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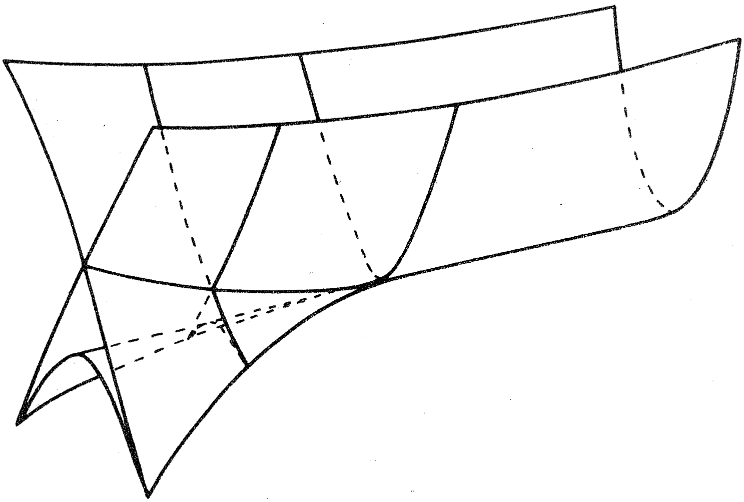

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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A surprising statistical phenomenon, a recent theorem of great generality and application, polygon constructions, the theory and practice of boomerang construction and throwing, a blind man's remarkable geometrical vision, and a debate in arithmetic. These form our offerings for this issue. Then there are also solutions to more of our problems, more letters to the editor and another match trick. We mollify an outraged typist and an enraged editor. Mathematics wasn't meant to be dull!

CONTENTS

THE FRONT COVER

M.A.B. Deakin, Monash University

Some years ago, Problem 1.2.1 (based on a 1975 H.S.C. question) involved exploring the way in which a quartic function altered its "shape" as the coefficients were changed. This type of question has come very much to the forefront of mathematics over the last ten years as a result of the rise of a discipline known as *Catastrophe Theory*. (See *Function*, Vol.1, Part 2.)

Our cover is one of the standard diagrams drawn in Catastrophe Theory; it relates to a slightly more complicated situation. We consider the quintic equation

$$y = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F \qquad (1)$$

and the shapes it may adopt. Equation (1) involves 6 *parameters* (constants whose values need to be known before we can plot the curve), but this number may be reduced by taking into account the fact that the *shape* of the curve does not depend on where the origin lies, nor on the scale we employ. This insight allows us to reduce Equation (1) to its standard, or canonical, form:

$$y = x^5 + ax^3 + bx^2 + cx \qquad (2)$$

involving only 3 parameters.

Equation (2) can give rise to three basically different shapes of quintic curve, with a number of "unbasic" intermediates. The basic types are:

(1) a graph with two maxima and two minima (Figure 1),

(2) a graph with one maximum and one minimum (Figure 2),

(3) a graph with no maxima or minima (Figure 3).

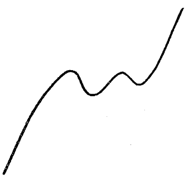

Fig. 1

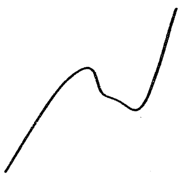

Fig. 2

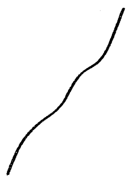

Fig. 3

Which of these shapes arises depends upon the values of a, b, c in Equation (2). It is possible to set up a three-dimensional coordinate system involving these parameters and to demarcate the three-dimensional space so spanned into three regions corresponding to the basic types of quintic. The cover diagram shows the result. The surface there drawn divides space into three regions:

(1) the tapering underbelly to the left,

(2) the region under the surface as a whole,

(3) the region above the surface as a whole.

The coordinates are placed on the diagram as follows. The first region comes to an end in the middle of the picture and this end-point we take as our origin. The a-axis runs tangentially to the right along the surface from this point; the b-axis is also tangential at the origin and is at right angles to the a-axis and points "inward"; the c-axis is perpendicular to both of these and points up the page.

Values of a, b, c corresponding to points in the first region give quintics of the first type (as in Figure 1). In the same way, regions 2, 3 correspond respectively with types 2, 3. Values corresponding to points on the surface give the additional types which we leave to the reader to discover.

The formula for the surface is complicated and has several forms. Here is one:

$$6b^2(3a^2 + 10c)(15b^2 + 3a^3 - 10ac) = c(30b^2 + 9a^3 - 10ac)^2.$$

The properties of the surface were first discovered by the French geometer *Bernard Morin*, who named it the *swallowtail* from the shape of the cross-section at the left. Morin is remarkable for his ability to visualize the properties of complicated geometrical objects. This ability is rare in any case, but quite unexpected in Morin's, as he has been blind since childhood. This front cover is thus a fitting one in this, the year of the disabled.

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UPDATE NO.1

A typist objected to the quote from Borel (*Function*, Vol.5, Part 3) which spoke of the "miracle of the typing monkeys" and then went on to discuss the situation in terms of an incarcerated French typist. Were we, she demanded, calling typists monkeys?

Sorry about that! Some background is in order. It was Sir Arthur Eddington in his book *The Nature of the Physical World* (1932) who introduced the typing monkeys. "If an army of monkeys were strumming on typewriters they *might* write all the books in the British Museum."

Borel presumably introduced the typist to overcome the obvious objection that the monkeys would probably jump up and down all over the typewriters and wreck them!

THE APPROXIMATE CONSTRUCTION OF REGULAR POLYGONS

J.W. Hille, SCV Frankston

The task of constructing a regular polygon of n sides inside a given circle using compasses and straight edge only (the "Euclidean" tools of classical geometry) has received considerable attention over a long period. For certain values of n (e.g. 3, 4, 5 and 6) the required solutions have been known since ancient times. At the age of seventeen, Gauss discovered that a regular polygon of p sides (p prime) is constructible using such tools if and only if p is of the form $p = 2^{2^n} + 1$ (Fermat numbers), i.e. $p = 3, 5, 17, 257, \ldots$. Conversely, the result indicated that for $p = 7, 11, 13, 19, 23, \ldots$ the construction was not possible.

In view of such a result it is not surprising that "approximate" methods of constructing the regular polygons were sought and this article concerns an analysis of a commonly employed procedure for obtaining approximately correct n-gons using Euclidean tools.

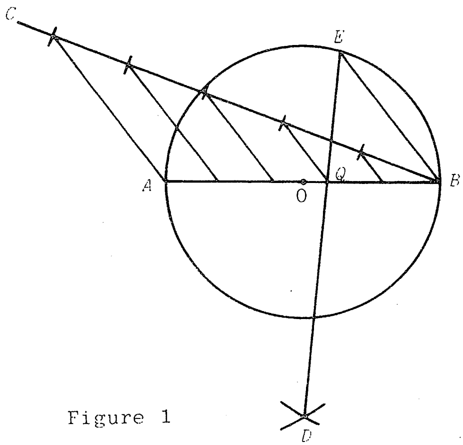

Figure 1

In this method (see Figure 1), the diameter AB is divided into n equal parts using parallels through corresponding equal intercepts on an arbitrary line BC drawn through one end of the diameter (a standard construction with Euclidean tools). Using AB as radius, circular arcs are described using A and B as centres respectively, the arcs intersecting at D. From D, line DE passing through the *second* intercept (Q) on the diameter meets the circle at E. EB is claimed to be the required side (approximately) of the polygon of n sides.

The accuracy of the method was put into question by an artist friend who found that "stepping-off" distance EB around the circle n times often led to rather large errors by the n^{th} side, no matter how carefully the construction steps were followed. He wondered if a theoretical analysis of the construction procedure would indicate the accuracy to be expected assuming perfect precision was possible. This led to an exercise in problem-solving which was of some interest to me since I had known of the method previously but had no direct knowledge of why or how well the construction worked.

Initial attempts using simple geometrical properties yielded little information so a trigonometrical approach was tried. The rather large algebraic expressions obtained were a bit "off-putting" so a third approach using the methods of coordinate geometry was tried. The approach centred on α, the angle subtended by side EB at the centre of the circle and I hoped to derive a formula of the form $\alpha = 2\pi/n \pm \beta$ where the "error" β could be stated (probably as some function of n). Surprisingly, the result eventually found was somewhat more complex and several computations were required to just check that $\alpha \approx 2\pi/n$ in each case. In what follows, d is the diameter, AB.

