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# FUNCTION

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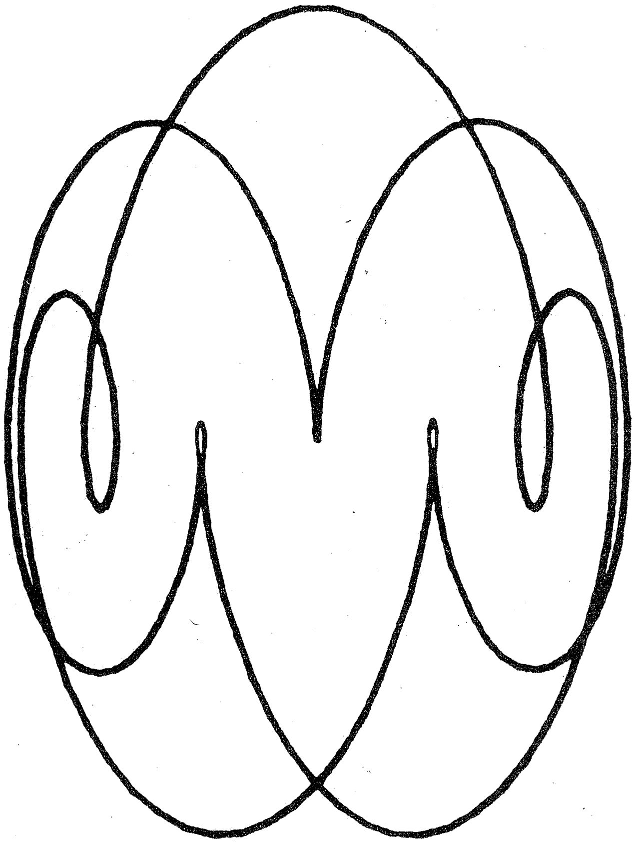

A SCHOOL MATHEMATICS MAGAZINE

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*Function* is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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This issue of *Function* contains much that we hope will interest you, but it disappoints the editors in one important respect - we have written or directly commissioned almost all of it ourselves. It would please us more if our readers were also our authors. One reader, A.D. Mattingly, who has just commenced university studies, contributes an interesting article on a topic of practical as well as theoretical importance. Another article, John Stillwell's, is included in direct response to a reader's request. We would, however, like much more 'feedback' as the year progresses.

Our leading article is another biography. It tells the story of one of mathematics' most celebrated (and notorious) prodigies, Evariste Galois. Had Galois been born in more recent times, he might have been classed as a Weatherman, a Yippie or a Red Guard. His outspoken anti-clericalism, his radical politics and his impatience with authority led to his work being almost completely ignored. It survives today largely because his teacher, L.P.E. Richard, and another great mathematician, Joseph Liouville, saw to it that his major discoveries were published (long after his death).

What would have happened in the history of mathematics if these two had not intervened? You may care to meditate that it is entirely possible that an equally great genius has died quite unrecognised.

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# THE FRONT COVER

M.A.B. Deakin, Monash University

The simple pendulum subject to an external simply periodic force obeys the equation (in suitably chosen units)

$$\frac{d^2y}{dt^2} + \sin y = K \sin \Omega t, \tag{1}$$

where $y$ is the deflection from the vertical, $t$ is the time in suitably chosen units, and $\Omega, K$ are positive constants.

Where $y$ is small, we may set $\sin y \simeq y$, to reach the (comparatively simple) equation of forced harmonic motion:

$$\frac{d^2y}{dt^2} + y = K \sin \Omega t. \tag{2}$$

A better approximation to $\sin y$ is $y - \frac{1}{6}y^3$, which leads us to study the equation

$$\frac{d^2y}{dt^2} + y - \frac{1}{6}y^3 = K \sin \Omega t, \tag{3}$$

known as *Duffing's Equation*, after the first mathematician to make a systematic study of it (in 1918).

Since that time, much more work has been done to explore the rich diversity of behaviour exhibited by equation (3). A commonly used technique is to plot $dy/dt$ against $y$. (In the case of simple harmonic motion, this produces the well-known *reference circle*.) The front cover picture is produced by the Duffing equation (3) with $K = 3$ and $\Omega = 2{\cdot}785\ 35$. (This last figure is three times the angular frequency that would be observed for $K = 0$, $y(0) = \pi/3$ and $y'(0) = 0$.) The initial conditions used in our calculation were $y(0) = 0$, $y'(0) = 0$.

