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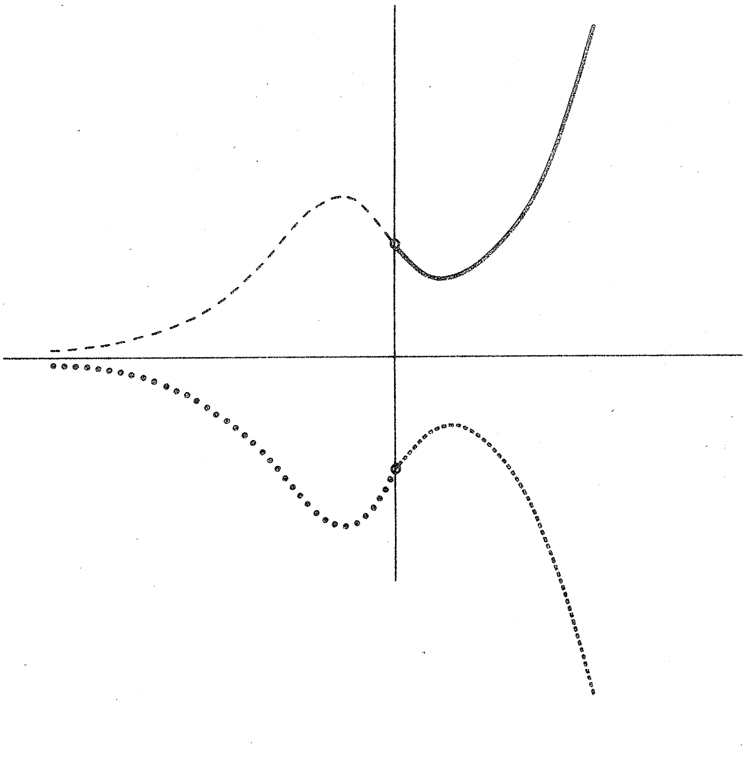

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Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Monash University's Foucault pendulum always attracts attention. It is, in fact, unique in its accuracy and its novel design. This issue explains the theory behind such pendulums and the difficulty inherent in getting them to work. Other articles tell the story of the colourful, if eccentric, genius Oliver Heaviside and reveal some of the hidden mysteries of the compound interest formula.

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THE FRONT COVER

We have had from two sources discussions of the graph $y = x^x$. A joint study by R. Lush and R. Warmington, then Year 11 students at Scotch College, and carried out some years ago at a Mathematics Summer School, reached us, as did a shorter and considerably more technical account from G.P. Speck of Wanganui Boys College, New Zealand.

The editors decided to summarise the findings of these different investigators. The results are presented here, and graphed on the front cover.

First, consider the sign of x. x may be positive, negative or zero. We do not discuss now the case $x = 0$. This case alone took four pages of Volume 5, Part 4.

Suppose then $x > 0$. For all such x, a positive value may be assigned to x^x. These values are those lying on the part of the graph in the *first* quadrant of our cover diagram. These form a continuous curve on the interval $0 < x < \infty$. As $x \to 0$, $x^x \to 1$ (as stated in *Function*, Vol.5, Part 4). As $x \to \infty$, $x^x \to \infty$ and does so very rapidly indeed. The minimum value of the function is to be found where $x = 1/e$; e being the base of the system of natural logarithms, an irrational number whose value is $2{\cdot}718281284\ldots$.

Still with x positive, consider four cases:

1. x is rational, $x = \frac{p}{q}$ (say) where p,q have no common factor, and q is even (and hence p is odd);

2. As above but q is odd and p is even;

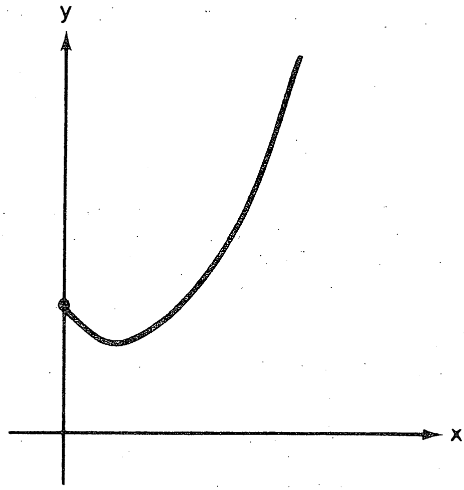
3. As in Case 1, but both p,q are odd;

4. x is irrational.

In Case 1, x^x has two values, one positive and one negative. For example $(\frac{1}{2})^{\frac{1}{2}} = \pm 1/\sqrt{2}$. For all x of the type described, there are two values, equal in magnitude but opposite in sign.

Consider, by contrast, Case 2. An example is $\left(\frac{2}{3}\right)^{2/3}$. Here only one real value $\left(\sqrt[3]{\frac{2}{3}}\right)^2$ is assigned, although in complex number theory, two other (complex) values are also given to the expression. For our purposes, however, x^x is one-valued if x is as described for Case 2. The same applies to Case 3. Case 4 can only be described in terms of some limiting process or other, and, as irrational numbers may be approached via sequences of numbers with either even or odd denominators, it is usual to assign only the positive value in this case.

Thus, when $x > 0$, the graph consists of the continuous curve at right and the mirror image of a subset of its points. We have indicated that only a subset are involved by using a discontinuous line, but clearly there can be no attempt at exact representation for the points of Case 1 are dense - they lie arbitrarily close, each to infinitely many others.



When $x < 0$, we need to consider the same four cases. But now it is in Case 1 that x^x does not exist, as negative numbers have no real square roots.

Case 2 numbers, like $-\frac{2}{3}$, give rise to positive values of x^x: $\left(-\frac{2}{3}\right)^{-2/3} = \left\{\left(-\frac{2}{3}\right)^{1/3}\right\}^{-2} = 1/\left(-\sqrt[3]{\frac{2}{3}}\right)^2 = 1/\left(\sqrt[3]{\frac{2}{3}}\right)^2$, and this is positive. Such numbers are graphed in the second quadrant.

Case 3 numbers, like $-\frac{1}{3}$, give rise to negative values of x^x. These are graphed in the third quadrant.

If, finally $x < 0$ and x is irrational (Case 4), x^x is not assigned a value.

The complete graph is thus that of the cover, where in three quadrants the graph is discontinuous but with points lying arbitrarily close together along it. The two "maxima" and the "minimum" of the third quadrant would be achieved at $x = \pm 1/e$, which is irrational. Thus, although points may be found arbitrarily close to the maxima (say), no points lie exactly on them.

You may notice that the behaviour of x^x is somewhat like that of the elementary trigonometric functions in the different quadrants. This was the topic Mr Speck set out to explore. It does need complex analysis, and so lies outside the scope of *Function*, but the topic leads to some very beautiful mathematics.

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THE FOUCAULT PENDULUM AT MONASH †

If you were to stand at the North Pole, you would rotate slowly with the earth and turn round through a full 360° once each day. Seen from above, you would rotate anti-clockwise, as the earth's rotation is transmitted (via friction) to such a person. But if, when you were there, you held a pendulum and let it swing, there would be no force whereby the earth could alter the plane of oscillation of the pendulum. It would therefore swing in a fixed plane while the earth turned beneath it.

We do not notice the rotation of the earth, except as an apparent motion of bodies such as the sun, the fixed stars and other heavenly bodies. To this list should be added the pendulum, since, to an observer regarding the earth as fixed, the plane of oscillation in this case appears to rotate 360° clockwise every 24 hours (or more precisely, every 23 hours 56 minutes and 4·091 seconds - the slight discrepancy is a contribution from the earth's revolution about the sun to the length of the solar day).

A similar analysis applies at the South Pole, except that the direction of apparent rotation is now anticlockwise. At intermediate latitudes, the time taken for the 360° rotation is $(24/\sin\lambda)$ hours, where λ is the latitude. (Here and in the rest of this article, ignore the small correction to the length of the day.) We may thus calculate that the plane of oscillation will appear to rotate $(15 \sin\lambda)°$ per hour.

These ideas were first put forward by the French physicist and mathematician, Leon Foucault (1819-1868) in 1851, who constructed such a pendulum in an attempt to demonstrate the effect he had predicted. Much more careful experiments were performed by the Dutch physicist Heike Kamerlingh Onnes in 1879 in the course of work for his Ph.D. degree (he was later to win a Nobel prize for his investigations in low temperature physics - notably his discovery of superconductivity).

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†This article was compiled by one of the editors (MABD) from the writings of the pendulum's designer, Dr C.F. Moppert, and appears with Dr Moppert's permission. Dr Moppert wishes us to point out that the process used to produce this article was also used in the case of the one on the Monash sundial (*Function*, Volume 5, Part 5).