Figure 2 shows how the diagram of Figure 1 was located on the Cartesian plane. With the centre O at $(0, \frac{d\sqrt{3}}{2})$, the intersection of the arcs drawn from A and B becomes the origin, EB subtends angle α at O and the second of the equal intercepts on the diameter, Q, has coordinates $(\frac{d}{2} - \frac{2d}{n}, \frac{d\sqrt{3}}{2})$.

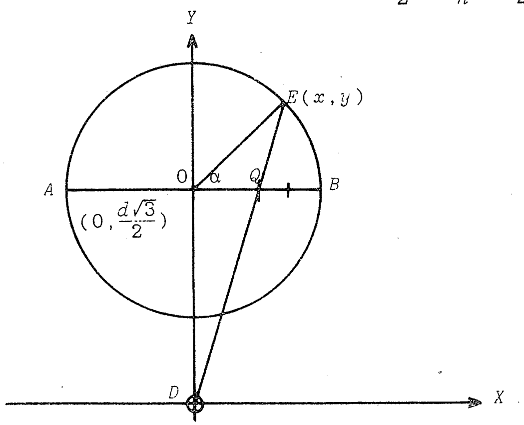

Figure 2

(i) Slope(DE)= $\frac{d\sqrt{3}}{2}$ / $d(\frac{1}{2} - \frac{2}{n}) = \frac{\sqrt{3}n}{n-4}$ so the equation of

DE is $y = \frac{\sqrt{3}n}{n-4}x$ or $\frac{x}{y} = \frac{n-4}{\sqrt{3}n}$ . . . . (1)

(ii) The circle has equation:

$$x^2 + (y - \frac{d\sqrt{3}}{2})^2 = \frac{d^2}{4},$$

and using (1) this gives

$$y^2 \frac{(n-4)^2}{3n^2} + y^2 - d\sqrt{3}y + \frac{3d^2}{4} = \frac{d^2}{4}$$

for the y-coordinate of E,

i.e. $(4n^2 - 8n + 16)y^2 - 3d\sqrt{3}n^2y + 3n^2d^2/2 = 0.$

This quadratic in y has, in general, two solutions given by

$$y = \frac{3\sqrt{3}n^2d \pm \sqrt{27n^4d^2 - 6n^2d^2(4n^2 - 8n + 16)}}{2(4n^2 - 8n + 16)}$$

but, for the point E, the diagram indicates that the larger solution involving the + sign is required and this value is given by

$$y = \frac{3\sqrt{3}n^2d + \sqrt{3}nd\sqrt{n^2 + 16n - 32}}{2(4n^2 - 8n + 16)} \quad . . . . \quad (2)$$

(iii) Now $\tan\alpha = (y - \frac{\sqrt{3}d}{2})/x$

$$= \frac{y}{x} - \frac{\sqrt{3}d}{2x} .$$

From (1), $\frac{y}{x} = \frac{\sqrt{3}n}{n-4}$ and, substituting (2) in (1) to evaluate x, we obtain

$$\tan\alpha = \frac{\sqrt{3}n}{n-4} - \frac{\sqrt{3}(4n^2 - 8n + 16)}{(n-4)(3n + \sqrt{n^2+16n-32})}$$

or $$\alpha = \tan^{-1}\left\{\frac{\sqrt{3}}{n-4}\left[n - \frac{4n^2 - 8n + 16}{3n + \sqrt{n^2 + 16n - 32}}\right]\right\}$$

. . . (3)

At first glance any resemblance to $\alpha = 2\pi/n$ seems remote but Table 1 indicates the results obtained using (3) for $n = 3$ to 30 together with the accumulated "error" for n sides.

It appears that the construction is theoretically quite accurate for small values of n with the cases $n = 3, 4,$ and 6 precisely so. This is not obvious (except for $n = 4$ where $\tan \alpha = \infty$) from an inspection of (3). The rather small errors involved up to about $n = 10$ probably justify the practical use of the method since few would use the method for larger n. It would, for example, appear more useful to bisect the side for a regular n-gon to obtain the $2n$-gon side where this was possible. The steadily increasing accumulated error reaches an interesting point at about $n = 24$ where both $\alpha°$ and $(360 - n\alpha)°$ are some 15° in magnitude - using the construction would lead to a situation where only 23 sides could be stepped off! The form of (3) sensibly indicates that, for large n, $\tan \alpha$ is approximately equal to $\frac{\sqrt{3}}{n}(n - \frac{4n^2}{3n+n})$, which equals 0 as required.

TABLE 1

n	$\alpha°$	$n\alpha°$	$(360 - n\alpha)°$
3	120	360	0
4	90	360	0
5	71.9535	359.7673	0.2327
6	60	360	0
7	51.5182	360.6276	-0.6276
8	45.1874	361.4990	-1.4990
9	40.2778	362.5003	-2.5003
10	36.3558	363.5581	-3.5581
11	33.1479	364.6273	-4.6273
12	30.4734	365.6811	-5.6811
13	28.2080	366.7041	-6.7041
14	26.2634	367.6879	-7.6879
15	24.5752	368.6283	-8.6283
16	23.0953	369.5241	-9.5241
17	21.7868	370.3754	-10.3754
18	20.6213	371.1835	-11.1835
19	19.5763	371.9501	-11.9501
20	18.6339	372.6773	-12.6773
21	17.7794	373.3673	-13.3673
22	17.0010	374.0222	-14.0222
23	16.2889	374.6443	-14.6443
24	15.6348	375.2355	-15.2355
25	15.0319	375.7979	-15.7979
26	14.4744	376.3333	-16.3333
27	13.9572	376.8434	-16.8434
28	13.4761	377.3299	-17.3299
29	13.0274	377.7942	-17.7942
30	12.6079	378.2377	-18.2377

In conclusion, it would be interesting to hear from any readers who may have found a purely geometrical (or any other) interpretation of the construction procedure or who have some knowledge of the origins of the method.

Further reading:

Interesting discussions concerning the mathematical interpretation of constructions possible with "Euclidean" tools are contained in:

1. Eves, H. *An introduction to the history of mathematics* (3rd edition), Holt, Rinehart and Winston, New York, 1969.

2. Courant, R. and Robbins, H. *What is Mathematics*, Oxford University Press, London, 1969.

A useful summary of general geometrical construction techniques is contained in:

3. Morris, I.H. *Geometrical Drawing for Art Students*, Longmans, London, 1958.

[*In the case of large values of* n *the relative error remains quite large. The relative error may be measured as*

$$\frac{\tan(\alpha - \frac{2\pi}{n})}{\tan \frac{2\pi}{n}}$$

It may be proved, using the author's Equation (3), that this tends to the limit

$$\frac{2\sqrt{3}}{\pi} - 1 \simeq 0{\cdot}1$$

as $n \to \infty$. *J.M.*]

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UPDATE NO.2

Our last issue twice criticised the mathematics writer, E.T. Bell. One editor (G.B.P.) wrote that "[he] has written attractively and at a level that can arouse interest and enthusiasm for sixth formers. We should direct readers to him - make him attractive. ... He has been one of the greatest stimulators of the study of the history of mathematics."

Here to redress the balance is Bell at his best. The source is *Mathematics - Queen and Servant of Science* (1951).

"[A] certain statistical investigation, full of high-powered mathematics, yielded the unexpected conclusion that vaccination against smallpox is a waste of time and money. This, of course, is the sort of applied mathematics that always delights one large section of the community, no matter how much it may annoy the medical profession. For my own part I have swallowed mathematics enough in my life to be immune to just one dose, and I shall continue to get smallpox vaccinations whenever I contemplate a vacation in any of the filthier parts of the North American continent.