The curve is known as *Murphy's Eyeballs*, and it was first found by Professor H.T. Davis of Northwestern University (Illinois). Our version was calculated by Sean O'Connor, a Research Assistant at Monash.

Murphy, after whom the curve is named, is not the editor who often prepares this page, but (to quote Davis) "the titular deity who presides over error in computing laboratories. Murphy is credited with the discovery of three propositions: (1) If anything can go wrong, it will; (2) Things when left alone can only go from bad to worse; (3) Nature sides with the hidden flaw. During this investigation Murphy's eyes were constantly upon these computations."

They certainly were, for Davis' version contains several inaccuracies. We believe our cover is a world first for *Function* - the first correct picture of Murphy's Eyeballs ever published.

# REVOLUTIONARY GENIUS, FAILED REVOLUTIONARY: EVARISTE GALOIS

Hans Lausch, Monash University

The theory of solubility of algebraic equations with all its beauty and depth bears the name of Galois. His tragic life and death have made Galois one of the most fascinating and attractive personalities in the history of mathematics.

Felix Klein[1], a famous German mathematician, alluded to the extraordinarily brief, but immensely productive period of less than three years which was left to Galois for his research: "About 1830 a new star flared up in the heavens of pure mathematics, with undreamed of brilliance, only to fade away too soon like a meteor." It was Galois' entirely new approach to algebra which, in the end, led up to the strong emphasis on structure in present-day mathematics.

Evariste Galois was born on 25th October 1811 in the small French town of Bourg-la-Reine near Paris. His father was the headmaster of a local boarding school and later on became mayor of his town. His anti-clerical attitudes exposed him to attacks and intrigues which ultimately led him to commit suicide when Evariste was eighteen.

During the twenties of the last century, France was in political turmoil. The end of the Napoleonic era and the restoration of the Bourbon dynasty had antagonized the Bourgeoisie who resented the recovery of the landowning classes, the royalists, and the Church after the long hiatus of the Revolution and the Napoleonic regime. Successor to Louis XVIII, who had died in 1824, was his brother Charles X. On 27th July 1830, however, the Paris uprising put an end to Bourbon rule and Louis-Philippe of Orléans was made "citizen king" as a result of a compromise between the barons and the bourgeoisie. Not unlike other politicians and rulers of the era - the Austrian chancellor Metternich was a typical representative of this style of politics and lent his name to the period between 1815 and 1848 - the new French regime relied on a huge army of police and secret agents to support and enact its policies.

Galois' childhood and youth has to be seen against this background as well as his father's republican sentiments. At the age of twelve, Evariste entered the Collège Louis-le-Grand, an establishment school aimed at educating future public servants for advanced positions. Galois' republican convictions led him to participate in a students' revolt. During the 1830 revolution when he was a student at the Ecole Normale Supérieure he and his fellow students were forcibly prevented from leaving their students' residences and thus could not join the uprising. Subsequently in a republican newspaper, Galois attacked the director of the Ecole normale for his opportunist behaviour of having turned republican only after the victory of the revolution. Thereupon Galois was expelled from the Ecole.

He then became a member of the National Guard's artillery section. This was a key section of the army, and probably there was a plan to infiltrate this unit by republicans. In 1831, at a republican gathering, Galois proposed a toast that was interpreted as a threat to the monarch's life, and he was arrested. Although released, he was again arrested some months later and sentenced to six months' jail. Shortly afterwards he became involved with a coquette and, under a code of honour, was unable to avoid a duel. On 29th May, 1832, the day before his duel, he wrote a "Letter to all Republicans", apologizing for dying not for the country but for "such a miserable cause".

Galois' other passion was mathematics. Frustrations seemed to be his destiny also in this field of activity. While imprisoned at Saint-Pélagie he received a letter from the Academy telling him that a manuscript he had submitted concerning the solubility of algebraic equations, was "not sufficiently clear" and "not sufficiently detailed" and would therefore be returned to him with the request to clarify and elaborate his exposition. The Academy's answer is quite understandable in the light of the extremely terse and almost aphoristic style of the paper and the then completely unexplored and extraordinarily difficult subject matter. However, two other manuscripts submitted by Galois at an earlier date had been simply lost by the Academy and Galois quite justifiedly regarded this as a manifestation of arrogance and mental inertia if not as intentional "forgetfulness". Embittered also by the atmosphere of prison confinement, he burst out "I swear that I do not owe any gratitude to the leaders in politics and in the sciences. I am indebted to the political leaders for having had to write all this in prison, a place which is hardly conducive to meditation. But I have to report on how frequently manuscripts get lost in the Academicians' drawers, even though I can hardly comprehend the absentmindedness of those who already have Abel's[2] death on their consciences."