In practice, the simple theoretical picture becomes much more complicated. Actual pendulums are typically in error by more than 15%, a quite substantial amount. The physicist Sommerfeld, writing in 1944, stated: "Foucault's experiments in 1851 and those of his countless successors gave only qualitive results". Dr C.L. Stong, writing in *Scientific American* (February, 1964) noted: "None of the Foucault pendulums [I] have examined (including the one on display at the Griffith Observatory in Los Angeles and the splendid installation that adorns the entrance hall of the United Nations General Assembly building in New York) betters the 15% error". Stong then describes a pendulum built by B.B. Bingham, which gave an average error of only 2% on most days and was considerably more accurate on some days (and presumably considerably less accurate on others).

Among the problems experienced by actual pendulums are these.

(1) Running Down - unless some driving force is supplied, a pendulum comes to rest in a reasonably short time - the driving force must not influence the rate at which the plane of oscillation rotates, for this would invalidate the experiment.

(2) Construction Details - the geometry of the top support can adversely affect the motion, as can kinks and other irregularities in the wire supporting the pendulum bob.

(3) Ellipsing - a major source of error, to be discussed further later in this article.

At Monash, two pendulums have been built, both designed and constructed by W.J. Bonwick and C.F. Moppert with help from a number of colleagues. Both of these incorporated a unique drive. At the bottom of the bob is a small permanent magnet. When the pendulum is at rest, this hangs directly above the centre of a circular coil of one foot ($\approx$ 30 cm) diameter. When the pendulum is set into motion, the magnet passes over the coil, and, as it does so, induces a small current. This in turn causes the discharge of a condenser, and this discharge gives the magnet a small kick exactly in the direction of its motion. The power consumption is a few watts. This same drive is now incorporated in a pendulum hanging in the McCoy building at the University of Melbourne.

The first pendulum was set up in the Electrical Engineering building at Monash, but later it was possible to build another in an otherwise disused lift-well near the entrance to the Mathematics Building. This second pendulum is the one now on display. It was officially set in motion on June 16, 1978, by the Chancellor, Sir Richard Eggleston, and has run since then with few interruptions. A typical run for this pendulum extends over some two months.

The accumulated angles are noted with about one reading per day. If these are plotted against time, the result should be a straight line, and so it is to remarkable accuracy; in one 58 day run, the deviation was less than 0·01 of 1%. This remarkable consistency showed that we had overcome the first two problems. The third has been rather more difficult. Let us now consider it.

It is actually rare for a pendulum to swing precisely in a plane. The curve traced out is, to a good approximation, that followed by a point moving around an ellipse which is itself rotating. An ellipse is described by two numbers a, b known respectively as the semi-major axis and the semi-minor axis. The rotation of this ellipse by an amount of $\Delta\phi_s$ per swing is known as "spherical precession" and has nothing to do with the rotation of the earth. It arises from quite independent dynamical considerations.

It may be shown that, if ℓ is the length of the pendulum, then

$$\Delta\phi_s = \frac{3\pi}{4}\frac{a}{\ell}\frac{b}{\ell} = \frac{3\pi\alpha\beta}{4} \tag{1}$$

where $\alpha = \frac{a}{\ell}$ and $\beta = \frac{b}{\ell}$. (α measures the angle through which the pendulum swings - the amplitude; β measures the angle through which it deviates from the ideal plane of oscillation.)

The effect we want to observe is the apparent rotation of $(15 \sin\lambda)^\circ$ per hour. This amounts to an angular change $\Delta\phi_F$ per swing, where

$$\Delta\phi_F = 2\pi\ \omega(\sin\lambda)\surd(\ell/g) \tag{2}$$

ω being the angular speed of the earth (in radians per second) and g the acceleration due to gravity.

The relative size of these two effects is given by

$$\frac{\Delta\phi_F}{\Delta\phi_s} = \frac{8\omega\ \sin\lambda}{3\alpha\beta}\surd(\ell/g)\ . \tag{3}$$

Formula (3) shows that a long pendulum is preferable (ours occupies a four-floor lift well) and that the experiment becomes difficult in low latitudes (i.e. near the equator).

Kamerlingh Onnes, who used a short pendulum in a vacuum, analysed his own results and one set of earlier ones to verify Equation (3) to within 1·5%. Later workers have checked the result to even greater accuracy.

However, these results apply to the free pendulum. Driven pendulums are subject to more complicated analyses. As driven pendulums run for months at a time, the number of swings and, in consequence, the accumulated spherical precession can become very great. For this reason, attempts are made to eliminate ellipsing in such pendulums. The arrangement adopted by all driven pendulums (apart from the Monash pendulum) is a device known as the Charron ring - a small fixed ring situated at a distance of some 20 cm below the point at which the pendulum is suspended.

The Charron ring, however, disturbs the motion of the pendulum in complicated ways, and although Charron himself devised a formula by means of which these could be allowed for, its accuracy is suspect.

Until recently, to eliminate ellipsing in the Monash pendulum a mechanical brake was used - a sponge rubber sleeve around the bob, which touched the rim of a circle cut in a chipboard plate at the maximum amplitude of the swing. The amplitude of the swing was made to be of precisely the right amount by adjusting the drive.

This arrangement allowed us to automate the measurements. Arranged at 15° intervals around the circular hole in the chipboard were contact wires. When the bob touched these, a small electrical pulse was recorded on a chart. This chart, moving at 1 cm/hour, thus showed a pen-recorded trace from which the angular velocity of the pendulum's plane of oscillation could be calculated.

As mentioned earlier, the plot of accumulated angle against time should be a straight line. This we have achieved at Monash to very high accuracy. The slope of this line should give the expected change of $(15 \sin \lambda)^\circ$ per hour. This, at the latitude of Monash is 9·2° per hour. The observed results, however, were less than this. We tended to get results of about 8·3° per hour. In other words, the pendulum was retarded.

Dr Moppert believes that this retardation was caused by the mechanical brake used to prevent ellipsing. By changing the input voltage of the drive observations showed that the higher the drive, the greater the retardation. This was as expected; however, there seemed to be no possibility of extrapolating the measured results to a degree when no touching would occur, although a theoretical analysis of the effect of the padding gave results that accorded reasonably well with observation.

Early in 1981, Drs Moppert and Bonwick introduced a new system. With the new setup, an electrical brake instead of a mechanical one is used. A magnet, moving over a copper surface experiences a retardation due to eddy currents induced in the copper. The new brake, a copper ring, is now installed and is being used in a new series of experiments, changing the voltage and the geometrical configuration.

Dr Moppert says that projects such as this one and the Monash sundial (*Function*, Volume 5, Part 5) appeal to him because they show that mathematics is not a sterile subject. Nonetheless, he adds, as a teacher, he feels it his duty to point out that science starts and ends with experiments. The theories are secondary.

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OLIVER HEAVISIDE F.R.S.

– THE TACTICAL ELECTRICIAN

C.H.J. Johnson, CSIRO

Oliver Heaviside was one of the most creative applied mathematicians of modern times. A man largely selftaught, from an early age he had set his mind in the direction of his later accomplishments. He was born at Camden Town, London, on 18th May 1850 and died at Paignton, Devonshire, on 3rd February 1925. He was an intense individualist with great confidence in himself and in his creative ability. His "applied Mathematics" - then hardly recognized as a subject - bore the same qualities of individualism and assertiveness. He made many contributions to physical and mathematical science. He recast Maxwell's theory of electromagnetism into a form more suited for application to physical problems. This enabled him to formulate and develop modern transmission-line theory and to solve many problems in electromagnetic theory: problems that were both technically important and new.

He is remembered by mathematicians for his contributions in two areas. First, his work on transmission lines led him to develop an "operational calculus" which, though not accepted at the time, did, in due course, influence the development of mathematics in this century. Second, he contributed greatly to the development of vector analysis - at the time most mathematicians used the more cumbersome "quaternion algebra" (see *Function*, Volume 5, Part 3).

He also made many other contributions: the introduction of moving charges into electromagnetic theory, the Heaviside unit step and unit impulse functions (the latter to emerge again much later in a quantum theory context as the Dirac δ-function) and the Heaviside ionospheric layer, whose existence he inferred from the properties of transmission lines.

His early life was spent "in miserable poverty", and, to continue in his own words, the effect "permanently deformed my future life". There is no doubt that it determined his attitude to society and to "established authority", but, proud and independent, he remained attached to the few close friends he made. His research made him a legend in his own lifetime. He made no money from his work, save the little from the publication of his books, although many of his results might have been patented, and, had they been so, might have given him a more than comfortable income. He cared little for the world's honours and died as he began - in poverty.