It may be a prejudice on my part, but I believe that many of the more loudmouthed leaders in education and a majority of stupidly led teachers should take out similar insurance against the less desirable consequences of indiscriminate intelligence testing. They might be less enthusiastic were they to swallow

cont. p.14

WINNING STRATEGIES

John Stillwell, Monash University

Rod Topor's article on computer chess (*Function* Vol.5, Part 2) touched implicitly on a surprising result which is worth stating explicitly:

Theorem. In any finite two-person game with perfect information, one of the players has a winning strategy. (The theorem is a recent one, having been first stated explicitly in 1943 by von Neumann and Morgenstern in their book *The Theory of Games and Economic Behaviour*.)

To cover games such as chess, draughts, and noughts and crosses, where draws are possible, we define a winning strategy to be one which protects a player from being beaten. (In this sense, both players in noughts and crosses have a winning strategy.) A two person game is one in which players I and II move alternately, and "perfect information" means that I and II both know the complete state of play at any stage.

I shall give two proofs of the theorem. The first is more concise but only gives *existence* of a winning strategy; the second describes how to find it.

Proof by induction. Any finite game has a *length*, which is the maximum number of moves which can occur before play ends. (To get a finite length for chess or draughts one has to agree to stop when any position has been repeated a certain fixed number of times.) We now show by induction on n that, in any game of length n, one of the players has a winning strategy.

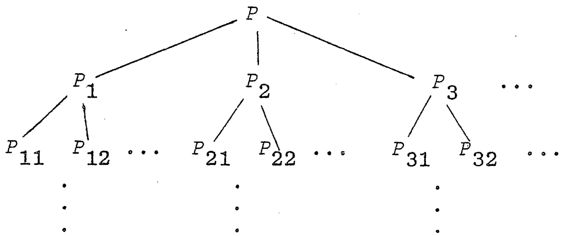
This is certainly true when $n = 1$, because then only player I has a move. If any move leads to a win for I or a draw, I's winning strategy is to play it. Otherwise, any move leads to a win for II, so II has a winning strategy (namely, sit and watch I lose!).

To complete the induction we have to show that if all games of length at most k have winning strategies, then so have all games of length $k + 1$. Let G be any game of length $k + 1$, and let $P_1, P_2, \ldots$ be the positions possible after I's first move. The continuations of G from these positions can be thought of as *new games* $G_1, G_2, \ldots$ of length at most k. Thus we can assume by induction that, in each of $G_1, G_2, \ldots,$ either I or II has a winning strategy.

But now a winning strategy for G follows in much the same way as for $n = 1$. If I has a winning strategy for any G_i, then he also has one for G - namely, make his first move into G_i, then play his strategy for G_i. Otherwise, II has a winning strategy for each G_i, and hence for G, since he can play his strategy for whichever G_i is entered by I's first move. The proof is now complete.

If you think the induction proof is a bit sneaky, since it doesn't actually reveal the strategy, the following proof will show what is really going on.

Proof by construction of the game tree. Let P be the initial position of G, let $P_1, P_2, \ldots$ be the positions possible immediately after P, let $P_{i1}, P_{i2}, \ldots$ the positions possible immediately after P_i, and so on. These positions are displayed in the *game tree*:



whose branches represent possible sequences of moves ("plays"). The endpoints of the branches are terminal positions, which can be inspected to find the winner of each play.

Put a label I on each endpoint which is a win for I, and II on each endpoint which is a win for II (so draws are labelled with both I and II). Now continue labelling upward by the following rules, which ensure that label I on position Q means that I has a winning strategy from Q, and label II on Q means that II has a winning strategy from Q. Q gets a label as soon as all the positions below it have been labelled, and the label is:

I if (i) it is I's move at Q and *some* position immediately after Q has label I

or (ii) it is II's move at Q and *all* positions immediately after Q have label I;

II if (iii) it is II's move at Q and *some* position immediately after Q has label II

or (iv) it is I's move at Q and *all* positions immediately after Q have label II.

Again, it is possible for Q to be labelled with both I and II. The important point is that *at least one* of the conditions (i) - (iv) must hold, so every position eventually gets a label.

It is also clear that whichever player has his label on Q has a winning strategy from Q by following his labels down the tree, i.e. moving to a position with his own label each time it is his turn. He cannot be "derailed" from a winning path, since his labels are on every available position whenever it is his opponent's turn. In particular, the player whose label is on the top position P has a winning strategy for the whole game. This completes the proof.

The strategy described in this proof is simple enough in concept, but completely out of computational reach for any interesting game. Games share this feature of computational difficulty with other natural problems discussed in my previous *Function* articles "Why mathematics is difficult" and "The two faces of coding theory". This is one of the reasons mathematicians and computer scientists are interested in them. In particular, it has been shown that computing a winning strategy for the Japanese game of *Go* (generalized to an $n \times n$ board) is at least as hard as solving any *NP*-complete problem.†

As mentioned in the above articles it is not yet *proved* that any of these problems is computationally infeasible, but if any proof is forthcoming, it will probably vindicate our feeling that games require *insight* more than computational power.

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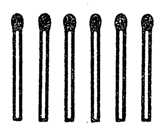
MATCH TRICK NO.15

To the right is one of the more difficult of the Match Tricks from the backs of the Bryant and May redhead series (supplied to us through the courtesy of the Wilkinson Match Company). Note that each of the six matches is to touch every one of the other five. Have a go at solving it - a thorough go - before looking up the solution, which is on p.32.

MATCH TRICK No. 15



Arrange 6 matches so that each match touches all the others.

†These remarks should be read in conjunction with the author's articles in *Function*, Vol.4, Parts 3, 5. See also the article by Hellman in *Scientific American*, August 1979.

WHICH SCHOOL?

G.A. Watterson, Monash University

WARNING: *The data in this article are entirely fictitious and any similarity to persons, living or dead, is purely coincidental.*

During my schooldays, I attended both State and Independent schools. Which type of school is better? The answer to such a question depends on what you expect a school to offer you. Would you emphasize academic instruction, or personal development, or sport, or social contacts, or ...? Even if you have a clear idea as to what a school *should* offer, it may not be easy to collect data which are relevant to the question, and even if you can, it may not be easy to interpret those data. This article is meant to point out one of the possible pitfalls in interpreting statistical data, not just in this educational context but more generally.

Suppose we decide to judge the performances of schools on the pass rates of their candidates in H.S.C. English; the higher the pass rate, the better. (Here, "pass" will be denoted by "*P*", and will mean a mark of 50% or more; a "fail" will be denoted by "*F*".) An English candidate will be denoted by "*S*" if he/she attended a State High School, and by "*I*" if he/she attended an Independent School. Suppose 80 candidates were studied, 40 from *S* and 40 from *I*, and their results were as given in Table 1.

Table 1

Pass rate depending on school type

	P	*F*	Total	Pass rate
S	24	16	40	60%
I	20	20	40	50%
	44	36	80	

On the basis of Table 1, you would conclude that it is better to go to *S* than to *I*, because the former has the higher pass rate.

But now suppose someone says that they think girls have a higher pass rate than boys, and maybe the sexes of the 80 candidates might be relevant. It turns out that exactly half were boys and half girls, and their results are as in Table 2.

Table 2

Pass rate depending on school type and sex

		P	*F*	Total	Pass rate
Males	*S*	3	7	10	30%
	I	12	18	30	40%
		15	25	40	
Females	*S*	21	9	30	70%
	I	8	2	10	80%
		29	11	40	

We see from Table 2 that indeed, girls do have the higher pass rate. But wait! What about the original question as to which type of school is better? Table 2 shows that, for *both* boys and girls, the pass rates at *I* schools are *better* than those at *S* schools. This is exactly the opposite conclusion to the one we drew from Table 1, in spite of the fact that we are basing our judgements on the results of the very same candidates!

What has happened here, of course, is that the inherently good English students (the girls) mostly went to State schools while the bad students (the boys) mostly went to Independent schools. One can't judge the performances of the schools fairly on the basis of Table 1, because they had, in considerable measure, different types of students to deal with.

On the basis of these (fictitious!) data, one might conclude that *I* schools were better than *S* schools for English. But wait again! Might there not be some other relevant factor, e.g. whether the schools were city or country schools? That could change our opinion again! For instance, Table 3 is quite consistent with Tables 1 and 2, and yet now, for *each* of the four categories of students, *S* results are better than *I* results, the opposite conclusion to Table 2, but back to the same conclusion as in Table 1!

