. When doing this with birthdays you can follow up with a reference to the famous rhyme, which I can never remember, "Monday's child ..." You can also check dates appearing in books (there is at least one mistaken one in the Sherlock Holmes' stories) and elsewhere; I leave other applications, both academic and social, to your imagination.

* * * * *

STEPHEN MURPHY WINS AUSTRALIAN BHP SCIENCE PRIZE

Stephen Murphy won this year's BHP Science Prize for excellence in scientific research by Australian school students. Stephen, at 12 years old, is the youngest winner of the prize. The prize included a gold medal, a cheque for $5000, and a trip arranged to the 38th International Science Fair at Puerto Rico, in May.

Stephen's research was an investigation into the glacial region of New Zealand's South Island. His interest was aroused three years ago into its many curious features : the milky water of glacial rivers, the odd places large boulders are dotted about, the mysterious circular cavities in gorges.

He drew up a research plan and, armed with appropriate instruments, on his last holiday in New Zealand, collected measurements of river flows, sedimentation rates, air and water temperatures. Samples were bought back to study under microscopes in his school lab. He then began analysing his data using computer simulation models. In particular his models enabled him to estimate the rate at which silt would build up behind dams on glacial rivers, important information to have when planning hydro-electric schemes.

PRIZES AND MEDALS

Michael A.B. Deakin, Monash University

The Nobel Prizes, announced annually, recognise excellence in Chemistry, Physics, Physiology and Medicine, Literature, Peace and (more recently) Economics. Except for the last of these, the money for the prizes comes from the estate of the Swedish chemist and industrialist *Alfred Bernhard Nobel* (1833-1896). Nobel studied, patented, and manufactured explosives, being most remembered as the inventor of dynamite. He funded his prizes from the wealth these activities brought him, and did so in the hope of leaving a lasting legacy of service and benefit to humanity.

Many fields of endeavour are not recognised, and among these is Mathematics. Why this is so is not really known. There does seem to be some evidence that Nobel at one point considered including Mathematics among the fields chosen for the awards.

One widely believed explanation for his change of heart (if in fact there was one) is his bad relations with the Swedish mathematician Mittag-Leffler. *Magnus Gustaf* (or *Gösta*) *Mittag-Leffler* (1846-1927) was a mathematician of some note, nowadays better remembered perhaps as an organiser and editor than for his technical contributions to Mathematics itself. He had studied under one of the very greatest of all mathematicians, the German *Karl Theodor Wilhelm Weierstrass* (1815-1897). Weierstrass made major contributions to calculus, geometry and approximation theory. He also changed for all time the standard of mathematical rigour and the concept of mathematical proof. Mittag-Leffler was one of many students who spread Weierstrass's influence and so helped to form the shape of modern mathematics.

Had there been a Nobel prize for Mathematics, it is just conceivable that Mittag-Leffler might, one year, have won it. Almost certainly, as the leading Swedish mathematician of his day, he would have played a part in administering it. So it could be that Nobel, if he were ill-disposed towards Mittag-Leffler, might have forestalled both these eventualities by deciding to scrap all ideas of a Nobel Prize for Mathematics.

The alleged quarrel between Nobel and Mittag-Leffler is supposed to have been the result of their rivalry for the affections of another mathematician: Sonya Kovalevsky. This story is in print : Solomon W. Golomb tells it in the journal Cryptologia (Jan.1980) and *Function's* Dutch counterpart, *Pythagoras* ran it in December 1983, and a translation of *Pythagoras'* article appeared in the Winter 1984 issue of the Belgian *Maths Jeunes*.

Let me tell you how the story goes.

Sonya Kovalevsky (1850–1891) was a Russian who escaped that country by the expedient of contracting a marrage of convenience and so being allowed to go abroad to enter a university. This device was the only one available at the time, and women seeking education had no other recourse but to adopt it. The couple did go abroad and, in large measure, drifted apart as such couples were expected to do – she to go to study with Weierstrass.

Her difficulties as a woman were enormous, and even Weierstrass's powerful help did not always win the day for her. Eventually she resumed her married life, returned to Russia and bore a daughter. The couple, however, were in deep financial trouble, and this phase of Sonya's life ended abruptly with the consequent suicide of her husband in 1883.

She turned, in this crisis, to Weierstrass and he contacted Mittag-Leffler, who offered her a lectureship, later upgraded to a professorship, in Stockholm. She thus became the world's second woman professor of Mathematics (after Marie Agnesi : see *Function Vol.*10, *Part* 4). Her contributions to mathematics were very significant, involving calculus, mechanics and the theory of Saturn's rings.

She also wrote several novels and was a political radical, especially in the area of women's rights. Even by today's standards, let alone those of 100 years ago, she would be classed as a radical feminist. Her early death was a great loss to the world.

Now this much is fact – but what of the story of her amorous liaisons with Nobel and Mittag-Leffler? Beyond the obvious – that the three were often in the same city at the same time, and so the men could have fallen out over the woman, there is very little evidence I have seen to support it. Certainly she was a colleague of Mittag-Leffler's and was in his debt for getting her her job, but that's not the same thing as having an affaire with him.

Earlier, indeed, she had been very close to Weierstrass, and malicious tongues had wagged, but it would seem that the gossip (which hurt Weierstrass very deeply) was unfounded and that the relationship was purely platonic. Certainly Mittag-Leffler, writing late in his life on the two and their relationship, leads one to this view. We may perhaps take the same view on her alleged affaire with Mittag-Leffler.

As for Nobel, although he never married, there ~~was~~ were a number of women in his life, and his official biography deals with these relationships in considerable detail. Nowhere, however, does it mention Sonya Kovalevsky at all!

If either Golomb or the anonymous author of the *Pythagoras* article had offered any evidence for the story, ~~that~~ it might perhaps be checked, but they don't, so I remain sceptical.

Anyhow, for whatever reason, there is no Nobel Prize in

Mathematics. This story has an interesting sequel.

John Charles Fields (1863–1932) was a Canadian mathematician who had a close mathematical association with Mittag-Leffler. Like Mittag-Leffler, he is best remembered today for the organisational and administrative work he did rather than for his actual mathematical research.

Every four years (apart from war-time interruptions) an International Congress of Mathematicians is held, and that for 1924 took place in Toronto and was organised by Fields. This congress left a surplus of funds, and Fields suggested that "International Medals" might be struck and awarded for excellence in mathematical research.

The fact of there being no Nobel Prize in Mathematics and of Fields' friendship with Mittag-Leffler has led to the speculation that Fields wanted to compensate for the lack, but, although the theory is an attractive one, again there is no real evidence for it.