After a somewhat interrupted schooling, Heaviside's formal education ceased, and his uncle, Sir Charles Wheatstone, who had made many contributions to "electric telegraphy", secured for him the job as telegraphic operator with a Newcastle telegraph company, where he remained from 1868 till 1874. There were many problems associated with telegraphy, mostly relating to circuits with batteries and resistances, and with switching within networks of these circuits. Heaviside dealt with many of these problems in a highly original manner, including the problem of duplex telegraphy where telegraphing can occur simultaneously in opposite directions on the same wire.

By the age of 24, he had published seven original papers and had made something of a name for himself. In 1874 he gave up his job as telegraphic operator and returned to London to live with his parents. For the rest of his life he remained unemployed. He continued his research on inductive effects in telephone systems, the telephone having been introduced in 1877. Heaviside considered in particular the problems of electrostatic and electromagnetic induction between two overhead parallel wires and to deal with these problems he taught himself, and used very effectively, much advanced mathematics.

In all these investigations, as in his later work, he saw right into the heart of the matter and was able to apply his mathematics in a highly tactical manner.

In 1873, James Clerk Maxwell, then professor of experimental physics in the University of Cambridge, published his great treatise on Electricity and Magnetism in which he set into mathematical form the experimental results of Faraday. These may be condensed into the statement that the electric field at any point in space can be described by two vectors (the electric force and the magnetic force) together with the field displacements they produce.

Heaviside read Maxwell's treatise and saw that it contained many of the ideas he needed to solve his own problems. Over the years 1885 through 1887 he reworked much of Maxwell's presentation and eventually published a series of papers under the general title "Electromagnetic Induction and its Propagation". In these papers Heaviside gave Maxwell's theory a new form. In particular, he removed from the discussion a number of old and irrelevant ideas which Maxwell had retained. He used vectors in place of the clumsier quaternions, and so structured the equations that they could be applied directly to new physical problems. His equations are in the form used today - perhaps they should be called the "Maxwell-Heaviside equations".

The theories of electricity before Maxwell had regarded electric charge as a kind of fluid, in fact two fluids - one for positive charge and another for negative charge. Maxwell introduced a completely new approach, suggested by the theory of elasticity, where electric displacement (which is present whenever there is an electric force) is the fundamental quantity, in exactly the same way that elastic displacement is the basic quantity in elasticity theory. At first he gave only a formal mathematical meaning to electric charge. However, much later, when electrons were discovered, he changed his position somewhat and eventually admitted charges into the theory. It was Heaviside who first incorporated electrons into the theory.

Heaviside's research into electric signalling and the theory of telegraphs, telephones and cables is one of his most lasting contributions. First of all, he derived the equation known today as the Equation of Telegraphy (sometimes named, in his honour, Heaviside's Equation of Telegraphy). This involved analysis of a transmission line with the four characteristics: resistance, inductance, capacitance and leakage. Previous accounts, neglecting inductance and leakage, had led to paradoxical and impractical results. For example, there were many "electricians" (as electrical engineers were then called) who seemed to imagine that the current, when it started its passage through the cable, knew just how far it had to go and adjusted its speed of propagation accordingly!

This was the state of the theory of transmission of signals along cables as it was given in Maxwell's treatise in 1873. However, the theory was adequate for the Atlantic cable, for example, since the rate of signalling was quite low. The first real advance in the theory came in 1876 when Heaviside, working with his developments from Maxwell's treatise, took into account effects due to inductance and leakage, as well as those of resistance and capacitance, and obtained his general "telegrapher's equation".

This led to the immediate resolution of one paradox, for the speed at which a wave propagates along the line turned out to depend very critically on the inductance, which had hitherto been neglected.

Such analyses led Heaviside to propose that inductance should be *increased* in long cables, particularly to avoid the distortion that arose with the frequencies employed by the human voice. Leakage also reduces distortion (as he showed), although, as we would expect, it weakens the signal. Thus, for Heaviside, improvement in cables could be brought about by increasing both the inductance and the leakage.

Although Heaviside's analysis was correct, it was not appreciated by the British Post Office and their Chief Electrician, W.H. (later Sir William) Preece, who regarded inductive effects as "harmful to telephony" and to be avoided at all costs. Heaviside wrote a number of papers criticising the blunders of the Post Office but the various journals refused to publish them, due to the influence of the Post Office engineers. It is clear that Preece had no understanding of the problem. The only thing remembered of this limited man is his ignorance. Heaviside's ideas were first put into practice in the U.S.A. in the 1890's by Pupin who, by inserting inductive coils along the line, made telephony possible across the American continent.

Heaviside continued to find it difficult to publish on transmission lines so he turned to other areas. In 1888 he published an investigation of the field of a moving charge, giving for the first time the well known result for the force on a charge moving in a magnetic field. This probably may be regarded as the first appearance of the "electron" in electromagnetic theory.

By 1890 Heaviside's sixteen years of work had brought little recognition from those in official positions but now the situation changed. Interest in Maxwell's theory was increasing and the broad truth of the theory of propagation on wires which Heaviside had worked out, on the basis of Maxwell's theory, was confirmed experimentally by Hertz, Lodge and others. Heaviside gradually came into his own. In 1891 he was elected to the Royal Society of London, although he hardly took the matter seriously. His papers on electromagnetic theory were collected and brought out in book form, but the sale was not very great. "They printed 750 copies", he wrote, "and had 359 copies left five years later." (Nowadays, his books are widely available in reprinted versions but this is a much belated development.)

However little his own books might be read he never ceased to read those of others. He read the great mathematical texts of the day. However, he had very little to do with that centre of mathematics, Cambridge University. "Good mathematicians, when they die, go to Cambridge", observed Heaviside. However, he was rather isolated and his work was hardly known there. Soon after 1890 when Heaviside was at last recognized he soon fell foul of the Cambridge pure mathematicians over his unorthodox mathematical methods, particularly his "operational calculus".

For many years, Heaviside had followed Boole and others in writing the "operator" D to mean $\frac{d}{dx}$ and then treating D as a quantity of familiar algebra. In many cases, this yields correct results very efficiently - in other cases it does not. Heaviside's approach to these matters was far from rigorous - he regarded mathematics as an "experimental science" and worked accordingly.

His work on equations related to the Equation of Telegraphy led him to consider expressions such as $D^{\frac{1}{2}}$, which at first sight seems to make no sense. Actually, he was not the first to do this, but he seems to have been unaware of previous work. He developed the theory from scratch to a point where he was able to use it in solving important and difficult problems, particularly in relation to heat conduction.

Although the solutions he derived in this way were, in point of fact, correct, his methods were not accepted by his mathematical contemporaries, particularly as the mood of the day was one in which formal rigour was beginning to be stressed. Heaviside again found difficulty in publishing - this time because of the influence of the Cambridge mathematicians. "Mathematics is an experimental science", wrote Heaviside in the heat of this debate, "and definitions come not first but later on."

Heaviside's mathematics still generate controversy. Whittaker, in 1950, saw his operational calculus as "[one of] the three most important mathematical advances of the last quarter of the nineteenth century", but this is perhaps rather generous praise. Certainly, however, it was influential, and a modern version of it (known technically as the Laplace Transform) is a basic component of all courses in engineering mathematics.

Besides his difficulties with the "scienticulists" and "mathematicians of the Cambridge or conservatory kind", as he described them, Heaviside had little money and less prospect of improving his circumstances. Eventually in 1896 the Government gave him a Civil List Pension of £120 per annum - hardly a fortune. He lived in seclusion in Devonshire from around 1896 till he died in 1925. He never attended scientific meetings and had constant money problems. In later years he considered the propagation of free electromagnetic waves round the earth where his earlier work on the guidance of electromagnetic waves by conducting wires suggested a physical solution to the problem. He said "There may possibly be a sufficiently conducting layer in the upper air. If so, the waves will, so to speak, catch on to it more or less. Then the guidance will be the sea on one side and the upper layer on the other." The permanently ionised layer in the upper air was later called the Heaviside layer. Its existence is one of the factors that make short-wave radio possible.

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FROM HEAVISIDE'S WRITINGS

Oliver Heaviside, whose story is told in the preceding article, was given to a colourful and direct form of writing that provides interesting reading and introduces invective into mathematics in a peculiar, intensely personal way. He spoke, for example, of the "go" of an argument.

At other times he let the reader into his private difficulties.

"How is it possible to be a natural philosopher when a Salvation Army band is performing outside; joyously, it may be, but not most melodiously?"

This is from §4 of his three-volume work *Electromagnetic Theory* (EMT).

For his unusual approach to mathematics itself, we have the testimony of these quotes:

"..we shall have, preliminarily, to work by instinct, not by rigorous rules. We have to find out first how things go in the mathematics as well as in the physics. When we have learnt the go of it we may be able to see our way to an understanding of the meaning of the processes..." [EMT §239].