Table 3

		P	*F*	Total	Pass rate
Country males	*S*	2	2	4	50%
	I	11	12	23	48%
		13	14	27	

Table 3

		P	*F*	Total	Pass rate
City males	*S*	1	5	6	17%
	I	1	6	7	14%
		2	11	13	
Country females	*S*	18	2	20	90%
	I	8	1	9	89%
		26	3	29	
City females	*S*	3	7	10	30%
	I	0	1	1	0%
		3	8	11	

What is the message of this article? It really has nothing to do with its title, but rather, to point out that if data are collected in an uncontrolled way, their interpretation may not be easy. The problem arises in many contexts, e.g. does smoking tend to cause lung-cancer, does drinking tend to cause road fatalities, do high wages tend to cause unemployment, etc. Without being able to do properly controlled experiments (as a scientist with good statistical training *should* do), a researcher can easily be misled as to what is important, and what is not, as a cause of some effect. The changing (indeed reversing) of percentages due to classifying data into more and more classes is known as "Simpson's paradox".

E.H. Simpson warned statisticians of this possibility in 1951 in an article he wrote in the *Journal of the Royal Statistical Society*. Of course in practice it would be more usual for data, such as in our Tables 1, 2 and 3, to lead to the *same* conclusion whichever table was studied, rather than to the apparently *contradictory* conclusions we have reached here.

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UPDATE NO.2 (CONT. FROM P.8)

a liberal dose of the higher mathematics, including the theory of probability, on which the useful art of intelligence testing is based. The same holds for most applications of mathematics to the actual world. Only a decently critical familiarity with *all* the assumptions underlying a particular mathematical formula can teach us what not to take too seriously when the formula presents us with an impressive-looking number. Mathematicians are not, as a rule, credulous; their clients almost invariably are."

MANY HAPPY RETURNS†

The boomerang is thought by most people to be an exclusively Australian artefact, as unique to the Antipodes as the kangaroo, the wombat and the duck-billed platypus.

However, this is not quite true. Although the Australian Aborigines have been throwing boomerangs for probably tens of thousands of years, they have been used in north-central Europe, India, Ancient Egypt and Central America. It was, however, in Australia that the boomerang reached the highest stage of development and provides the oldest example of applied aerodynamics in the hands of man. Today the boomerang has been defined as a *curved, two-bladed throwing stick which when thrown returns to the thrower.* In fact this type of boomerang, popularly regarded as a hunting weapon, has never been a killer. It was its close relative, the *kylie* or curved throwing stick that was used by the Aboriginal hunters. This longer, heavier example of the curved aerofoil was the real hunting weapon. It was not meant to come back in flight but to fly along close to the ground, spinning on an unerring course for up to 200 metres and cutting a swath a metre across. The boomerang *per se* was a sporting implement, a trainer of skills by means of which young men could practise their hunting weapon throwing technique.

Boomerangs come in a variety of sizes, ranging in wingspan from about 10cm to 1m, although these extremes can be considered more as curiosities than practical boomerangs. A 30 to 60cm wingspan is more usual. The materials from which they are manufactured are equally diverse, wood being the more common material for the traditional Australian weapons, although one example in the collection of the Museum of Western Australia in Perth is made of iron! (Well, steel ships float!). Modern sporting boomerangs are made from just about anything available including plywood, duralumin and acrylics: fibreglass laminates are becoming popular.

But why does a boomerang fly, and why does it perform in the way it does? The explanation lies in the cross-sectional shape of the arms and in the fact that the boomerang spins. On close examination a boomerang is seen to be something other than a bent stick of flat section: the arms are flat only on their lower surfaces. The upper surfaces are curved, forming a blunt leading edge tapering to a thin, sharp trailing edge.

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†This article is reprinted, with permission, from TAA's inflight magazine *Transair* (March 1981). For more on boomerangs, see the book *All about Boomerangs* by Lorin and Mary Haws (publ. Hamlyn, 1975), and Felix Hess' article in *Scientific American* November, 1968.

This shape is that of a classical aerofoil, although the curved contours are not symmetrical as in a pair of aeroplane wings. Rather it resembles a three-bladed propeller. Lift is thus generated by this asymmetric wing section in a similar way to that in which lift is produced by an aeroplane wing. The same laws of aerodynamics apply *as far as lift is concerned.* The fact that the boomerang is launched on a spinning flight gives it stability but it is also due to the rotational mechanics of the spinning that the boomerang returns to the thrower.

The boomerang is thrown by gripping the end of one of the arms and flicking it forward *in an almost vertical position.* Its initial flight is therefore straight out in front of the thrower and in an upright position. When thrown by a right-handed thrower it leaves the hand spinning very fast in a counter-clockwise (seen from the curved side) direction, the flat side of the boomerang should be to the thrower's right, the plane of the boomerang is almost vertical, being inclined at an angle of almost 60 degrees from the horizontal. The direction of throw is straight ahead, and is neither upward nor downward.

When a wing is at the top of a rotational cycle its speed through the air equals the speed imparted by the spinning added to the forward speed of the boomerang as a whole; conversely, when the same wing is at the bottom of a rotational cycle its speed through the air is less because the speed of rotation now opposes the forward speed of the boomerang as a whole.

Thus, although both wings of a boomerang are contoured to obtain lift from moving air, a wing at the top of its rotational cycle generates more aerodynamic lift (because of its greater speed through the air) than it does at the bottom of the cycle.

Because the boomerang begins its flight in an almost vertical plane, the stronger aerodynamic force at the top of the rotational cycle tries to *lift* the top of the boomerang into a position even closer to the vertical, thrusting anti-clockwise against the flat under-surface. However, this does not happen, or even begin to happen, because of a characteristic of spinning bodies known as *gyroscopic precession.* This counteracts the tendency to push the boomerang into a vertical plane by twisting the axis of rotation instead, and the boomerang veers off to the left rather than going straight ahead. (This is the same effect which causes a child's spinning top to *wobble* as the pull of gravity tries to overcome the gyroscopic effect of the spinning.)

As the boomerang turns to the left it begins to experience another force, an upthrust on its under-surface, because it is inclined at an angle to the direction of the airstream. The effect of this is fairly direct: the boomerang begin to rise. As it does so, its forward speed decreases slightly, as does the speed of rotation of the wings, for the energy to raise the boomerang has to come from somewhere. While all this is going on the original forces which caused the boomerang to veer to the left are still acting, so that the boomerang arcs its way to the left in a sweeping and slightly rising curve. At the top of this curve the blades are spinning noticeably more slowly and the direction of travel has changed remarkably from what it was originally, being about at right angles to the original direction. In effect, the boomerang is now on its way back to the thrower. The boomerang, having lost forward air-speed in climbing, no longer experiences up-thrust from the air pushing against its under-surface; so begins to lose altitude. This in turn makes the rotation speed increase, though not sufficiently to create a thrust towards the vertical as was the case when the flight began. Instead, the aerodynamic force generated by the rotation is expressed in an extra lift to the wings, so that near the end of the flight the boomerang is rotating in a nearly horizontal plane - floating at a fairly constant height until its forward speed has diminished to zero and its plane of rotation is indeed horizontal. At this point the boomerang hovers down slowly into the waiting hands of the thrower. To sum up, therefore, a good boomerang travels horizontally (for most of the time) in a circle and, on returning will hover before dropping to the ground at the end of its flight.

Now that it has been explained that the boomerang is launched in spinning flight, why should it be bent as it is? The elbow angle of most boomerangs is within 20° of the apparently theoretically ideal of 109°. It doesn't fly or glide like an aeroplane, so the swept wing effect is not due to the same considerations involved in aircraft wing design. The answer lies in a phenomenon known as the rotational inertia which is the inherent ability of an object to keep spinning for a long time. Rotational inertia is measured by a quantity known as the "moment of inertia". Different geometric shapes have different moments of inertia. The best shape is a thin circular rim, like a bicycle tyre; one of the worst is a straight configuration like a wing spinning about its midpoint. Thus the shape of a boomerang is a compromise between these two forms, one with a high moment of inertia but no aerodynamic lift, and the other with good aerodynamic lift but poor spinning ability.