Fields' proposal had still not been implemented by the time of his death in 1932, but his will left further funds and later that year his proposal was accepted. The first medals were awarded in 1936.

Medals (between two and four of them) have been given at each of the congresses since then. The next was not held till 1950, but they have continued unbroken since then (although the 1982 congress very nearly didn't happen, due to political developments in Warsaw – it went ahead a year late).

Whatever the history of the matter, mathematicians regard these medals as being the mathematical equivalent of a Nobel Prize. There are, however, some important differences. There is no cash award, as with the Nobel Prizes, and the rule has grown up that the recipient is to be under forty years of age. (The Nobel Prizes have no such restriction.)

The last congress was held in 1986, and three medals were awarded. The recipients were Michael K. Freeman, an American, Simon K. Donaldson, an Englishman, and Gerd Faltings, a German. Freeman and Donaldson won their medals for work on the topology of four-dimensional spaces, and Faltings was honoured for his proof of Mordell's Conjecture. (See *Function Vol. 7, Part 5.*)

One final irony. Fields, in his will, stipulated that the medals should not bear "in any way, the name of any country, institution, or person". They are universally known today as the "Fields Medals".

* * * * *

GOON SHOW

Moriarty: 'How are you at Mathematics?'

Neddy Seagoon: 'I speak it like a native.'

TARTAGLIA AND CARDANO†

John Stillwell, Monash University

Niccollò Tartaglia

Niccollò Tartaglia was born in Brescia in 1499 or 1500 and died in Venice in 1557. The name "Tartaglia" (meaning "stutterer") was actually a nickname, and his real name is believed to have been Fontana. Tartaglia's childhood was scarred by poverty, following the death of his father, a mail courier, around 1506, and injuries suffered when Brescia was sacked by the French in 1512. Despite taking refuge in the cathedral, Tartaglia received five serious head wounds, including one to the mouth which left him with his stutter. His life was saved only by the devoted nursing of his mother, who literally licked his wounds. Around the age of 14, he went to a teacher to learn the alphabet, but ran out of money for his lessons by the letter k. This much is in Tartaglia's own sketch of his life. After that, the story goes, he stole a copybook and taught himself to read and write, sometimes using tombstones as slates for want of paper.

By 1534 he had a family and, still short of money, he moved to Venice. There he gave public mathematics lessons in the church of San Zanipolo, and published various scientific works. The famous disclosure of his method for solving cubic equations occurred on a visit to Cardano's house in Milan on 25 March 1539.

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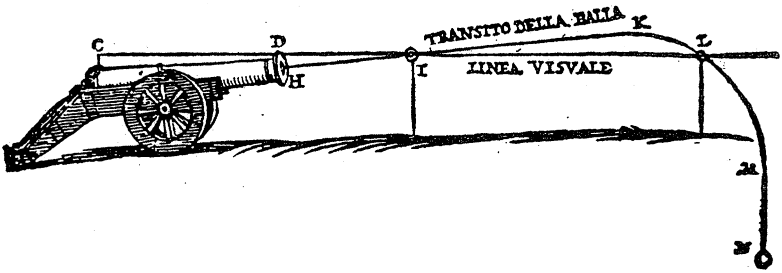
† Extracted from a book on the history of mathematics being written by John Stillwell.

When Cardano published it in 1545, Tartaglia angrily accused him of dishonesty. Tartaglia claimed that Cardano had solemnly sworn never to publish the solution, and to write it down only in cipher. Ferrari, who had been an 18-year old servant of Cardano at the time, came to Cardano's defence, declaring that he had been present and there had been no promise of secrecy. In a series of 12 printed pamphlets, known as the *Cartelli* (reprinted by Masotti [1947]), Ferrari and Tartaglia traded insults and mathematical challenges; the two finally squared off in a public contest in the church of Santa Maria del Giardino, Milan, in 1548. It seems that Ferrari got the better of the exchange, as there was little subsequent improvement in Tartaglia's fortunes. He died alone, and still impoverished, 9 years later.

Apart from his solution of the cubic, Tartaglia is remembered for other contributions to science. It was he who discovered that a projectile should be fired at 45° to achieve maximum range. His conclusion was based on incorrect theory, however, as is clear from Tartaglia's diagrams of trajectories, e.g. see the picture below, reproduced from Tartaglia's works.



Tartaglia's Italian translation of the *Elements* was the first printed translation of Euclid in a modern language, and he also published an Italian translation of some of Archimedes' works.

Girolamo Cardano often described in English books by the anglicised name name Jerome Cardan, was born in Pavia in 1501 and died in Rome in 1576. His father Fazio was a lawyer and physician who encouraged Girolamo's studies, but otherwise seems to have treated him rather harshly, as did his mother Chiara Micheri, whom Cardano described as "easily provoked, quick of memory and wit, and a fat, devout little woman". Cardano entered the university of Pavia in 1520 and completed a doctorate of medicine at Padua in 1526.

He married in 1531 and, after struggling until 1539 for acceptance, became a successful physician in Milan. So successful, in fact, that his fame spread all over Europe. He evidently had a remarkable skill in diagnosis, though his contributions to medical knowledge were slight in comparison with those of his contempraries Andreas Vesalius and Ambroise Paré. Mathematics was one of his many interests outside his profession. Cardano also secured a niche in the history of cryptography for an encoding device known as the Cardano grille and in the history of probability, where he was the first to make calculations, though not always correctly.

The violence and intrigue of Renaissance Italy soured Cardano's life just as much as Tartaglia's, though in a different way. An uncle died of poisoning, attempts were made to poison both Cardano and his father (so Cardano claimed) and in 1560 Cardano's oldest son was beheaded for the crime of poisoning his wife. Cardano, who believed his son's only fault was to marry the girl in the first place, never got over this calamity. He could no longer bear to live in Milan and moved to Bologna. There he suffered another blow when his protegé Ferrari died in 1565 - poisoned by his sister, so it was said. In 1570 Cardano was imprisoned by the Inquisition for heresy. After a few months he recanted, was released, and moved to Rome.

In the year before he died, Cardano wrote *The Book of My Life* which is not so much autobiography as self-advertisement. It contains a few scenes from his childhood, and returns again and again to the tragedy of his oldest son, but most of the book is devoted to boasting. There is a chapter of testimonials from patients, a chapter on important people who sought his services, a list of authors who cited his works, a list of his sayings he considered quotable, and a collection of tall stories which would have done Baron Von Münchhausen proud. Admittedly, there is also a (very short) chapter "Things in which I have failed" and frequent warnings about the vanity of all earthly things, but Cardano invariably tramples all such outbreaks of humility in his rush to admire other facets of his excellent self.