"Shall I refuse my dinner because I do not fully understand the process of digestion? No, not if I am satisfied with the result." [EMT §225].

These opinions are *not* today's accepted wisdom, but they did lead Heaviside to some interesting explorations.

DIFFERENCE OF INTEREST RATES OR

HOW TO MAKE YOUR FORTUNE BY INVESTMENT

G.B. Preston, Monash University

If a businessman will offer you 15% compound interest, per annum, on money that you lend him and your bank will lend you money at 10% compound interest per annum, what is the difference between these interest rates? On the face of it, obviously 5%. However, does this mean, that by borrowing money from the bank and lending it to the businessman you can effectively earn 5% on what you borrow (neglecting any questions of taxation)?

In fact, your effective rate of earning depends on how many years the money is lent for.

Suppose an amount of \$1 is borrowed from the bank and lent to the businessman. Interest payable to the bank at the end of one year is $\$\frac{10}{100}$ and interest you receive from the businessman is $\$\frac{15}{100}$; so your gain at the end of one year is $\$\frac{5}{100}$, and hence you gain 5% on what you borrow. All very simple.

Suppose, however, that the same amount is now borrowed and lent for an agreed period of two years, at the same annual rates of interest with settlements to be made only at the end of two years. At the end of two years the amount owed to the bank is $\$\left(1 + \frac{1}{10}\right)^2$, since interest for the second year is due on the total amount $\$\left(1 + \frac{1}{10}\right)$, namely the original amount borrowed plus interest due. Similarly, the businessman owes you, at the end of two years, $\$\left(1 + \frac{15}{100}\right)^2$. The difference between these is $\$(0{\cdot}1125)$. If this is equivalent to a rate of return of $r\%$ compound interest per annum, then

$$\$\left(1 + \frac{r}{100}\right)^2 = \$(1{\cdot}1125).$$

To solve this for r, take logarithms:

$$2\ \log\left(1 + \frac{r}{100}\right) = \log\ 1{\cdot}1125,$$

whence, by tables, or by hand calculator,

$$r \approx 5{\cdot}475.$$

Thus if the arrangement, as described, with the bank and the businessman, lasts for two years, you gain more than the 5% interest rate difference between the rate at which you lend and the rate at which you borrow. In fact you earn 5·475%.

This rate increases with the number of years for which the arrangement or contract is made. Calculations for differing numbers of years are in the following table.

No. of years of contract	Rate of interest at which borrowed	Rate of interest at which lent	Rate of interest earned
1	10	15	5
2	10	15	5·475
3	10	15	5·966
4	10	15	6·468
5	10	15	6·974
10	10	15	9·383
15	10	15	11·266
20	10	15	12·550
50	10	15	14·739

It appears that the effective interest rate you get from your transaction is tending to 15% as the length of the contract increases; in other words, provided the contract is for a long period, the eventual cost of borrowing the money you invest becomes negligible.

This may be proved. If n is the number of years for the contract, and r is the effective rate of interest (note that, as exhibited in the above table, the value of r depends on the number of years n), then the original \$1 will have increased to $\$\left(1 + \frac{r}{100}\right)^n$ at the end of n years. But the businessman pays you back $\$\left(1 + \frac{15}{100}\right)^n$ and you owe the bank $\$\left(1 + \frac{10}{100}\right)^n$, so that your profit at the end of n years is

$$\$\left(\left(1 + \frac{15}{100}\right)^n - \left(1 + \frac{10}{100}\right)^n\right) .$$

So the \$1 borrowed has become

$$\$\left(1 + \left(1 + \frac{15}{100}\right)^n - \left(1 + \frac{10}{100}\right)^n\right)$$

at the end of n years. Since this equals $\$\left(1 + \frac{r}{100}\right)^n$, we have the equation

$$1 + \left(1 + \frac{15}{100}\right)^n - \left(1 + \frac{10}{100}\right)^n = \left(1 + \frac{r}{100}\right)^n .$$

It may be shown that as n gets larger and larger, the value of r satisfying this equation increases steadily and approaches 15.

There is nothing special about the interest rates 10 and 15; all that is required is that the investment interest rate is greater than the rate at which the money is borrowed. We now give a demonstration of this for the general case.

So, suppose that the two interest rates involved are α and β, with $\alpha > \beta$, so that β is the rate of compound interest paid on the money borrowed and α is the rate of compound interest returned on your investment. Then, as for the special case just discussed in which $\alpha = 15$ and $\beta = 10$, the effective rate r of compound interest, obtained from an n-year contract of the type described, is given by the equation

$$1 + \left(1 + \frac{\alpha}{100}\right)^n - \left(1 + \frac{\beta}{100}\right)^n = \left(1 + \frac{r}{100}\right)^n \quad . \qquad (1)$$

We now show, using the fact that $\alpha > \beta$, that the solution r to this equation tends to α as the number of years n gets larger and larger.

To see this we simply need to use the fact that, if $0 < \delta < 1$ then $\delta^n \to 0$ as $n \to \infty$ (i.e. δ^n tends to 0 as n gets larger and larger).

Let us simplify the notation by setting $a = 1 + \frac{\alpha}{100}$, $b = 1 + \frac{\beta}{100}$ and $x = 1 + \frac{r}{100}$. The condition $\alpha > \beta$ is equivalent to $a > b$, and the equation (1) can be written

$$1 + a^n - b^n = x^n. \qquad (2)$$

Ths solution to this equation is

$$x = \sqrt[n]{1 + a^n - b^n}.$$

Here a and b are constants, and so x depends on n, the number of years. What we want to investigate is how x depends on n, in particular we want to find out what happens to the value of x as n increases.

We begin by rearranging equation (2). Divide by a^n, and equation (2) becomes

$$\left(\frac{1}{a}\right)^n + 1 - \left(\frac{b}{a}\right)^n = \left(\frac{x}{a}\right)^n \quad . \qquad (3)$$

Since $a > 1$ and $a > b$, we have $0 < \frac{1}{a} < 1$ and $0 < \frac{b}{a} < 1$. So $\left(\frac{1}{a}\right)^n \to 0$ as $n \to \infty$ and $\left(\frac{b}{a}\right)^n \to 0$ as $n \to \infty$. Thus the left-hand side of (3) tends to the value 1 as $n \to \infty$; hence the right-hand side also tends to 1, in other words

$$\left(\frac{x}{a}\right)^n \to 1 \quad \text{as} \quad n \to \infty. \qquad (4)$$

Now observe that, since $b > 1$, $b^n > 1$, and so, using equation (2),

$$x^n = a^n - (b^n - 1)$$
$$< a^n.$$

Thus $\left(\frac{x}{a}\right)^n < 1$, whence also

$$\frac{x}{a} < 1.$$

But for any positive number $s\,(< 1)$,

$$s > s^2 > \ldots > s^n > \ldots .$$

Hence

$$\left(\frac{x}{a}\right)^n < \frac{x}{a} < 1, \tag{5}$$

holds for each n and the corresponding solution x of (2).

We have seen in (4) that $\left(\frac{x}{a}\right)^n \to 1$ as $n \to \infty$. Hence, since by (5), for each solution x, $\frac{x}{a}$ is squeezed between $\left(\frac{x}{a}\right)^n$ and 1 we conclude that

$$\frac{x}{a} \to 1 \quad \text{as} \quad n \to \infty \quad . \tag{6}$$

Recalling that $x = 1 + \frac{r}{100}$ and $a = 1 + \frac{\alpha}{100}$, this means that

$$r \to \alpha \quad \text{as} \quad n \to \infty.$$

In other words, as claimed, we have shown that the effective rate of return on the transaction approaches the rate of interest you get on your investment, as the length of the contract gets longer and longer.

Hence, to make your fortune, you need no capital to invest. All you need to do is to borrow from a bank at the bank's rate of interest and then invest what you borrow at a higher rate of interest. The businessman you invest with will be glad that he need not repay capital, or interest, until a number of years have elapsed. You will of course find it more difficult to persuade a bank manager to allow you to repay your loan from the bank, together with all interest due, only after the lapse of a number of years. However, should you find a willing bank manager, then after the appropriate number of years, the net return to you will be effectively at the rate of interest the businessman pays you: relatively, the interest you have to pay the bank has become negligible.

Compound interest effects are common in the physical world. We now consider a quite different situation involving varying compound interest rates. Let us look at the relative birth and death rates of different species living in a particular region.