The lengths of the arms, their thickness, and the wing contours can all be varied so as to produce the sort of flight desired. Long wings rotate relatively slowly and short ones spin fast. Thick wings can produce more aerodynamic lift, but they also generate more drag in the air and may slow down the spin disastrously (the result is a boomerang which is very lively at first but which runs out of steam after a few seconds). Conversely, thin wings do not lend themselves to fast turnabouts but retain their spin more efficiently. Thus, performance depends upon arm design.

The elbow angle of the boomerang also has an important bearing upon the moment of inertia. Within reasonable limits, the more acute the elbow angle the higher will be the moment of inertia; conversely, the more obtuse the angle the lower the moment of inertia for wings that are otherwise the same.

An interesting feature of boomerangs is that they can be either left or right-handed depending upon whether the thrower is a left-handed or right-handed person. Structurally, left-handed boomerangs are mirror images of right-handed ones and, as is to be expected, they fly in almost exactly opposite directions to their right-handed counterparts. They spin in a *clockwise* direction leaning 30° to the left of vertical and land on the thrower's right. Throwing a left-handed boomerang in the normal right-handed way gives spectacular although short-lived results. Since the aerofoil surfaces of a boomerang operate effectively only when the boomerang is spinning in the correct direction, and since this sort of throw makes them spin in the opposite direction, the boomerang goes out straight, then apparently goes haywire, fluttering wildly to the ground like a one-winged bird.

Like frisbee, kite-flying and similar activities, boomerang construction and throwing has become a popular sport. Competitions and record attempts are organized on a world-wide basis, although the epicentre of such activity tends to be here in Australia. Boomerang throwing competitions can be quite exciting events and the possibilities for a unique throw to be recorded for posterity are very real. The following World Records give some idea of the honours to be won:

Greatest distance thrown: 146.3 metres (Frank Donnellan, Centennial Park, Sydney).

Most consecutive single-handed catches: 36 (Denis Maxwell, Dinley, Australia).

Accuracy: 11 straight catches without moving either foot (Robert Boys, Merrian, Kansas, U.S.A.).

Shortest range: 1.85 metres (Bevan Rayner, Sydney, using a 40 centimetre boomerang).

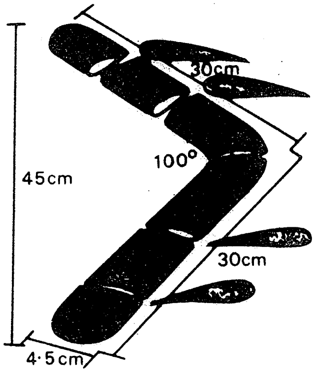
Most consecutive catches: 129 (John McMahon, South Padre Island, Texas, U.S.A.).

Best feat with feet: Joe Timbrey of La Perouse, Australia, has on many occasions, one of which was in the presence of Queen Elizabeth II, thrown a boomerang and caught it with his bare feet on the return.

To get fun from boomerangs one can simply buy one and gain expertise in throwing it, but much more satisfaction can be gained by making one's own boomerang. Tuning and trimming its flight also leads to a much fuller understanding of its aerodynamic properties.

As previously stated, a variety of materials can be used to make boomerangs; however to begin with a relatively simple one can be made by using marine grade plywood. Marine grade ply is preferable as it is resistant to damp and therefore less susceptible to warping, for warping will alter the shape of the aerofoil and thereby the flight characteristics of the finished article. Marine grade plywood is also dense which means a relatively heavy boomerang. Draw the outline of a boomerang on to a sheet of ply having at least 5 laminations and about 8mm thickness.

The angle of the arms should be as near as possible to the theoretically ideal angle of 109°. First cut out the boomerang blank; the arms should then be shaped to their aerofoil section with a rasp or coarse sand-paper; that is, rounded on the top surface and flat on the bottom surface, with a blunt leading edge and a sharp trailing edge. The arms must have the cross-sectional shape of a classic aerofoil providing lift much as the classic aerofoil does.



Remember that, since the boomerang is launched on a spinning flight, the outside vee formed by the arms will consist of one leading edge and one trailing edge. Similarly, the inside vee has the same characteristics but on opposite arms. Some dimensions are given on the illustration, but are probably not critical as long as the general proportions are respected. Final finishing is achieved with fine sandpaper, but before doing so a test flight can be made to see how the basic shape flies. If its performance does not come up to expectations, continue shaping the arms in order to modify the aerofoil sections.

Remember that no two boomerangs have exactly the same flight characteristics: it may be necessary to adjust one's throwing technique slightly to adapt to a particular boomerang. If the boomerang is capable of an exact return without wind assistance it is by definition a success. Don't forget that these instructions concern basic methods of making a *right-handed* boomerang. A left-hander can be made if required by shaping the arms to a mirror image of that illustrated. Contrary to popular belief, the boomerang is usually thrown from a *vertical* position with the concave vee section formed by the arms facing in the direction of flight, that is away from the thrower, if the shaping was done properly! This is not the only way to throw, but no doubt anyone who gets into boomerangs will find the other methods by trial and error.

It may be seen that boomerangs are by no means hit and miss affairs. They fly according to strict rules of physics and aerodynamics and their flight path can be predicted and/or modified at the construction stage. To prove the point here are two examples of extraordinary boomerang throwing. Felix Hess, a Dutch mathematician, once worked out the theoretical trajectory of a particular boomerang by pure mathematics and concluded that he should be able to throw it around the Washington Monument, a building which has a base of 16.9 metres square. Hess constructed his boomerang and threw it successfully to circumnavigate the said Monument.

The late Frank Donnellan of Parramatta had such confidence in the performance of his boomerangs that he used to throw a boomerang whilst standing blindfolded and would allow the boomerang to hit an apple balanced on top of his head. Even William Tell would have raised his hat!

So, if you have become bored with frisbees, kites or other flying objects, try a boomerang. And if you get fed up with it too you can always throw it away - I suppose!

$0^0 = 1$?

M.A.B. Deakin, Monash University

In my final year at high school, I had an argument with my maths teacher over the value of 0^0. He said that it was equal to 1, whereas I held it to be meaningless just as $\frac{0}{0}$ is - that is to say that we may assign to it any value we please. It perhaps says something for my mathematical ability that I was, by and large, right. But it also speaks for my arrogance at the time that I now do not remember the grounds on which he argued his side of the case - or even, for that matter, if he had any.

He was, however, in good company. I read recently that *Euler*, one of the greatest mathematicians of all time, held the same view. At first I was suspicious of this claim and so I looked it up - it is correct. If you have access to Euler's collected works, you may check it also - the very first volume, page 65, with a footnoted disclaimer by the 20th century editor.

This seems to have been the first attempt to discuss the matter and the discussion is brief but enlightening - for Euler did give a reason for holding the view that he did. He argued as follows:

> *For every non-zero* a, $a^0 = 1$. *It is therefore reasonable to* define 0^0 *to be 1 also.*

In so arguing, Euler employed a form of discussion still used today. For example we know that, when m, n are positive and $m > n$, $a^{m-n} = a^m/a^n$. We thus *define* this relation to hold also when $m = n$ and so reach $a^0 = 1$, the relation Euler used.

Euler's view was unchallenged until the 1820's, when another mathematical great, *Cauchy*, stated (in a text-book on calculus) the modern view that 0^0 may take any value at all. There he gives no reason for this opinion, but in a slightly later set of lecture notes, he states that 0^0 has the value $\frac{0}{0}$ - that is to say we do not assign it a definite value.

There the matter rested until, in 1830, a mathematician called *Libri* reiterated Euler's argument, and apparently he was expressing the popular view. For Cauchy's opinion had no support at all until 1834, when an anonymous author (he called himself S., and I have no idea who he was) wrote a brief article on the other side of the debate. Here S. queries the "almost unanimous opinion of all geometers, in the works that have come to my notice". Elsewhere he notes that "Cauchy is the only author in whose work I have found [my] opinion".