As for the quarrel with Tartaglia, *The Book of My Life* is almost silent. Among the authors who have cited him, Cardano lumps Tartaglia with those for whom he "cannot understand by what impertinence they have managed to get themselves into the ranks of the learned". Only at the end of the book does Cardano concede that "in mathematics I received a few suggestions, but very few, from brother Niccoló".

YELLOW LIGHTS

Michael A.B. Deakin, Monash University

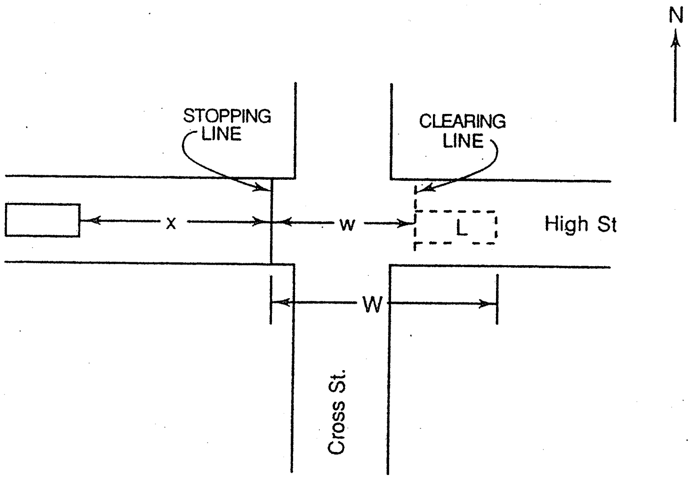
Many readers of *Function* will be learning or thinking about learning to drive a car. And among the many skills this entails is judging what to do if the light you are approaching changes from green to yellow. This is an important question not only for the driver but also for the traffic engineer who designs the lights in the first place. It is a problem that has attracted quite a lot of mathematical attention, and here I would like to give some of the analysis involved.

The problem was first addressed in the immediate post-war years (late '40s) in the U.S. but the first really thorough and accurate study was by Gazis, Herman and Maradudin, three engineers with the General Motors Corporation. This was published in 1960 in the journal *Operations Research*. Their paper is in the main a theoretical analysis of the issues involved. It was followed by an observational study by two other General Motors researchers, Olson and Rothery, who, from camera recordings of driver behaviour, estimated the values of the various parameters used in the theoretical analysis given by the first group. This second study was also published in *Operations Research* in 1961.

In 1962, another theoretical analysis appeared, independently of the others. This is a little more accessible, being rather less technical. It appeared in the *American Journal of Physics* and its author was Howard Seifert of Stanford University. These early studies set the basis for the analysis, and by 1981, Fred Watts of The College of Charleston, in South Carolina, was using the situation for laboratory classwork for his students. (He described this in *The Physics Teacher* for that year.) These discussions are summarised also by Jearl Walker in *Scientific American* (March 1983) along with a lot more interesting questions on traffic lights. (One of these is very like Problem 3.4.1 that appeared in *Function* in 1979.)

So if you would like to see more on this topic, there is no shortage of reading material!

You will notice that all the above references are American, and this is important in one key respect. Look at Figure 1, showing the interesection of High St and Cross St (let us call them).



In America, a car travelling east along High St, as shown, must clear the intersection completely before the light it is facing turns red. For, in most such instances, the moment the High St light turns red, the Cross St light turns green and so cars enter the intersection travelling North. (Remember Americans drive on the right as you read this diagram.)

In Australia, matters are different and we'll get to that, but let's analyse the simpler American situation first.

Figure 1 gives some of the basic notation. As the car approaches the intersection, the light turns yellow when it is x metres short of entering. The width of the intersection is w metres and the length of the car is L metres. Thus, to completely clear traffic in Cross St, the car has to travel $w + L (= W)$ metres.

Consider first the case in which the car is to stop. This must be done within the distance x. Suppose the car is travelling at speed u. First the driver must react to the signal. This takes about three-quarters of a second, during which the car continues to travel at speed u. The distance travelled during this period is called the "thinking distance"; it varies with u (see Table 1) and will be denoted by D_1.

When the driver registers the yellow light, the brakes are applied and the vehicle is brought to a halt. This takes a further distance D_2 and this distance varies as the square of u. In fact

$$D_2 = u^2/(2a_1) \quad ,$$

where a_1 is the deceleration brought about by the brakes. Values for D_2 are also given in Table 1. (Table 1 is a metric version of a table in Fred Watts' paper.) D_2 is called the "stopping distance".

Speed u (kph)*	Thinking Distance D_1 (m)	Stopping Distance D_2 (m)
30	6	5
40	8	8
50	10	12
60	13	16
70	15	23

Table 4.

* For convenience we list these values as kph, but in theoretical work we use ms^{-1}. 30 kph = 8.33 ms^{-1}, etc.

So, if the car is to stop in the available distance, we must have

$$x \geq D_1 + D_2 . \qquad (1)$$

If Condition (1) cannot be met, then the car must attempt to drive through the intersection and reach the other side of Cross St before the light turns red. The yellow light lasts, let us say, T seconds, and so, if the speed u is maintained, it must be high enough to allow the car to travel $x + W$ metres in T seconds, i.e.

$$u \geq \frac{x + W}{T} \qquad (2)$$

If u does not satisfy Inequality (2), then the driver must accelerate to get across in time. The acceleration will commence at a point $x - D_1$ metres to the left of the intersection. Thereafter the speed will increase, the acceleration being a_2 ms^{-2}. a_2 is, to complicate matters, a function of u. (We are here ignoring a less important complication, that the thinking distance D_1 is slightly longer in the case of acceleration than it is for braking - at least, for most drivers, most of the time.)

The time taken to travel D_1 m at speed u ms^{-1} is D_1/u s, so there is now available a time

$$t = T - \frac{D_1}{u}$$

seconds in which to complete the crossing. The distance travelled in this time is known to be

$$s = u(T - \frac{D_1}{u}) + 1/2\ a_2\ (T - \frac{D_1}{u})^2$$

and so we require, if the crossing is to be successful

$$x + W - D_1 \leq u(T - \frac{D_1}{u}) + 1/2\ a_2\ (T - \frac{D_1}{u})^2 . \qquad (3)$$

(Inequality (2) is the special case of this condition for which $a_2 = 0$.)