Suppose, for simplicity, there are two different species of animals competing for food in a certain game reserve. They do not attack each other, but depend on the same vegetation for food. Suppose that the vegetation suffices to keep alive and healthy only a fixed total number P of the members of both species each year. Suppose further that the net percentage rate of increase per year of species S_1 is α and that of the other species S_2 is β, with $\alpha > \beta$. Here, by net increase we are measuring the difference between total births and total deaths, from natural causes, in a year. In order to ensure that the animals in the game reserve keep healthy, each year game wardens cull the populations of the two species, killing members of either species at random until the total population is reduced to P.

Suppose that, at the beginning of a particular year, there are A members of species S_1 and B members of species S_2, so that

$$A + B = P.$$

At the end of the year, the number of members of species S_1 will have increased in size to $A(1 + \frac{\alpha}{100})$, while the number of species S_2 will have increased to $B(1 + \frac{\beta}{100})$. The total number of animals of the two species is then P_1, say, where

$$A\left(1 + \frac{\alpha}{100}\right) + B\left(1 + \frac{\beta}{100}\right) = P_1.$$

Culling then takes place, so that each species is reduced in size in the proportion P/P_1. At the end of culling, the first species has A_1 members, where

$$A_1 = (P/P_1)A\ (1 + \frac{\alpha}{100}),$$

and the second species has B_1 members, where

$$B_1 = (P/P_1)B\ (1 + \frac{\beta}{100}),$$

and where, consequently,

$$A_1 + B_1 = \mathrm{P}.$$

In general, if the population before culling, at the end of the n-th year, is P_n, and A_n and B_n are the populations of the first and second species, respectively, after culling, then similar considerations show that

$$A_n = \frac{P^n}{P_1P_2\ldots P_n}\, A\left(1 + \frac{\alpha}{100}\right)^n,$$

and

$$B_n = \frac{P^n}{P_1P_2\ldots P_n}\, B\left(1 + \frac{\beta}{100}\right)^n$$

Hence
$$\frac{B_n}{A_n} = \left(\left(1 + \frac{\beta}{100}\right)^n \Big/ \left(1 + \frac{\alpha}{100}\right)^n\right)(B/A)$$
$$= (b/a)^n(B/A),$$

setting $a = 1 + \frac{\alpha}{100}$ and $b = 1 + \frac{\beta}{100}$. Since $\alpha > \beta$, $a > b$ and so $\frac{b}{a} < 1$. Hence $(b/a)^n \to 0$ as $n \to \infty$.

Thus the effect of the culling procedure described will be, in the long run, to exterminate entirely one of the species from the game reserve.

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THE SIEVE OF ERATOSTHENES

Write down the sequence of odd numbers, beginning with 3:

3 5 7 9 11 15 17 19 21 23 25 etc.

Begin with the number 3, skip the next two numbers and cross out the next; skip the next two and cross out the next, and so on:

3 5 7 ~~9~~ 11 13 ~~15~~ 17 19 ~~21~~ 23 25 etc.

Now move on to the 5, skip the next four numbers and cross out the next; skip the next four and cross out the next, and so on:

3 5 7 ~~9~~ 11 13 ~~15~~ 17 19 ~~21~~ 23 ~~25~~ etc.

Continue with the 7, crossing out every 7th number.

The 9, being already crossed out, is not used, but we move on to the 11, now crossing out every 11th number and so on. This process leaves uncrossed only the odd prime numbers 3, 5, 7, 11, ... It is due to Eratosthenes, a mathematician of the second century B.C. The name "sieve" derives from the fact that on successive runs, we cross out, or let out of the sequence, multiples of the primes and retain at the end of the process only the prime numbers, much as a sieve separates (say) gravel from sand.

This is the process used by Colin Wright in his search for primes (p.20).

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MENTAL WIZARDRY

To find the square of any two-digit number ending in 5 (say 65), take the first digit (6) and multiply it by one more than itself (7) and adjoin 25 to the product (42) to get the required square (4225).

Can you explain why this trick works?

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LETTERS TO THE EDITOR

THE AMAZING CONVERGENTS

Consider the set of fractions

$$\frac{1}{1},\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \frac{41}{29},\ \frac{99}{70},\ \frac{239}{169},\ \ldots$$

in which each numerator (or denominator) is twice the preceding numerator (or denominator) increased by the numerator (or denominator) before that. E.g. $41 = 2 \times 17 + 7$ and $70 = 2 \times 29 + 12$.

These fractions converge to $\sqrt{2}$ and are derived from a *continued fraction*. (See *Function*, Vol.4, Part 4; Vol.5, Parts 2, 4.) They are termed the *convergents* of the continued fraction representing $\sqrt{2}$.

The digits making the successive convergents arise in another truly remarkable way. Consider the numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, ... These are the *triangular numbers* and each is of the form $\frac{1}{2}n(n + 1)$ for integral n. Consider also the numbers 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, ..., which have the formula $m(m + 1)$ and which I call the *oblong numbers*.

Can a triangular number also be an oblong number? To answer this, we seek integers m, n such that

$$\tfrac{1}{2}n(n + 1) = m(m + 1).$$

This equation may be written

$$(2n + 1)^2 = 2(2m + 1)^2 - 1,$$

an equation of the form

$$x^2 = 2y^2 - 1,$$

whose solutions (x,y) are (1,1), (7,5), (41,29), (239,169), ...

These solutions are, respectively, the numerator and denominator of the first, third, fifth, etc. fractions in the sequence of convergents.

We may now deduce the values of n, m and thus find the sequence of numbers that is both triangular and oblong. The first five are 6, 210, 7140, 242556, 8239770. Each of these factorises as the numerator and the denominator of *two successive convergents*:

$$
\begin{aligned}
6 &= 1.1.2.3 \\
210 &= 2.3.5.7 \\
7140 &= 5.7.12.17 \\
242556 &= 12.17.29.41 \\
8239770 &= 29.41.70.99.
\end{aligned}
$$

We may similarly ask if a triangular number may also be square, i.e. we wish to solve the equation

$$\tfrac{1}{2}n(n + 1) = m^2.$$

This equation reduces to

$$(2n + 1)^2 = 2(2m)^2 + 1$$

or

$$x^2 = 2y^2 + 1.$$

The solutions (x, y) are here (3,2), (17,12), (97,70), ..., and these pairs arise from the second, fourth, sixth, etc. convergents of the sequence. The numbers that are both square and triangular are $1 = (1.1)^2$, $36 = (2.3)^2$, $40816 = (5.7)^2$, etc., where the factorisations again relate to the convergents.

Garnet J. Greenbury,
123 Waverley Road,
Taringa, Queensland, 4068.

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TWIN PRIMES

I have been using the sieve of Eratosthenes (*see p.18, Eds*) in a computer search for large twin primes. One recent run found fifteen primes beginning with 117 959, and ending with 118 127. In this list were two pairs of twin primes: (117 977, 117 979) and (117 989, 117 991). These are particularly interesting in view of the fact that the four primes involved are consecutive. No primes lie between 117 979 and 117 989.

Colin Wright,
Science student,
Monash University.

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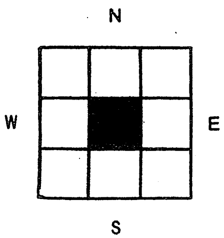
MULTIPLADDS

I enclose a copy of my book MULTIPLADDS, a collection of puzzles designed to provide a number of exercises in number facts and basic logic. In each puzzle, the three numbers in each side of the large square multiply together to give a common answer. Various patterns in the addition facts for each side are also specified.

Ken Montgomery,
P.O. Box 178,
Nunawading, Victoria, 3131.

[Mr Montgomery's book is available from him at the given address. We regret that we do not have a price, but Mr Montgomery would presumably supply this information. As a sample, we reprint one of the more interesting MULTIPLADDS below.]

<u>No. 43</u>



Put the numbers: 1, 2, 2, 3, 4, 6, 6 and 12 into the spaces so that:

(a) the three numbers in each side of the large square multiply together to give the same answer.

(b) the east side adds to one more than the north side, the west side adds to three more than the east side *and* the south side adds to five more than the east side.

\+ + + + + + +

MULTIPLADDS

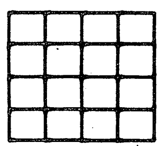
∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

THE VERY BEST MATCH PUZZLE

Readers of Volume 5 of *Function* will be familiar with our discussions of the Match Tricks on the back of recent series of Bryant and May matchboxes. These are supplied to us, with solutions, courtesy of the Wilkinson Match Company.

We print at right the most interesting of these from the mathematical point of view (and incidentially the most difficult also). Analogous puzzles can be produced for squares of different sizes and some relevant results are discussed on p.29, where the solution is given.

MATCH TRICK No. 23



What is the smallest number of matches you can remove so that no square of any size is left?

PROBLEM SECTION

We begin by solving some of the problems from previous issues.