And S. argues his case. He proceeds as follows:

$$a^{m-n} = \frac{a^m}{a^n} , \qquad (1)$$

and so, when $a \neq 0$, $a^0 = 1$, *but if, in Equation (1), we put* $a = 0$, *and* $m = n$, *we get* $0^0 = \frac{0}{0}$, *which is undefined.*

However, S. was immediately challenged by *Möbius* (of Möbius strip fame) who published a proof of the equation

$$0^0 = 1.$$

The proof he attributed to an earlier (1814) unpublished work by the geometer and analyst *Pfaff* (still remembered for work on partial differential equations). Möbius had been one of Pfaff's students. Pfaff's proof, as revised by Möbius, essentially gives the result

$$\lim_{x \to 0} x^x = 1, \qquad (2)$$

which is true, although I will not give either their or any other proof in the course of this article. (You may care to explore the behaviour of x^x with a calculator.)

However, S. was not to take this lying down. He replied - as did another author, who remained completely anonymous, that Equation (2) did not settle the problem. Suppose we choose

$$\lim_{x \to 0} x^{f(x)}, \quad \text{where} \quad \lim_{x \to 0} f(x) = 0 , \qquad (3)$$

or, for that matter,

$$\lim_{x \to 0} [F(x)]^{f(x)}, \quad \text{where} \quad \lim_{x \to 0} F(x) = \lim_{x \to 0} f(x) = 0. \qquad (4)$$

For example, taking logarithms to base e and remembering that $\lim_{x \to 0} \log x = -\infty$, we have

$$\lim_{x \to 0} \frac{a + x}{\log x} = 0,$$

but , as $x = e^{\log x}$,

$$\lim_{x \to 0} x^{\frac{a + x}{\log x}} = \lim_{x \to 0} e^{(\log x)(\frac{a + x}{\log x})} = \lim_{x \to 0} e^{a+x} = e^a ,$$

which may take any positive value at all (depending on the value of a). This example came from Anon., who used the form (3). S. gave, using the form (4), the example

$$\lim_{x\to 0}\left(e^{-\frac{1}{x}}\right)^{x} = e^{-1} \neq 1$$

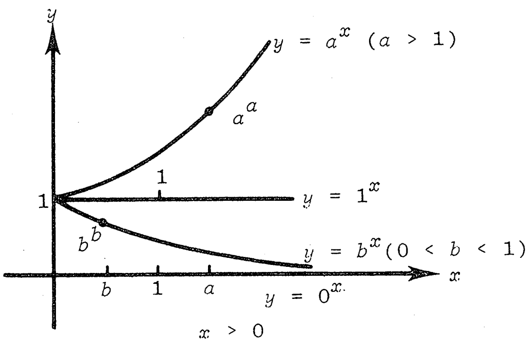
and

$$\lim_{x\to 0}\left(e^{-\frac{1}{x}}\right)^{2x} = e^{-2} \neq 1,$$

etc. (Actually, in this last example, S. got $\frac{1}{e^{\frac{1}{2}}}$. I sincerely hope this was a misprint.)

The argument continued, but Cauchy's view gained ground, largely due to arguments such as these. Oddly enough, when I had my debate with my teacher, I used much more elementary considerations. One could, it seemed to me, argue (as I now know Euler did) that $a^0 = 1$ for all non-zero a, and hence put $0^0 = 1$; on the other hand, one could say $0^a = 0$ for all positive a and $0^a = \infty$ (or, if you prefer, does not exist) for all negative a. So it seemed to me that the safest thing to do was to leave it undefined.

A related point is made by the graphs of Figure 1. $a^x \to 1$ for $a > 0$ but if $a = 0$, the limit is zero, if approached via positive x.



The matter still generates discussion today. In 1970, a mathematician called *Vaughan* argued for the equation $0^0 = 1$. His main ground was this:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$
$$= x^0 + x^1 + x^2 + x^3 + \ldots,$$

a form which is valid if $-1 < x < 1$. So, put $x = 0$ and get

$$\frac{1}{1} = 0^0 + 0,$$

giving $0^0 = 1$. Actually, Libri had used a similar argument 140 years earlier.

The value we assign to 0^0 depends upon the assumptions we make in setting up the definition; these in turn depend upon the purpose we have in mind.

$\frac{0}{0}$ may be assigned values (by means of calculus) in specific contexts as the result of specific limit operations. Much the same is true of 0^0, but there is, if you like, a "more preferred" value in this instance.

Numerous commentators have remarked that although (as we have seen) there are cases where the limits involved in Forms (3) and (4) are not equal to 1, these are rather difficult to find. Usually, the limit works out to be 1. This observation was explained in 1954 by the U.S. mathematician L.J. Paige, whose result may be given fairly simply.

THEOREM. *If, in Expression (3), $f(x)$ possesses a derivative at $x = 0$, the limit is one.*

A relatively simple proof (not that used by Paige) may be constructed using the Pfaff-Möbius Theorem (Equation (2)). It runs as follows.

PROOF. Let the value of the derivative $f'(0)$ be a. Then, by definition of the derivative,

$$\lim_{x\to 0} \frac{f(x)}{x} = a.$$

So, for sufficiently small x, $f(x)$ may be approximated to arbitrary accuracy by ax. Hence

$$\lim_{x\to 0} x^{f(x)} = \lim_{x\to 0} x^{ax} = \lim_{x\to 0}(x^x)^a = 1^a = 1.$$

Notice that Anon.'s counterexample, with $f(x) = \frac{a+x}{\log x}$, does not fall within the purview of the theorem, as $\log x$ has no derivative at $x = 0$. S's counterexample is a little more complicated, as we need to put it into the form (3) before we can analyse it. When this is done, we see that it becomes equivalent to

$$\lim_{x\to 0} x^{-\frac{1}{\log x}} \left[= \lim_{x\to 0} e^{(\log x)(-\frac{1}{\log x})} = e^{-1}\right],$$

and once again $\log x$ has no derivative at the origin.

In 1978, two other mathematicians, Baxley and Hayashi, looked at Form (4) and generalised Paige's result. The details are, however, technical and I will omit them. This is where the matter currently rests.

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LETTERS TO THE EDITOR

GREENBURY'S THEOREM

Greenbury's Theorem is a generalised extension of Cohen's First Theorem (*Function* Vol.5, Part 2, April 1981).

Form a sequence by the rule:

$$T_m = 2n\,T_{m-1} + T_{m-2}\,,\quad T_0 = 0$$

$$T_1 = 1$$

$$T_2 = 2n(1) + 0 = 2n$$

$$T_3 = 2n(2n) + 1 = 4n^2 + 1$$

$$T_4 = 2n(4n^2 + 1) + 2n = 8n^3 + 4n$$

$$T_5 = 2n(8n^3 + 4n) + 4n^2 + 1 = 16n^4 + 12n^2 + 1$$

$$T_6 = 2n(16n^4 + 12n^2 + 1) + 8n^3 + 4n = \text{ etc.}$$

Right-angled triangles are now calculated by the Pythagorean Triple:

$$a = x^2 - y^2$$

$$b = 2xy$$

$$h = x^2 + y^2$$

where y and x are consecutive members of the sequence; a and b the sides about the right angle; h the hypotenuse.

Greenbury's Theorem states that $b = n.a \pm 1$ where $n = 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, \ldots$ to infinity and $nh/a \sim \sqrt{n^2 + 1}$, when n is an integer, and $nh/a \sim \sqrt{(n + \frac{1}{2})^2 - 1}$, when n is not an integer.

To explain line 1 in the table opposite, let $n = 1$.