There is a third constraint that also must be satisfied if the motorist is to keep within the law. The final speed attained, which is $u + a_2(T - \frac{D_1}{u})$, must not exceed the speed limit, V (say). That is

$$u + a_2(T - \frac{D_1}{u}) \leq V . \qquad (4)$$

The constraints applying to the driver are that Inequality (4) must hold, together with either (1) or (3).

One of the problems highlighted by the various authors is that traffic lights ae sometimes engineered in such a way that this is not always possible. The dilemma this creates is one that all the papers referenced earlier discuss at some length. Fred Watts, for example speaks of the absurdity of having "a law which cannot be obeyed", while Gazis, Herman and Maradudin speak of "man-made systems, man-made laws and human behaviour [not being] always compatible."

I thought to test this in Melbourne and chose the intersection of High st and Chapel st. For an eastbound car on High St, $W \simeq 20$ m and $T \simeq 3$ s. A reasonable value for L is 4 m. Suppose the car to be travelling at the speed limit : 60 kph, or 16.67 ms^{-1}. Then Inequality (4) requires $a_2 = 0$. We must therefore satisfy either Inequality (1) or Inequality (2).

Inequality (1) gives (from Table 1)

$$x \geq 29 ,$$

while Inequality (2) gives (from the figures quoted)

$$x \leq 26 .$$

Continued on page 22.

BIG NUMBERS

Alasdair McAndrew

Footscray Institute of Technology

Big numbers have a great fascination, to the mathematician and non-mathematician alike. In this article, we shall look at some particular numbers, and where they occur in mathematics. But first, let us consider how to construct big numbers.

The simplest and most common way of getting a big number is to raise one number to the power of another. For example, we can write down things like 3^5 or 10^{100}. In this way we can very easily obtain numbers too large to have any physical significance (if we assume that the largest number with any physical significance is 10^{87}, which gives an approximation to the number of sub-atomic particles in the entire known universe). The number 10^{100} has been given the name "googol" - see the book "Mathematics and the Imagination" by Kasner and Newman.

We can get even bigger numbers by allowing not just one number and one exponent, but a chain of exponents, such as

$$3^{4^5} \quad \text{or} \quad 10^{10^2}$$

- the last number, you may notice, is just the googol again. When dealing with such chains, the convention is that we work from the top down, rather than from the bottom up. Take the number

$$3^{4^5}$$

for instance.

If we work from the bottom up, we get the number

$$(3^4)^5 = (81)^5 = 3486784401.$$

This is still small enough to be printed. However, if we work from the top down, we get

$$3^{(4^5)} = 3^{1024} = 3.7339 \times 10^{488}$$

which is a number of 489 digits – already this is too big to make any sense as a number printed in its glorious entirety.

Now let us look at some particular numbers, and where they occur.

Skewes' number

This number comes from the realm of number theory (seems fitting!); in particular the distribution of prime numbers. A prime number, you recall, is a number with no integer divisors other than itself and 1; for instance 5, 17 and 53 are prime numbers. The number of prime numbers less than a given integer x is denoted by $\pi(x)$. So $\pi(10) = 4$, as the only prime numbers less than 10 are 2,3,5 and 7 – the number 1 is not considered to be prime.

One of the great results of nineteenth century mathematics was a proof that $\pi(x)$ was approximately

$$\int_0^x \frac{1}{\log t}\,dt;$$

this integral is known as $\mathrm{li}(x)$. For many years after this it was thought that $\pi(x)$ was always strictly less than $\mathrm{li}(x)$, but in 1913 a proof by a mathematician named Littlewood demonstrated that for some large number x, $\pi(x)$ was greater than $\mathrm{li}(x)$. However, nobody as yet knew how big this x would have to be. This is where the mathematician Skewes comes into the story. His result concerned the finding of an upper bound for x; that is a number X for which $\pi(x)$ is greater than $\mathrm{li}(x)$ for some x less than X. And now, folks, here is this number:

$$X = 10^{10^{10^{34}}}.$$

In describing this wondrous number, the mathematician G.H.Hardy said proudly "I think that this is the largest number which has ever served any useful purpose in mathematics." He goes on to give some idea of the size of Skewes' number.

"The number of protons in the universe is about 10^{80}, and the number of possible games of chess is much larger, perhaps

$$10^{10^{50}}$$

(in any case a second order exponential). If the universe were the chessboard, the protons the chessmen, and any interchange in the position of two protons a move, then the number of possible games would be something like the Skewes number."

In any case, this number certainly earns for itself the criterion *big*.

Graham's number

Before describing this magnificent number, we shall first describe the notation needed. This so-called arrow notation was developed by Donald Knuth in an article "Mathematics and Computer Science: Coping with Finiteness" in *Science* in 1976. It is an extension of the idea of exponentiation, and works as follows:

The expression $x{\uparrow}n$ means x^n. That is, one arrow means we multiply the first number by itself as many times as the second number. When there are two arrows, for example $3{\uparrow\uparrow}4$, we "arrow" the first number with itself as many times as the second number. Thus, $3{\uparrow\uparrow}4 = 3{\uparrow}(3{\uparrow}(3{\uparrow}3))$. In standard notation:

$$3{\uparrow\uparrow}4 = 3^{3^{3^{3}}}.$$

This innocent looking number equals

$$3^{3^{27}} = 3^{7,625,597,484,987}$$

which is a number of over three thousand billion (one billion = 10^9) digits when written out fully. Then three arrows work similarly: $3{\uparrow\uparrow\uparrow}4 = 3{\uparrow\uparrow}(3{\uparrow\uparrow}(3{\uparrow\uparrow}3))$. To get an idea of the extraordinary size of this number, let's try to evaluate it.

To start with, $3{\uparrow\uparrow}3 = 3{\uparrow}(3{\uparrow}3) = 3{\uparrow}3^3 = 3{\uparrow}27 = 3^{27} =$ 7,625,597,484,987. Putting this number into the expression for $3{\uparrow\uparrow\uparrow}4$ we get $3{\uparrow\uparrow}(3{\uparrow\uparrow}762...987))$. Then the expression $3{\uparrow\uparrow}762...987$ is equal to $3{\uparrow}(3{\uparrow}(3{\uparrow}...3)...))$, where there are 762...987 3's altogether. That is, written in exponential form, we would get a tower of exponents 7,625,597,484,987 levels high. But this number, huge though it is, is just $3{\uparrow\uparrow}(3{\uparrow\uparrow}3))$. Let's give this number the symbol ¶. So the number we really want is $3{\uparrow\uparrow}$¶. This, of course, is merely an exponential tower of 3's – ¶ levels high! This number is certainly so huge as to be completely indescribable in anything but abstract mathematical terms.