SOLUTION TO PROBLEM 5.4.1

This problem read:

Let p be a prime greater than 3 and consider p consecutive integers. Square them and add them up. Prove that the result is divisible by p.

The special case $p = 5$ constituted Problem 5.1.4. The result is not true for $p = 3$, and it is clearly false for $p = 2$.

For any other prime p, $p = 6k \pm 1$ for some natural number k, since all other possibilities allow for division by either 2 or 3 (or both).

We now use the trick that Wen-Ai Soong used in her solution of Problem 5.1.4. She called her five consecutive numbers $a - 2$, $a - 1$, a, $a + 1$, $a + 2$. This must be generalised, but this is easily done. Our p consecutive numbers are

$$a - \tfrac{1}{2}(p - 1),\ a - \tfrac{1}{2}(p - 3),\ \ldots,\ a - 1,\ a,\ a + 1,\ \ldots,$$
$$a + \tfrac{1}{2}(p - 3),\ a + \tfrac{1}{2}(p - 1).$$

These p numbers are to be squared and added. We have

$$\begin{aligned}
&[a - \tfrac{1}{2}(p - 1)]^2 + [a - \tfrac{1}{2}(p - 3)]^2 + \ldots + (a - 1)^2 + a^2\\
&\qquad + (a + 1)^2 + \ldots + [a + \tfrac{1}{2}(p - 3)]^2 + [a + \tfrac{1}{2}(p - 1)]^2\\
&= (a^2 + a^2 + \ldots + a^2 + a^2 + a^2 + \ldots + a^2 + a^2) +\\
&\qquad 2\{1^2 + 2^2 + \ldots + (\tfrac{1}{2}(p - 3))^2 + (\tfrac{1}{2}(p - 1))^2\}\\
&= pa^2 + 2\{1^2 + 2^2 + \ldots + (\tfrac{1}{2}(p - 3))^2 + (\tfrac{1}{2}(p - 1))^2\}
\end{aligned}$$

since there are p terms in all. Notice that the terms not included in the sum cancel out in pairs: $-2a$ with $2a$, etc.

Now the sum of squares of the first n integers is given by the formula

$$1^2 + 2^2 + \ldots + n^2 = n(n + 1)(2n + 1)/6.$$

(For a derivation of this result, see *Function*, Vol.1, Part 3.)

Thus

$$\begin{aligned}
1^2 + 2^2 + \ldots + (\tfrac{1}{2}(p - 1))^2 &= \tfrac{1}{2}(p - 1)\tfrac{1}{2}(p + 1)p/6\\
&= p(p^2 - 1)/24.
\end{aligned}$$

But now recall that $p = 6k \pm 1$. So $p^2 - 1 = 36k^2 \pm 12k$, and $(p^2 - 1)/24 = (3k^2 \pm k)/2$.

If k is even this is clearly equal to an integer (say m).

If k is odd, a slightly more involved calculation shows that it is still integral. Again, call it m.

Thus the sum of our p consecutive squares reduces to $pa^2 + pm$, and this is clearly divisible by p.

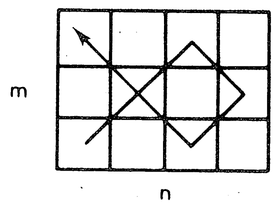
MORE ON PROBLEM 5.4.2

This concerned the scheduling of women drawing water from a village pump and to this extent was solved in our last issue. However, we asked a further question - how to schedule the women if two pumps are available? We leave this open for the present - indeed, we might equally ask: how to schedule m women if $n(<m)$ pumps are available.

SOLUTION TO PROBLEM 5.4.3

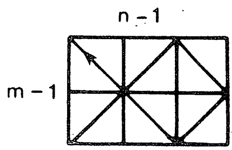
A billiard table is n units long and m units wide and a billiard ball has a diameter of one unit. The ball is fired from a corner at 45° to the sides. It is pocketed when it next visits a corner. If we imagine the table to be coloured like a chequerboard with squares of one unit on a side, how many squares does the ball visit in all?

This problem was solved by Vincent and Wen-Ai Soong of Casuarina, N.T. They first noted that the *centre* of the billiard ball moves through the centre-points of the squares, and these lie on an $(n - 1) \times (m - 1)$ rectangle. Thus the ball's path (Figure 1) may be replaced in our analysis by the path of the centre (Figure 2).



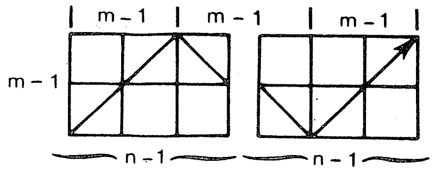
[1]

Vincent and Wen-Ai now write: "Repeat the lattice unit until the ball strikes a pocket. Note that this makes no difference to the number of squares visited. It just makes things more clear."



[2]

Figure 3 shows their reasoning. Note its relation to Figure 2. The argument may be viewed in another way.



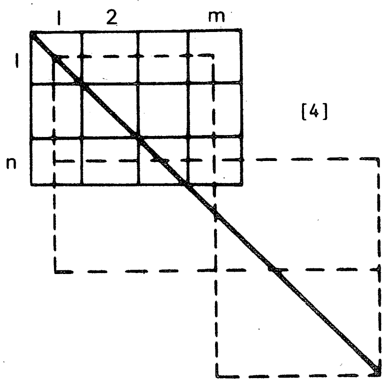
[3]

Imagine a mirror placed at the right-hand edge of Figure 2. When the ball is reflected off the right cushion, it appears in the mirror to be going straight on.

We now take up Vincent and Wen-Ai's argument again. "We want", they write, "the ball to reach one corner of the $(n - 1) \times (m - 1)$ lattice. But the ball always strikes the upper or lower sides of every $(n - 1) \times (m - 1)$ lattice (assuming, as drawn, $m < n$). Hence we require the lowest common multiple of $(m - 1)$ and $(n - 1)$." After being struck, the ball visits this many squares, and adding in the initial square, we get:

total number of squares visited = $1 + \text{L.C.M.}(n - 1, m - 1)$.

The argument may be slightly simplified by use of Figure 4. Here all reflections are removed and the path of the ball becomes a straight line. The formula derived by Vincent and Wen-Ai then appears in a very elementary way.



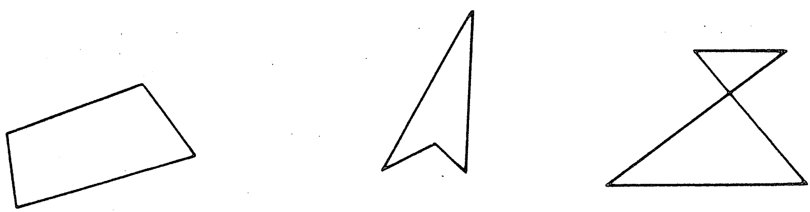
[4]

Note that if h is the highest common factor of $n - 1$, $m - 1$, then $\text{L.C.M.}(n - 1, m - 1) = (n - 1)(m - 1)/h$.

SOLUTION TO PROBLEM 5.5.1

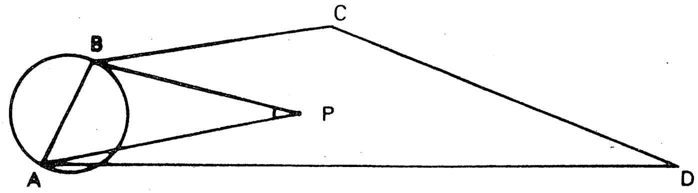
$ABCD$ is a quadrilateral. Circles are drawn on each of AB, BC, CD, DA as diameter. Let P be any point in the interior of the quadrilateral. Show that P lies on or within at least one of the four circles.

Quadrilaterals are of three types: convex, re-entrant and twisted (Figures a, b, c below).



a *b* *c*

We consider the convex case and leave the others as exercises to the reader (they are easier). Let P be a point inside the quadrilateral $ABCD$. (See the diagram below.)



In order for P to be outside the circle on AB, we must have $\measuredangle APB < 90°$. In order for P to be outside the other three circles we would need $\measuredangle BPC < 90°$, $\measuredangle CPD < 90°$, $\measuredangle DPA < 90°$. But at least one of these is impossible as

$$\measuredangle APB + \measuredangle BPC + \measuredangle CPD + \measuredangle DPA = 360°.$$

MORE ON PROBLEM 5.5.3

This was the traffic problem discussed in the last issue. If a car skids along a road, its stopping distance is determined by the magnitude of the retarding force. This, in the case of a skid on an incline, is partly due to friction and partly to gravity. Let the car travel up a slope of inclination θ and let the coefficient of friction be μ. g is the acceleration due to gravity.