If $n = 1$, the sequence is 1, 2, 5, 12, 29, 70,... . The sides of the right-angled triangle are:

y	x	$x^2 - y^2$	$2xy$	$x^2 + y^2$
1	2	3	4	5
2	5	21	20	29
5	12	119	120	169
12	29	697	696	985
29	70	4059	4060	5741

It can be seen that $b = 1.a \pm 1$ alternately. Also

$$\left.\begin{aligned} h/s &= 5/3 = 1{\cdot}6666 \\ h/s &= 29/20 = 1{\cdot}45 \\ h/s &= 169/119 = 1{\cdot}4202 \\ h/s &= 985/696 = 1{\cdot}4152 \end{aligned}\right\} \text{ where } s = \min(a,b)$$

$$h/s = 5741/4059 = 1{\cdot}4144$$

which is approaching $\sqrt{1^2 + 1}$.

n	Sequence	Triangles	Lt h/s
1	1, 2, 5, 12, 29	3, 4, 5; 20, 21, 29	$\sqrt{2}$
$1\frac{1}{2}$	1, 3, 10, 33, 109	6, 8, 10; 60, 91, 109	$\sqrt{3}$
2	1, 4, 17, 72, 305	8, 15, 17; 136, 273, 305	$\sqrt{5}$
$2\frac{1}{2}$	1, 5, 26, 135, 701	10, 24, 26; 260, 651, 701	$\sqrt{8}$
3	1, 6, 37, 228, 1405	12, 35, 37; 444, 1333, 1405	$\sqrt{10}$
$3\frac{1}{2}$	1, 7, 50, 357, 2549	14, 48, 50; 700, 2549, 2451	$\sqrt{15}$
4	1, 8, 65, 528, 4289	16, 63, 65; 1040, 4161, 4289	$\sqrt{17}$
$4\frac{1}{2}$	1, 9, 82, 747, 6805	18, 80, 82; 1476, 6643, 6805	$\sqrt{24}$
5	1, 10, 101, 1020, 10301	20, 99, 101; 2020, 10101, 10301	$\sqrt{26}$
.			
.			
.			

It can easily be seen that $b = n.a \pm 1$ in each case.

Garnet J. Greenbury,
123 Waverley Road,
Taringa, Queensland, 4068.

[As with Cohen's letter, Mr Greenbury's has been rather heavily cut in the editing process. The omitted sections were concerned particularly with more on continued fractions and the various available methods of triad generation. Eds]

BLYTH'S PARADOX

A statistician rates three types of pie at his restaurant as follows: (ratings of 1 to 6; 1 denoting minimal, 6 denoting maximal satisfaction):

Apple pie - always a 3 score

Blueberry - 56% of the time scores a 2
22% of the time scores a 4
22% of the time scores a 6

Cherry pie - 51% of the time scores a 1
49% of the time scores a 5.

Every day the restaurant offers both apple and cherry, the statistician chooses apple because 51% of the time he will gain more satisfaction.

[Were the restaurant to offer both blueberry and cherry, the statistician would choose blueberry, because 61·78% of the time he will gain more satisfaction.

Were the restaurant to offer both apple and blueberry, the statistician would choose apple, because 56% of the time he will gain more satisfaction.]

Occasionally the restaurant offers blueberry in addition to apple and cherry. Yesterday was such a day. When the statistician entered the waitress said: "Shall I bring your apple pie?"

Statistician: "No. Seeing that today you also have blueberry, I'll take the cherry pie."

The statistician's order is based on the reasoning that when he has a choice of all three, the cherry will provide more satisfaction 38·22% of the time, the blueberry 33·22% of the time, and the apple only 28·56% of the time.

per Francine McNiff,
Monash University.

[We note that the statistician does not maximize the expected value of his satisfaction, but rather his chances of eating the best available type of pie. The paradox owes its origin to this. See, for a related case, Michael Morley's letter on Tattslotto Systems in Function, *Vol.5, Part 2. Ms McNiff, a sub-dean of the Monash Law School, tells us that this paradox was once a part of their Problems of Proof course. Neither she nor we know who Blyth was. Eds]*

GUIDANCE?

A certain careers office advises students on the basis of their answers to a single question: "A normal elephant has four legs. If the trunk is also counted as a leg, how many legs does the elephant have?"

If the student answers "Under the stated hypothesis, it has five legs" he is counselled into mathematics.

If he says "That is an exceptional situation: on the average, an elephant has four legs" he is headed towards statistics.

If he says "A trunk is not a leg, and merely saying it is does not reclassify it: the elephant has four legs" he is advised to become a zoologist.

If he says "If the trunk counts, so should the tail, and it has six legs" then he is good at juggling facts and is aimed towards the social sciences.

If he answers "You said a normal elephant has four legs but you did not state that this one was normal: I would prefer to withhold my opinion for lack of evidence" he is advised to consider law.

If he says "That's a good question!" he is counselled into teaching.

T. Gilham,
R.A.A.F. Academy

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PROBLEM SECTION

SOLUTION TO PROBLEM 5.1.1

This peculiarly capitalist problem came, we are told, from a Russian problem book.

Three poor woodcutters, stranded in the bitter winter, seek shelter in an abandoned cottage. "I", said the first, "have 5 logs of wood to help keep us warm". "And I", said the second, "have 3". "Alas", said the third, "I have no wood, but I have 8 kopeks to repay you for allowing me to share your fire".

How should the 8 kopeks be distributed between the first two woodcutters?

We received two incorrect solutions but nothing else. Here is how the Russians solve the problem, according to our informant.

Call the woodcutters A, B and C. C pays 8 kopeks for 1/3 of the heat. The total value of the heat is thus assessed at 24 kopeks. Thus each log is worth 3 kopeks. Thus

> A donates 15 kopeks' worth of timber and receives 8 kopeks' worth of heat
>
> B donates 9 kopeks' worth of timber and receives 8 kopeks' worth of heat.

A is thus 7 kopeks out of pocket and B 1 kopek, and that is how C's 8 kopeks should be divided up. (Note that A and B are using C's kopeks to even out their own contributions.)

Karl Marx, we presume, is turning in his grave.

SOLUTION TO PROBLEM 5.1.2

This problem, originally proposed by A.K. Austin of the University of Sheffield, first appeared in the U.S. journal *Mathematics Magazine* in 1971, as "Quickie Problem" 503. It read (in our slightly altered version):

A boy, a girl and a dog go for a walk down the road, setting out together. The boy walks at a brisk 8 km/h, while the girl strolls at a leisurely 5 km/h. The dog frisks backwards and forwards between them at 16 km/h. After one hour, where is the dog, and in what direction is it facing?

Austin's solution is that the dog may be anywhere between the two and facing in either direction. He justifies this answer by letting "all three reverse their motion until they come together at the starting point at the starting time". This solution was accepted by Martin Gardner, the *Scientific American* columnist, when he ran the problem in July and August of 1971. This, however, was not the end of the matter.

Four objections were published in December 1971 by *Mathematics Magazine*. The tenor of these may be gauged from this extract from one of them, by M.S. Klamkin of the Ford Motor Company. "The dog would have a nervous breakdown attempting to carry out his program ... let the initial starting distance between the boy and the girl be [not zero, but arbitrarily small] ... the number of times the dog reverses becomes arbitrarily large in a finite time [as this distance approaches zero]."

Gardner, relying on a letter from Professor Wesley Salmon, attempted to answer the objections, the other three of which remark that whereas the telescoping motion envisaged in the solution is well-defined, the expanding motion envisaged in the problem is not. Gardner and Salmon embark on a lengthy analysis to meet this point. It may be boiled down to this: that there exists an expanding motion corresponding precisely to every telescoping one and the dog need only follow any one of these.

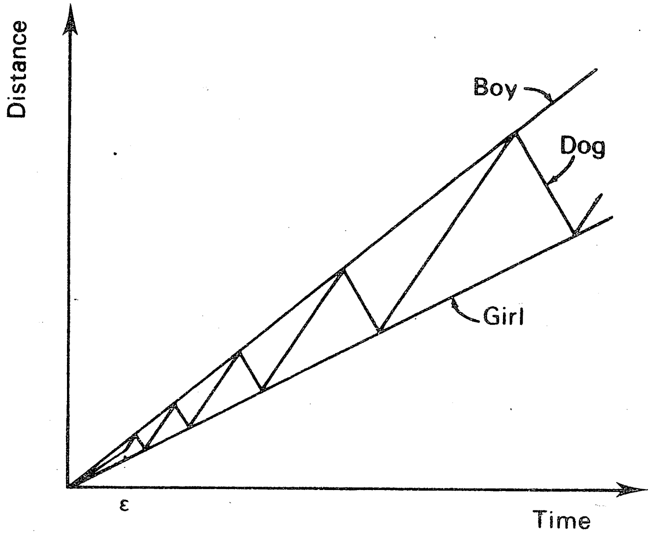
The following analysis seems simpler and more to the point. Suppose the dog to dawdle at a speed v km/h (where $5 < v < 8$) for time ε, and, having done so, then frisks at 16 km/h according to specifications thereafter. This the dog may do for any positive ε, no matter how small. Moreover, v is arbitrary, within the limits specified, and the dog may "break" toward either the boy or the girl.