Now we shall describe the problem which led to Graham's number. This problem is in an area of mathematics called graph theory. A graph, in this context, is nothing to do with axes and curves, but just a collection of dots with lines joining some or all of them. Here are some examples of graphs:

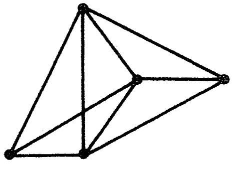


fig 1

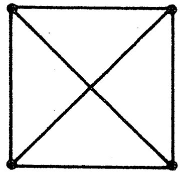


fig 2

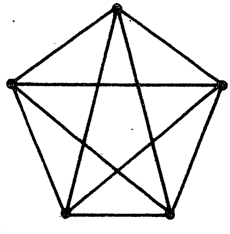


fig 3

A graph with every one of its points connected by a line to every other point is called a *complete* graph, and is denoted K_n, where n is the number of points in the graph. So in the diagram above, figures 2 and 3 are the graphs K_4 and K_5 respectively.

Now imagine an ordinary cube – or rather, the skeleton of a cube – in space. We can consider it as a graph on eight points. If we join all the points diagonally, what we end up with is the complete graph on eight points with a cubic structure. This particular graph is made up of a number of copies of K_4 – there are six K_4's as the faces, and four more going through the middle of the cube. Now here is a puzzle for you: Can you colour every line of this cube graph either red or blue in such a way that none of these ten K_4's is all of the one colour? (It can be done.) Such a one-coloured K_4 is said to be *monochromatic*.

The same sorts of graphs exist in higher dimensions, and for dimensions four and five it has been shown that it is possible to colour all the lines of these hypercubic complete graphs with red and blue in such a way that there is no monochromatic K_4. Here now is the big question: For what number n is there an n-dimensional graph (of the hypercubic sort) such that no matter how you colour the lines red and blue there will always be at least one monochromatic K_4? The exact answer is as yet unknown, but a mathematician named Ronald Graham has found an upper bound for n. The following diagram explains how to get this upper bound.

$$\left.\begin{matrix} 3\uparrow\uparrow\uparrow\uparrow 3 \\ \overbrace{3\uparrow\uparrow\,.\,.\,.\,.\,.\,\uparrow\uparrow 3} \\ \\ \overbrace{3\uparrow\uparrow\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,\uparrow\uparrow 3} \\ \vdots \\ \overbrace{3\uparrow\uparrow\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\,\uparrow\uparrow 3} \end{matrix}\right\} \text{64 layers}$$

The number we start with is $3\uparrow\uparrow\uparrow\uparrow3$ – you can show that this is equal to $3\uparrow\uparrow\uparrow$¶, where ¶ is the number we dealt with in describing the arrow notation (and you thought $3\uparrow\uparrow\uparrow4$ was big!).

This gives the number of arrows in the next row, which number gives the number of arrows in the next row, and so on until we have a column of numbers 64 rows deep. The last number in this sequence is Graham's number.

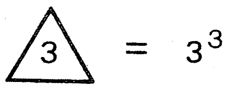
Now that is a *big* number, a *very* big number.

However, experts in the field claim that the answer to the hypercube problem is in fact – wait for it – six!

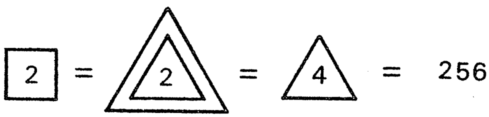
Some less serious numbers

The two numbers we have considered are serious in the sense that they occur in formal proofs. However, there are some nice numbers which have no known mathematical usefulness; they are just big. Here is yet another way of constructing big numbers.

A number inside a triangle means that number raised to the power of itself. So



Then a number inside a square is that number inside that many triangles. So



Continuing in this way we say that a number inside a pentagon is that number inside that many squares. So 2 in a pentagon is 2 in two squares, which is 256 in one square, which is 256 in 256 triangles, which is 256^{256} in 255 triangles, and so on. This number has the name *mega*. The *moser* (named after Leo Moser, a Canadian mathematician) is defined to be 2 inside a megagon.

But this is pure mathematical whimsy.

It may seem, at this point, that we've been cheating in not actually writing down the decimal expansions of any of the numbers we have been considering. Space, of course, is a major problem here. For example, to print Skewes' number would require (in terms of book size) about as many copies of the Encyclopaedia Brittanica as there are possible games of chess. This means that there is no way of printing this number and fitting it inside the known universe. However, there are some examples of handsome big numbers (including one of 6,421 digits) in the "Mathematical Games" column of *Scientific American*, August, 1967).

A final note

All these numbers are finite integers, and in the words of Donald Knuth: "Almost all numbers are larger than this." I like to think of these extraordinary numbers as helping us get an understanding of the set of integers. Most of us (including me) are used only to dealing with small finite numbers; the rest we dump conveniently into the "infinite set of integers", without realizing what "infinite" really means. When we see that this set contains numbers of the unimaginable magnitude of Graham's number or the moser, we begin to see that "infinite" is literally beyond our comprehension. Mind you, there are such very good mathematical tools for dealing with the infinite that the literal grasp of the concept of infinite is not an issue.

* * * * *

Continued from page 16.

So there is a (small, 3m long) region for which it is impossible to meet all the constraints. A car 28 m (say) from the corner of Chapel St cannot pull up in time, nor can it clear the intersection (unless it exceeds the speed limit) before the light turns red.

Were this intersection in the US, it would be classed as poorly engineered, although it is not as bad as some quoted in the references. Here in Australia, however, we organise our lights differently. After the High St light turns red, there is a further two second delay before those in Chapel St turn green and thus it is safe to cross even if the last part of the crossing is done in the red.

Jearl Walker recalls an experience of his own in which "I found myself facing a yellow light with neither the space to stop nor the acceleration to race through before the red light came on. I was saved from the possibility of a collision only by a delay in the light system : the green light for the perpendicular traffic came on about a second or so after the yellow light ended."

Because such delays are standard in Australia, we have fewer poorly engineered lights than does the U.S. Nevertheless, perhaps a little advice to younger drivers may not go astray here. I routinely change down a gear when approaching a light. This has the effect of increasing both a_1 (and so decreasing D_2) and a_2 (and so making Inequality (3) easier to satisfy). Howard Seifert also remarks: "Since most yellow lights are on for a more or less standard time interval, and most street widths lie within a limited spread, it becomes possible to develop an intuition concerning the possibility of a successful run-through. Most mature and still-surviving drivers have developed such intuition." To which I would add, "Take things slowly till you do".