The retardation in this case may be calculated to be $g(\mu \cos\theta + \sin\theta)$. When θ is 0, i.e. the road is flat, this reduces to $g\mu$. The defendant's case succeeds if the deceleration on the slope is less than that on the flat, i.e. if

$$\mu \cos\theta + \sin\theta < \mu$$
$$\text{or} \quad \mu(1 - \cos\theta) > \sin\theta.$$

This inequality reduces to $\mu > \cot\frac{\theta}{2}$ after some trigonometric manipulations. Now $\cot\frac{\theta}{2}$ decreases from infinity to the value 1 when $\theta = 90°$, the largest possible value. For practicable slopes, $\cot\frac{\theta}{2}$ exceeds 3. Most coefficients of friction lie below 1, values of about 0·4 being common, and although there are cases where $\mu > 1$ (Aluminium on Aluminium is an example: $\mu = 1{\cdot}4$), they do not occur on the roads. We learn from the Victorian branch of the Motorcycle Riders Association that the maximum value achieved in these circumstances is 0·85.

Thus the defendant should have lost.

This problem is based on one published in *American Mathematical Monthly* in 1963. The version published there was criticised for (in essence) its unrealistic value of μ. We had hoped to find a more believable case of the effect, but, alas, it is not possible.

We do not yet give the solution of the delightful

PROBLEM 5.5.2

To settle a point of honour, three men, A, B and C engage to fight a three-cornered pistol duel. A, a poor shot, has only a 30% chance of hitting his target; C is somewhat better, his chance of a hit being 50%; B never misses. A however has first shot. B, if he survives, fires next; then C; then A again, etc. However, if a man is shot, he takes no further part in the contest either as a marksman or a target. What should A's strategy be?

We hope readers will give this some very earnest thought and send the best of it to us.

Here now are some new problems.

PROBLEM 6.2.1 (From the 1982 Australian Mathematical Olympiad.)

A tosses $n + 1$ fair coins and B tosses n fair coins. What is the probability that A throws more heads than B?

PROBLEM 6.2.2 (From the same source.)

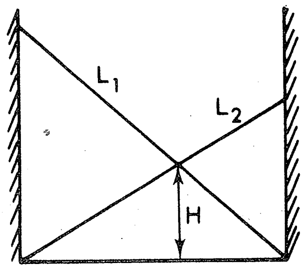
ABC is a triangle and the internal bisector of the angle A meets the circumcircle of ABC at P (as well as at A). Q and R are similarly defined in relation to B and C respectively. Prove that $AP + BQ + CR > AB + BC + CA$.

PROBLEM 6.2.3 (From the same source.)

Let $p_1 = 2$ and if $n \geqslant 2$ define p_n to be the largest prime divisor of $p_1p_2 \cdots p_{n-1} + 1$. Prove that $p_n \neq 5$ for any value of n.

PROBLEM 6.2.4

The ladders diagrammed at right are of lengths L_1, L_2 ($L_1 > L_2$) respectively. They cross at a point whose distance above the baseline is H. What is the distance between their feet?



In the version submitted to us, the following figures were given: $L_1 = 3m$, $L_2 = 2m$, $H = 0{\cdot}8$.

The problem is a famous one and we wish you luck.

PROBLEM 6.2.5

Three cabinets each contain two drawers. In each drawer a gold or a silver coin has been placed and the following information is supplied: *One cabinet contains two gold coins, another two silver, and the third one of each.*

A drawer is opened at random and is found to contain a gold coin. The other drawer in the same cabinet is then opened.

What is the probability that it too contains a gold coin?

PROBLEM 6.2.6

Let ABC be any triangle and P, Q two distinct points inside it. Find the shortest path from P to Q subject to the condition that the path must hit *each* side of the triangle.

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UPDATING TWO STORIES

A recent article of great interest and eminent readability is Jeremy Bernstein's profile of the computer scientist and A.I. (artificial intelligence) expert Marvin Minsky in *The New Yorker* (Dec. 14, 1981).

Among many other things, it gives more information on two matters discussed in *Function*. One concerns the alleged proof by a computer of a theorem in elementary geometry: the so-called *Pons Asinorum*, or Asses' Bridge (*Function*, Vol.3, Part 3). According to the story, H. Gelernter and N. Rochester programmed a computer to find proofs for Euclidean theorems and were surprised to find a proof that, although known to others, was not known to them at that time.

Bernstein quotes Minsky as claiming that he (Minsky) devised the basis for the program and, using hand simulation rather than a machine, discovered the proof which Gelernter later obtained by writing Minsky's routines out as a computer program and running it.

The article also has some interesting comments on the history of computer chess (described in *Function*, Vol.5, Parts 1 and 2). There are also articles on this topic in the journal *Chess in Australia* (August and September, 1981) by David Levy and Kevin O'Connell, who introduce their new programming concept "Philidor" (named after an early chess theorist).

"Philidor" is supposed to be programmed in a new way that incorporates strategic, as opposed to merely tactical, ideas. The games included do lend some credence to this view. Regrettably, the articles are rather in the nature of advertisements and the programming concepts are kept secret for commercial reasons.

MATHEMATICAL PRODIGIES

In a recent *Letter from America* (broadcast in Victoria on March 28, 29 by 3AR), Alistair Cooke tells the story of the youngest woman ever to win a Rhodes Scholarship: the 18-year old Nina Teresa Morishige, a first generation Japanese-American.

Nina has just graduated from the Johns Hopkins University in Baltimore and not only is her graduation some four years early, but she graduates not merely as a Bachelor of Science, but also as a Master of Science, with an average mark of 100%.

Her IQ, measured when she was four years old, was 171, but more to the point are her subsequent achievements. Scholastically, these have been in mathematics and related areas. Her specialties at this stage of her development are Real Analysis (one of the branches of advanced calculus), Differential Topology (the application of calculus to topology, or qualitative geometry) and Electromagnetic Theory.

While studying at Johns Hopkins, she studied Physics, Latin and Computer Science at another university. (Of Computer Science, she is quoted as saying that it is "boring and furthermore lacks the philosophical appeal and the rigour of advanced mathematics" - a judgement, she may, of course, modify.)

Furthermore, she plays the piano, flute and violin, the first of these so well that she won, at age 15, a national competition and appeared as a soloist with the Oklahoma Symphony Orchestra. Her sporting career has included winning (three times in a row) the U.S. Junior Golf Championship. She also plays chess and was vice-captain of Johns Hopkins' fencing team. Fear of injury to her hands led her to abandon softball.

Rhodes Scholarships are taken up at the University of Oxford which also has just admitted its youngest student ever, the 10-year old Ruth Lawrence. Ruth's field, like Nina's, is mathematics, and indeed her progress in that field is actually ahead of Nina's. There are, however, fewer details on Ruth's achievements available to us.

Most accounts of Ruth's admission to Oxford stress, pessimistically, the likelihood that she will come to nothing! They cite as precedent the story of Oxford's last mathematical prodigy, John Nunn, who gained a Ph.D. before he was 21 and is "now unemployed and reduced to competing in chess tournaments".

What such articles appear to overlook is that (a) Nunn is a grandmaster at chess, (b) his unemployment is surely an indictment of British society rather than of Nunn himself.

There have been several remarkable prodigies in the history of mathematics. We told (*Function*, Vol.1, No.1) the story of one, Louis Pósa. Norbert Wiener (of Cybernetics fame) was another, but the most famous of all is C.F. Gauss who deduced the formula

$$1 + 2 + \ldots + n = \tfrac{1}{2}n(n + 1)$$

when he was only seven years old.

Other feats fall short of the prodigy class, but are nonetheless remarkable. In *Function*, Vol.5, Part 2 we gave the 13-year old Jeanette Hilton's argument that

$$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2.$$

Of course, not all the world's great mathematicians began as prodigies. There are great "late developers" among them too.

SOLUTION TO MATCH TRICK NO.23

In the diagram given, 40 matches are arranged so as to make up 16 squares of side 1, 9 squares of side 2, 4 squares of side 3 and 1 square of side 4, a total of 30 squares.

Simpler versions of the problem would use

(a) 4 matches to make 1 square
(b) 12 matches to make 5 squares
(c) 24 matches to make 14 squares,

and then, of course, there are more complicated versions such as

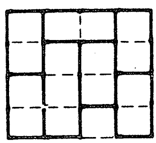
(d) 60 matches to make 55 squares.

(Can you derive formulae for the general case?)

It is not difficult to show that in Case (a) we need to remove one match to break up the (only) square, in Case (b) we need to remove 3 matches to break up all (5) squares and in Case (c), a little harder, 6 matches to break up all (14) squares. This would suggest that in the case given here, 10 matches need to be moved.

However, this solution is incorrect. We print at right a solution, put out by the Wilkinson Match Co. (Bryant and May) to show that the result may be achieved by removing only 9 matches.