Although the dog is cheating, in that it does not obey its instructions to the letter, we will never catch him at it for ε may always be made small enough to evade the limit of precision of any detecting device.

In this analysis, it is the arbitrariness of ε, v and the direction of the "break" that accounts for the arbitrariness in the solution.

The velocity-time graph opposite shows what we have in mind.

Note that this analysis meets Klamkin's objection and also the other objection, since, for given v, ε, the expanding motion is defined, and this corresponds exactly to the well-defined nature of the telescoping motion, once the position and direction of travel are specified.



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From p.30:

First estimate is $\frac{1000}{25}$ = 40 months.

Interest on \$1000 for 40 months = $\$333\frac{1}{3}$.

New principal = $\$1333\frac{1}{3}$.

Second estimate is $\frac{1333\frac{1}{3}}{25}$, which rounds up to 54 months.

Interest on \$1000 for 54 months = \$450.

New principal = \$1450.

Third estimate is $\frac{1450}{25}$ = 58 months.

Interest on \$1000 for 58 months = $\$483\frac{1}{3}$.

New principal = $\$1483\frac{1}{3}$.

Fourth estimate is $\frac{1483\frac{1}{3}}{25}$, which rounds up to 60 months.

Interest on \$1000 for 60 months = \$500.

New principal = \$1500.

Fifth estimate is $\frac{1500}{25}$ = 60 months.

Thus the answer to the problem is 60 months.

SOLUTION TO PROBLEM 5.1.3

This problem, which arose in practice, read as follow$.

Suppose a debt of $1000 incurs simple interest of 10% p.a. The borrower can repay a maximum of $25 per month. How long will it take to pay the debt off? This problem is easily reduced to a simple equation, but here is how some accountants do it. (See calculation on p.29.)

Does this method always work, and if so, why?

The answer is "yes". To justify this, however, we need to consider the problem in its full generality. Suppose $\$P$ are borrowed and repaid in monthly instalments. The simple interest rate is $R\%$ p.a. The borrower can repay a maximum of $\$M$ per month. Now let us attack the problem by the standard method.

Suppose the borrower repays the loan over n months. His total payment is $\$nM$ and this must equal the total debt incurred which is $\$(P + PRn/1200)$. Hence

$$nM = P + PRn/1200,$$

which, after manipulation, becomes

$$n = \frac{1200P}{1200M - PR} . \tag{1}$$

There is, however, a complication. Although, in the example given, $n = 60$, an integer, is readily found by use of Equation (1), this case is not typical. More usually Equation (1) will give a fractional answer, in which case, we round up to get the true value of n. For simplicity, write $a = P/M$ and $b = R/1200$. Also write $]x[$ for "the smallest integer greater than or equal to x". Then the formula for n becomes

$$n =]a/(1 - ab)[, \tag{2}$$

and we note that $ab < 1$ is the condition the debt be paid off at all.

We now put the accountants' method into this notation. This yields: $n_0 =]a[$, $n_{i+1} =]a(1 + bn_i)[$. We wish to show that $n_i \to n$.

To do this, note first that $n_0 \leqslant n$. We shall now prove three properties of the sequence $\{n_i\}$:

(a) if $n_i < n$, then $n_{i+1} > n_i$

(b) if $n_i < n$, then $n_{i+1} < n + 1$

(c) if $n_i = n$, then $n_{i+1} = n$.

Proof of (a): If $n_i < n$, then $n_i < a/(1 - ab)$. (This follows at once from Equation (2) for, if $a/(1 - ab)$ is integral, then it is n; if it is not integral, the next integer up is n, and n_i (which is integral) is at most $n - 1$). But now, by rearrangement, $n_i < a + abn_i \leqslant n_{i+1}$.

Proof of (b):

$$\begin{aligned} n_{i+1} &< a(1 + bn_i) + 1 \\ &\leqslant a(1 + b(n - 1)) + 1 \\ &= a(1 + bn) - ab + 1 \\ &< a + abn + 1 \\ &\leqslant n + 1, \text{ since } n \geqslant a/(1 - ab). \end{aligned}$$

Proof of (c): First $n_{i+1} = a(1 + bn) \geqslant a(1 + ab/(1 - ab)) = a/(1 - ab) > n - 1$. But also we have $a < n(1 - ab) + 1$, which yields, by rearrangement, $n_{i+1} = a(1 + bn) < n + 1$. Thus $n_{i+1} = n$.

Thus the sequence begins with $n_0 \leqslant n$, and, by (a), increases (unless $n_0 = n$, in which case, by (c), we are done), until it reaches n (since, by (b), it cannot "overshoot"). Then, by (c), once it reaches n, it stays at that value. This completes the proof.

SOLUTION TO PROBLEM 5.1.4

The problem was to show that the sum of the squares of any five consecutive integers is always divisible by five. Ms Wen Ai Soong of Casuarina, N.T., solved this by noting that the integers could always be written as $a - 2$, $a - 1$, a, $a + 1$, $a + 2$ and then the sum of their squares reduces to $5a^2 + 10$, which is indeed divisible by five.

SOLUTION TO PROBLEM 5.2.1

Ms Wen also solved this problem: to find the digits x, y for which $x028y$ is divisible by 23.

Writing $a|b$ for "a divides exactly into b", Ms Wen first established the result:

$$a|(b+c),\ a|b \Rightarrow a|c\ .$$

She then noted that the condition stated became

$$23|(10^4x + 280 + y),$$

which, by her earlier result, reduced to

$$23|(4 + y - 5x).$$

Examining each of the digits 0 to 9 in turn as values for x then gave her this set of solutions

$$(x,y) \in \{(1,1),(2,6),(6,3),(7,8)\}.$$

[*The dinner was held on the 10/2/81 and this problem gave the date, as the other three solutions were all unfeasible.* Ed.]

As usual, we conclude with some more problems.

PROBLEM 5.4.1

Let p be a prime greater than 3 and consider p consecutive integers. Square them and add them up. Prove that the result is divisible by p. (Compare this with Problem 5.1.4; why is the theorem not true for $p = 3$?)

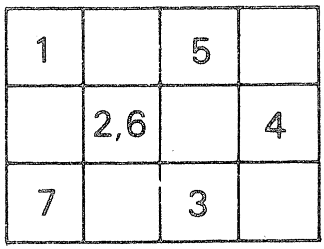
PROBLEM 5.4.2

We have had a number of problems from Russian problem-books. This one, we are told, is from a recent Chinese examination.

Ten women in a village go to collect water at the village pump. Some take only a short time, others rather more and yet others even more. How should one schedule them in order to minimize the total woman-hours spent at the pump?

Now do the same problem for the case in which two pumps are available.

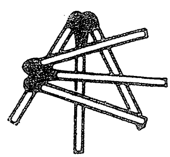
PROBLEM 5.4.3



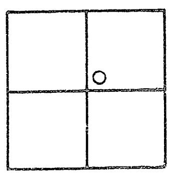
The diagram shows a billiard table with a chess-board diagram on it. A ball is cued out at 45° from the top left square and is pocketed when it reaches a corner. How many squares will it visit if the table has $m \times n$ squares in all? Notice that if a square is visited twice, it is counted twice, and also that the bounce at the sides is perfect. In the example diagrammed, $m = 4$, $n = 3$, and s, the total number of squares visited, is 7.

MATCH TRICK NO. 15 (CONT.)

SOLUTION



At right is the solution supplied by the Wilkinson Match Company. A number of variants on this basic configuration are possible, but note that all involve two "planes" of matches each containing three matches forming a triangle with extended sides. (In the diagram at right, the triangle is formed by the heads of the three matches in each plane.)



Note that if matches were perfect rectangular cuboids (which they are not) we could arrange 8 matches so that each touched all the others. Form two bundles of four (as seen, end on, at right) and bring them into juxtaposition, so that O and O' (the corresponding point on the second bundle) are in contact. All 8 cuboids then touch at this common point.

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