INVERSE FUNCTIONS

Graham Baird, Melbourne College of Advanced Education

Throughout this note f will denote a function which has an inverse f^{-1}. Many calculus texts point out that the graphs of f and f^{-1} are related by reflexion through the line $y = x$. If students are encouraged to think in this way the essential geometical relationships between f and f^{-1} are lost.

It is more instructive to introduce a new set of axes XY so that the X-axis is the y-axis and the Y-axis is the x-axis. The equation of the curve $y = f(x)$ (with respect to the xy axes) has the equation $Y = f^{-1}(X)$ (with respect to the XY axes). A typical illustration of this set-up is given in Fig.1.

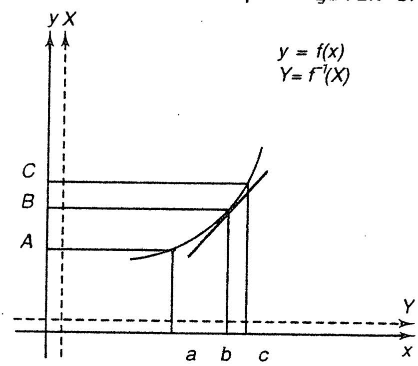


Fig 1

It is now evident that

(i) $$\left.\frac{dY}{dX}\right|_{X=3} = 1 \Big/ \left.\frac{dy}{dx}\right|_{x=6}$$ and, if the function has the form given in Fig.1,

(ii) $$\int_A^C f^{-1}(X)\,dY = Cc - Aa - \int_a^c f(x)\,dx$$

Fact (ii) is not as well-known as it should be and can be used to compute the integrals of many inverse functions. For example, if $y = \sin x$ we have (see Fig.2)

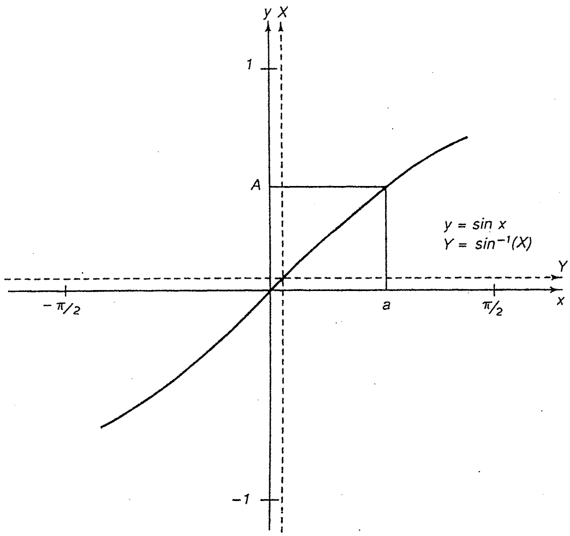


Fig 2

$$\int_0^A \sin^{-1} X dX = Aa - \int_0^a \sin x \, dx$$

$$= Aa + \cos a - 1$$

$$= A \sin^{-1} A + \sqrt{(1 - A^2)} - 1$$

It follows that the indefinite integral of $Y = \sin^{-1} X$ is

$X \sin^{-1} X + \sqrt{(1 - X^2)}$.

* * * * *

CORRECTION to *Function*, Vol.10, Pt.5, p.7.

The table in the article Chinese gambling games in NSW in 1891, by Frank Hansford-Miller, has an error in it. The entry £3.6s.8d., in the Prize Money column, should be replaced by £83.6s.8d. The reference in the text on p.8 is correct. [The mistake was made by the editors.]

* * * * *

THE ANALYTICAL ENGINE OF CHARLES BABBAGE

Peter Kloeden, Murdoch University

Mathematical tables used to be a common sight in secondary school mathematics classes. With their long, soporific lists of numbers, the values of trigonometric and logarithmic functions, they are rarely considered awe-inspiring. Yet it was the calculation of such numbers, with polynomials of high degree being used to approximate the given functions, that preoccupied many mathematicians during the sixteenth and seventeenth centuries, some of whom prided themselves on their arithmetic prowess. You may well ask: why did they bother? The answer is simple: these numbers were crucial for maritime navigation, and consequently for trade, colonization and the development of world empires.

In these days of electronic computers, this all seems rather mundane. For example, suppose you want to calculate a list of values for the polynomial

$$f(X) = 1 - \frac{1}{2} X^2 + \frac{1}{24} X^4 - \frac{1}{720} X^6$$

for systematically increasing values of X, say 0, 0.0001, 0.0002, 0.0003, ... etc. (This polynomial approximates cos X for small values of X). You could do this with a very simple program in which you define $f(X)$ as above, insert a value of X and then print out the corresponding value of $f(X)$. Then, just by adding a DO LOOP to your program, you could print out an entire table of $f(X)$ values corresponding to the systematically increasing X values. Pretty easy! In fact modern electronic computers and calculators have such programmes built into them. But you should not forget that electronic computers have been in common usage for only about 25 years, and in the class room for less than 10 years.

Mechanical devices to help with arithmetic calculations have been in use for over a thousand years, for example the Chinese abacus with its sliding beads to help with additions, carrying digits and so on. In the Western world, calculating machines with geared wheels were developed, firstly by Pascal in France to do additions and subtractions, and then some time later by Leibniz in Germany to handle the conceptually harder multiplications and divisions. Refinements of these machines were made over the next two centuries, but the underlying principles remained the same. They saved a person doing repetitious calculations a lot of tedium (and errors), but still required physical and mechanical movements for each step of the calculation, as well as decisions by the person. You may be surprised to learn that most of the calculations needed for the development of the atomic bomb in America during World War II were done on such mechanical calculating machines.

Actually, a calculating machine that could be programmed to do repetitious calculations and to make decisions without human intervention had been conceived and constructed more than one hundred years before World War II. This was the "Analytical Engine" of the English mathematician Charles Babbage (1792-1871). He got the idea from the Jacquard looms which had been developed in France to weave patterns on cloth and on tapestries. These had a long paper loop with holes punched in patterns across it in rows. The positions of the holes in a given row activated mechanical devices in the loom, which moved warp threads forwards or backwards as required for a single pass of the shuttle pulling the woof thread through the opening of the warp threads. The loop then rotated to the next row, the loom rearranged the warp threads and the shuttle made its next pass across the loom. Eventually the loop came back to its starting row and the pattern being woven repeated itself.