SOLUTION



Even this, though, is not the end of the problem. How do we know there is not an 8-match solution? We could get a computer to check all $\binom{40}{8}$ possibilities for us, but here is a simpler proof due to Derek Holton, one of *Function*'s editors.

Imagine each small square (of side 1) coloured black or white to make up a 4×4 checkerboard. Next, note that the largest (4×4) square must be broken, so that at least one of the outer boundary matches must be removed. We may suppose, without loss of generality, that it comes from a white square.

Now consider the set of 8 small black squares. *No two of these have any match in common.* It follows that to break all these we must remove 8 matches. Thus, at a very minimum, we need to move $1 + 8 = 9$ matches. That this minimum is achievable is proved by the diagram.

There seems to be no formula known for the general case. An interesting investigation would be the design of efficient computer techniques for giving the minimum in any particular case.

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THE JOY OF MATHEMATICS

We reprint below an excerpt from a talk entitled "The greatest happiness of the greatest number" by Professor E.M. Patterson, Dean of Science at the University of Aberdeen. The text was prepared by Professor Patterson for delivery at the annual conference of the Mathematical Association (U.K.) in April 1981. It was read to the Association by Dr Roger Wheeler, Professor Patterson being unable to attend. Four days earlier, his wife Joan, also a mathematician, had died of the cancer that beset her last years of life.

The full text of the address appears in the *Bulletin of the Institute of Mathematics and its Applications* (October, 1981) and our excerpt is reprinted by arrangement with the Institute.

"I do not know at what stage of my life I began to experience genuine pleasure in mathematics. At school, most enjoyment was indirect; it came from the satisfaction of achievement, from a piece of work completed and sometimes, if I were lucky, actually praised, or from solving a problem. Success was all the more enjoyable if it signalled a reversal of fortune, with earlier feelings of inadequacy or failure overcome.

Certainly when I was an undergraduate in Leeds I was discovering real enjoyment within mathematics. A striking example was that of mathematical analysis. My first encounter with this was discouraging. I met it in all its glory at the beginning of my second year and I did not understand it one bit. I thought that the lecturer was making a great deal of fuss about very little. However I do not surrender easily and I had by that time learnt never to trust my own hasty judgements. I tried hard to find out what it was all about and in due course patience and perseverance were rewarded: the pennies began to drop. By the time I was in my final year, analysis was for me a most attractive subject. It was orderly, logical, consistent and had about it an air of aptness. Within it there were surprises; intriguing examples showed that things should never be taken for granted; for instance a function could be everywhere continuous and nowhere differentiable. The system of real numbers was full of fascination. The rational numbers could be put into one-one correspondence with the positive integers, but the real numbers could not. The continuum hypothesis was something to turn over in the mind and marvel at. Down to a very basic level the discovery that π was not, after all, equal to $\frac{22}{7}$ was a great moment. That other important transcendental number e is at the centre of some splendid elementary mathematics, which I still find thoroughly enjoyable when lecturing to first-year students. A friend of mine who was an engineering student used to revel in the fact that $e^{\pi i}$ is -1; so complicated a number is, after all, so simple.

Sometimes what appeals most is the sheer simplicity of an argument. The first of my two illustrations of this is well known, but that does not detract from the enjoyment. If twenty-seven teams enter a knock-out competition, how many fixtures are there? (It is assumed that each is continued until a definite result is obtained.) Answer: twenty-six. More generally, if there are n teams then there are $n - 1$ fixtures, because the competition has one winner and everyone else loses exactly once.

The second illustration of attractive simplicity is the theorem that for any continuous function defined on a circle there is at least one pair of diametrically opposite points at which the values of the function are the same. To see this, start at any point on the circle, take the difference between the value of the function there and at the diametrically opposite point; then move the two points around the circle continuously, keeping them diametrically opposite, until they have changed places. The difference in the values of the function varies continuously, but we end up with the negative of the starting value and so somewhere the difference must be zero. A consequence of this result is that if the temperature varies continuously over the earth's surface then on any great circle we can find at least one pair of antipodal points at which the temperatures are the same. In fact, a generalisation of the above circle theorem, known as the Borsuk-Ulam theorem, shows that on the earth's surface we can find at least one pair of antipodal points at which the temperatures are the same and the pressures are the same."

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FROM A CORRESPONDENT

Kim Dean, of the Urban Campagna, U.S.A., last wrote to us two years ago of Dr Dai Fwls ap Rhyll's winning of the Prix le Bon for his reflections on gravity. According to Dean, Dr Fwls has turned his attention to the foundations of mathematics - with some unexpected and even alarming conclusions.

"The famous Gödel theorem put paid to Hilbert's dream of establishing the self-consistency of mathematics", writes Dean. "Post Gödel", he continues, "it was impossible to hold the view that the mathematics we have grown to know could ever prove its own consistency. But this is not to say that it could not be found to be *inconsistent*."

Dr Fwls' results are expected to be as important to the foundation of mathematics as his earlier work was to the structure of theoretical physics. He has shown no less than the fundamental *inconsistency* of the whole of arithmetic!

The upshot of his investigations is still being studied. Already sources close to President Reagan have suggested that all mathematics departments in schools, colleges and universities throughout the U.S.A. will be forced to close their doors and some professors fear the lawsuits that will inevitably follow under the U.S. legal code.

Dr Fwls, in his capacity as expert witness for the U.S. Internal Revenue Service, is almost certain to be paid a large proportion of the billions of dollars likely to be claimed in retrospect from professors and teachers of mathematics throughout the nation.

U.S. mathematicians, faced with the loss of lifetimes' accumulations of salary and consultation fees, have attempted to offset the Reagan-Fwls challenge. Still, they are pessimistic, despite a nationwide fundraising campaign by the American Mathematical Association (AMA).

Professor X. Wisehead, speaking for the Association, said, "I mustn't give away the best of our position, but frankly, we have little comeback. We'll be instructing our lawyers to argue that if mathematics is inconsistent, then so, *a fortiori*, is law. Our further argument will be that if arithmetic is inconsistent, then, if the IRS reckon we owe them money, we can equally well claim they have overtaxed us for a quite indeterminate number of years."

Kim Dean writes that Mrs Thatcher's Britain is anxiously awaiting the outcome of the pending U.S. case. He suspects that Australia will follow suit if Mr Fraser is in power in the n years (n being large) required for this country to catch up with the overseas developments.

As with Dr Fwls' earlier work, his argument is easily presented. He defines

$$s = 1 - 1 + 1 - 1 + 1 - 1 + \ldots,$$

and then sets out to calculate s.

He argues first:

$$\begin{aligned} s &= (1 - 1) + (1 - 1) + (1 - 1) + \ldots \\ &= 0 + 0 + 0 + \ldots \\ &= 0. \end{aligned}$$

But then he has:

$$\begin{aligned} s &= 1 - (1 - 1) - (1 - 1) - \ldots \\ &= 1 - 0 - 0 - 0 - \ldots \\ &= 1. \end{aligned}$$

And, as if this were not enough, he sets:

$$\begin{aligned} s &= 1 - (1 - 1 + 1 - 1 + 1 \ldots) \\ &= 1 - s, \end{aligned}$$

from which,

$$2s = 1 \quad \text{or} \quad s = \tfrac{1}{2}.$$

Thus $0 = 1 = \frac{1}{2}$ and arithmetic is inconsistent. Never has the $- \times - = +$ rule been put to more mischievous use.

MONASH SCHOOLS' MATHEMATICS LECTURES, 1982

Monash University Mathematics Department invites secondary school students studying mathematics, particularly those in years 11 and 12 (H.S.C.) to a series of lectures on mathematical topics.

The lectures are free, and open also to teachers and parents accompanying students. Each lecture will last for approximately one hour and will not assume attendance at other lectures in the series.

Location: Monash University, Rotunda Lecture Theatre R1. The Rotunda shares a common entry foyer with the Alexander Theatre. For further directions, please enquire at the Gatehouse in the main entrance of Monash in Wellington Road, Clayton. Parking is possible in any car park at Monash.

Time: Friday evenings as below; 7.00 p.m. to 8.00 p.m. (approx.).

Program: The remaining talks are:

June 4 "How Aeroplanes Fly". Mrs B.L. Cumming.

June 18 "The Mathematics of the Rubik Cube". Dr J.C. Stillwell.

July 2 "Chaos - Fluctuations in Populations". Dr G.A. Watterson.

July 16 "Formation of the Solar System". Dr A.J. Prentice.

July 30 "Two Circles Intersect at Four Points!". Dr C.F. Moppert.

Enrolment: There will be no enrolment formalities or fees. Just come along!

Further Information: Dr C.B.G. McIntosh: (03) 541 2607