It took Charles Babbage several decades to adapt this idea to the mechanical calculating machines then available. Besides having to invent new mechanical devices to do things which are easy for a human but difficult to get a machine to do (for example, storing a number to be reincorporated in the calculation at a later stage) Babbage also had to develop new methods for improving the accuracy in cutting and turning metal on lathes. This was because inaccuracies in the sizes of the gear wheels could lead to errors in the calculations, which would otherwise be noticed by a person doing the calculations step by step on a more shoddily built machine. The British Government of his day, in a surprisingly enlightened display of support for the sciences (perhaps they knew about trade deficits too?) generously supported Babbage to the tune of £20,000, which was an enormous sum in those days. This paid for his equipment and for his technicians. A machine, called his "Analytical Engine", was eventually built and could be programmed to carry out calculations as had been intended. It is now on display in the British Museum in London, but was never really used except for demonstrations. Perhaps the concept was too far ahead of its time. The concept was however not forgotten, in fact came back with a vengeance in the 1940's with the development of programmable electric computers. The British taxpayers reaped their reward much earlier, as the high precision metalworking techniques machines and skilled metal workers, that came from Babbage's workshop gave Britain a decisive lead in the early days of the Industrial Revolution. Technological spin-off?

While small compensation for the misery suffered by millions, the Second World War was a catalyst for some spectacular advances in science and technology. The design of aircraft, the development of the atom bomb and of numerical methods for long-term weather forecasts all required voluminous numerical calculations, far beyond the capacity of human operated calculating machines. It was during this that programmed electrical computers were constructed and used. The first was made by Konrad Zuse an aircraft designer with the Heinkel Aircraft Factory in Warnemünde near Rostock. He used it for his own aerodynamical calculations, but otherwise it received little interest; the then government of his country is not remembered as being particularly enlightened! After the war, Zuse's ideas were taken over by IBM, then a calculating and business machine manufacturer. Meanwhile in America, the famous mathematician (and a founding father of computer science) John von Neumann led a team which also made an electrical computer. This was like Zuse's in that it involved rooms full of valves, and by today's standards was primitive. During my recent visit to America, I met a mathematician who did numerical weather forecast calculations for von Neumann on such a computer. He said he always could tell where the computer was in the program by seeing which valves were glowing. He also said that a program "crash" was literally that, and so was a "bug", a moth in the works!

* * * * *

PROBLEMS

Solution to Problem 10.5.1, by David Shaw, Geelong West Technical School. The problem was to show that $e^{\pi} > \pi^{e}$ without computation.

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots (x > 0)\ , \text{ so } e^{x} > 1 + x\ .$$

$$\pi > e\ , \text{ so } \frac{\pi}{e} > 1\ , \text{ i.e. } \frac{\pi}{e} - 1 > 0\ .$$

$$\text{Hence } e^{\frac{\pi}{e} - 1} > 1 + \frac{\pi}{e} - 1$$

$$\text{i.e. } e^{\frac{\pi}{e} - 1} > \frac{\pi}{e}\ ,$$

$$\text{i.e. } e^{\frac{\pi}{e}} > \pi$$

$$\text{whence } e^{\pi} > \pi^{e}\ .$$

* * * * *

Two correspondents, John Barton and David Shaw, pointed out that Problem 10.5.2 probably had a slip in it. As presented, it has the solution that $x + y$ has maximum value 50, but two of the conditions on x and y are then superfluous. Try instead the following problem, the one probably intended.

PROBLEM 11.1.1.

If $x \geq 0$, $y \geq 0$ and

$$\begin{aligned} -2x + y &\leq 50 \\ 3x + 2y &\leq 300 \\ x + y &\geq 50 \\ \text{and} \quad x &\leq 90\ , \end{aligned}$$

what is the maximum value of $x + y$?

DISTRIBUTION OF PRIME NUMBERS

Arnulf Riedl, Year 12, Stawell High School

Recently I examined the distribution of prime numbers on my computer. First I used a method called the *Sieve of Eratosthenes* to find the primes greater than 2 and less than a number *N*. This works by systematically excluding multiples of previously discovered primes.

I implemented this as a BASIC program SIEVE.BAS which I give here.

```
10 CLS
20 INPUT "Prime numbers under (n)?";N:DIM A(N+2):Z=3:S=INT(N/2)+1
30 FOR P=1 TO S:A(P)=Z:Z=Z+2:NEXT P:PRINT "2 ";:FOR P=1 TO INT(SQR(S))+1
40 IF A(P)=0 THEN 100
50 PRINT A(P);" ";
60 FOR Z=P+1 TO S
70 IF A(Z)=0 THEN 90
80 IF A(Z)/A(P)=INT(A(Z)/A(P)) THEN A(Z)=0
90 NEXT Z
100 NEXT P
105 CLS
110 OPEN "o",#1,"primes"
120 FOR P=1 TO S
130 IF A(P)=0 THEN 160
140 PRINT A(P);" ";
150 PRINT#1,A(P)
160 NEXT P
170 CLOSE#1
```

Next I used a program PRIMETABS.BAS to create a file of primes. Here is my program.

```
10 ON ERROR GOTO 180
20 INPUT N
30 DIM A(N)
40 CLS
50 OPEN "i",#1,"primes"
60 FOR P=1 TO N
70 INPUT #1,A(P)
80 NEXT
90 CLOSE#1
100 REM ******************************
105 X=1:Y=1
110 FOR P=1 TO N
120 IF A(P)=0 THEN END
130 LOCATE X,Y
140 IF Y=73 THEN Y=1:X=X+2 ELSE Y=Y+6
145 IF X>=24 THEN FOR T=1 TO 1000:NEXT :CLS:X=1
150 PRINT A(P)
160 NEXT P
170 END
180 RESUME NEXT
```

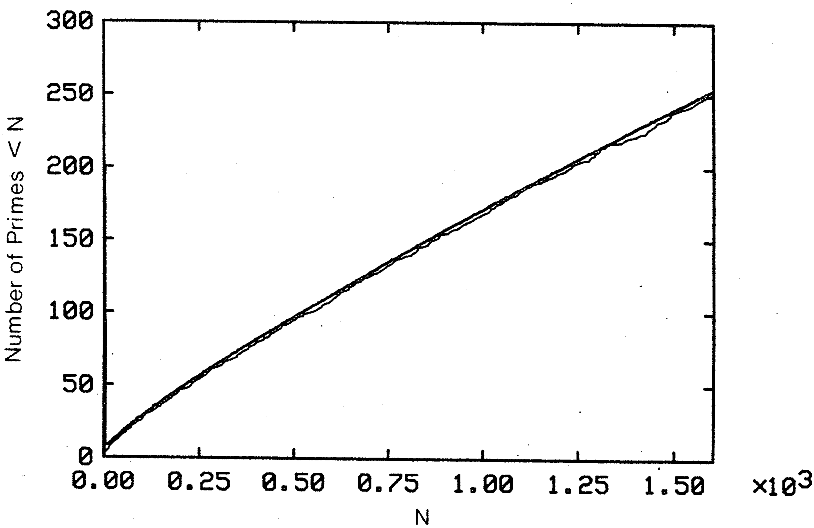
I used this file as input for a new program PRIMEGRA.BAS. This did a number of things. First it counts the number of primes greater than 2 and less than N. This number $P(N)$ is graphed and so is an estimate by the mathematician Legendre. Legendre's formula is

$$P(N) \approx \frac{N}{\ln N - 1.08366} \quad ,$$

where ln stands for the natural log.

Here is the result. Legendre's formula gives a slight overestimate and a smoother curve but it is very accurate.

A Plot of the Distribution of the Prime Numbers



```
5 DIM A(1600)
10 CLS
20 ON ERROR GOTO 1010
30 CLS
40 OPEN "I",#1,"PRIMES"
50 FOR P=1 TO 1600
60 INPUT #1,A(P)
70 NEXT P
75 ON ERROR STOP
80 SCREEN 2
90 DRAW "BM60,0"
100 DRAW "M+0,159"
110 Q=INT(16*1.8):W=Q
120 FOR T=19 TO 3 STEP -2
130 LOCATE T,3:PRINT W;
140 W=W+Q
150 NEXT T
160 DRAW "M+600,0"
170 DRAW "BM60,158"
180 FOR T=60 TO 600 STEP 27
190 PSET STEP(32,0)
200 NEXT T
210 LOCATE 2,7:PRINT "Y"
220 LOCATE 21,76:PRINT "X"
230 LOCATE 21,7:PRINT "0"
240 U=1
250 FOR T=11 TO 71 STEP 4
260 LOCATE 21,T:PRINT U:U=U+1:NEXT T
270 LOCATE 23,39:PRINT "N * 100"
280 LOCATE 3,12:PRINT "NO. OF PRIMES"
290 LOCATE 5,14:PRINT "BELOW N."
300 DRAW "BM63,160"
310 FOR L=160 TO 0 STEP -9
320 PSET STEP (0,-16)
330 NEXT L
340 DRAW "BM60,159"
350 FOR A=1 TO 580
360 L=A/32*100
370 Y=-L/(LOG(L)-1.08366)
380 DRAW "BM60,159"
390 PSET STEP (A,Y/1.8)
400 NEXT A
410 REM *******************************************
450 N=1
500 FOR A=16/6 TO 1600 STEP 16/6
505 L=A/32*100
510 IF A(N+1)<L THEN N=N+1:GOTO 510
600 Y=N
610 DRAW "BM60,159"
620 PSET STEP (A,-Y/1.8)
630 NEXT A
1000 END
1010 RESUME NEXT
 A(N+1)<L THEN N=N+
```

LETTERS TO THE EDITOR

I would like to comment on G.A.Watterson's article *Card Shuffling* in the October 1986 issue. In particular, I refer to the concluding section on the 'perfect' riffle shuffle.

With an even number of cards, the first and last cards retain their positions in the pack in the course of the shuffles. With an odd number of cards (divided so that there is one more card in one division than in the other), only the first card retains its position. The computer program is easily modified to include the shuffling of an odd pack and to print out the order of the cards after each shuffle. If we allocate position 1 to card 2, it is seen that in successive shuffles it takes up positions 2, 4, 8, ... until the power of 2 exceeds the number of cards in the pack. The position of card 2 after n shuffles is given by $2^n \pmod N$ if N denotes an odd number of cards in the pack and $2^n \pmod{N-1}$ if N denotes an even number.

The number (x) of shuffles required to return the pack to its original order is the least positive solution of the congruences

$$2^x \equiv 1 \pmod N \quad \text{if} \quad N \text{ is odd}$$

$$\text{or} \qquad 2^x \equiv 1 \pmod{N-1} \text{ if } \quad N \text{ is even .}$$

For example, for modulus 19 the residues for $2^1, 2^2, 2^3, \ldots$ may be set out as follows:

Index	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Residue	2	4	8	16	13	7	14	9	18	17	15	11	3	6	12	5	10	1

The index corresponds to the number of the shuffle and the residue corresponds to the position of card 2 which can be seen to return to its original position after 18 shuffles.

A similar argument may be applied to the position of any card in the pack. In general, if c is the number of the card, then its position after x shuffles is given by $(c-1)2^x$ (mod N if N is odd or mod $N-1$ if N is even).

A program for generating residues of 2^x (mod N or $N-1$) and for counting the number of shuffles is shown below. It will produce the desired result more quickly than the program in the article.

It follows from the above that a pack of 51 cards will perform in the same way as a normal pack of 52 in shuffling back to original order after 8 riffles.

```
   10 INPUT "NUMBER OF CARDS";N
   20 Q=N/2
   30 IF Q=INT(Q) THEN N=N-1
   40 S=1;C=0
   50 S=S*2
   60 IF S>N THEN S=S-INT(S/N)*N
   70 PRINT STR$(S);SPC(1);:C=C+1
   80 IF S=1 THEN 100
   90 GOTO 50
  100 PRINT
  110 PRINT "NUMBER OF SHUFFLES ";C
  120 END
>RUN
NUMBER OF CARDS?52
2 4 8 16 32 13 26 1
NUMBER OF SHUFFLES 8
```

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Since the time of its beginnings in Egypt and Mesopotamia some 5,000 years ago, progress in mathematical understanding has been a key ingredient of progress in science, commerce, and the arts. We have made astounding strides since from the theorems of Pythagoras to the set theory of Georg Cantor. In the era of the computer, more than ever before, mathematical knowledge and reasoning are essential to our increasingly technological world.

The application of mathematics is indispensable in such diverse fields as medicine, computer sciences, space exploration, the skilled trades, business, defense, and government. To help encourage the study and utilization of mathematics, it is appropriate that all Americans be reminded of the importance of this basic branch of science to our daily lives.

Ronald Reagan

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