Galois' results were revolutionary, but still based on tradition. While attending the Collège, Galois had familiarized himself with the then popular text "Eléments de géometrie" by A.M. Legendre[3]. He quickly displayed his mathematical genius. Within a few weeks he had fully absorbed the contents and methods of this rather difficult book and, advised by his appreciative teacher, he began to read original research papers. Amongst other authors he studied J.L. Lagrange[4] and C.F. Gauss[5]. At an early age, he also became aware of N.H. Abel's results on algebraic equations.

After completing his studies at the Collège, Galois intended to take up his studies at the famous Ecole polytechnique which was then the undisputed European mathematical centre and, at the same time, a fortress of republicanism. His plans were frustrated: twice Galois failed the entrance examination; on the first occasion he found out that the Collège had not sufficiently prepared him, on the second occasion he thought the questions were silly, refused to answer them, and as legend has it, threw a wet sponge at the examiner's face.

So Galois entered the so-called preparatory school which was an offshoot of the Ecole normale that had been established during the Revolution in 1795. Its function was that of a secondary teachers training college. It was here and after his expulsion that his ideas about the solubility of algebraic equations began to take shape.

In order to give the reader some idea of the problems Galois tackled in such an ingenious way, let us start with a linear equation $ax + b = 0$, $a,b$ being numbers (rational, or real, or complex), $a \neq 0$. Then $-\frac{b}{a}$ is a solution. We observe that this expression involves only the coefficients $a$ and $b$, and the operations of subtraction and division. Proceeding to quadratic equations $ax^2 + bx + c = 0$, $a \neq 0$, the solutions are of the form $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This expression involves only the coefficients $a,b,c$, the four fundamental operations (addition, subtraction, multiplication, and division) and taking (square) roots. For equations of degree 3 and 4, solution formulae of a similar structure though a little more involved, consisting of expressions involving the coefficients, the fundamental operations, and taking (square, cubic, fourth) roots, had been obtained a long time before Galois. P. Ruffini[6] and N.H. Abel, - the latter was first to give a complete and flawless proof in 1826 - showed that the algebraic equation $a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$, $a_n \neq 0$, had a solution formula for general $a_0, \ldots, a_n$, in terms of these coefficients, the fundamental operations, and taking roots of any order, if and only if $n \leqslant 4$. This came somewhat as a surprise, and is a famous example of the fact that a mathematical problem need not always have a solution, or at least not the expected one. As usual, the answering of one question immediately raised other problems. The obvious question was now: What if we specify the coefficients? That $a_5 x^5 + a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0$ has no "nice" solution, has to be accepted, but we also know that $x^5 + a_0 = 0$ does have "nice" solutions, namely $\sqrt[5]{-a_0}$; note that $\sqrt[5]{\phantom{x}}$ has, in general, five different meanings. So what can be said about the solutions of, say,
$x^5 + 3x^4 - 6x + 15 = 0$?

In 1830, soon after Galois had entered the preparatory school, three of his papers appeared in print which when compared with his later revolutionary work, were comparatively insignificant and elicited no response. One further paper submitted to the Academy was lost there, probably through A.L. Cauchy's[7] negligence, to share the fate of an important paper by N.H. Abel.

It appears that early in 1831 Galois had made the decisive breakthrough. He wrote another paper "On the conditions of solubility of equations by roots" and submitted it to the Academy. On 31st March, 1831, Galois turned on the President of the Academy after having been left in the dark about the fate of his manuscript. Referring to the problem he had solved in his paper he wrote: "... Since this problem has so far appeared to be extremely difficult if not impossible to solve, the [Academy] Committee must have assumed *a priori* that I could not have solved it; mainly because my name is Galois, and also because I was a student....". The Academy's reply reached Galois as late as October, 1831, when he was imprisoned. His manuscript was returned with the remark that his paper was "neither sufficiently clear nor sufficiently elaborate to ... enable judgement on its correctness .... . One has therefore to wait until the author produces a complete version of his paper before one can come to a final opinion."

Neither was Galois willing to rewrite his paper nor did he have sufficient time and opportunity to do so. At the end of April, 1832, he was released from prison, and on 30th May the fatal duel took place. On the eve of the duel, he wrote a letter to his friend Auguste Chevalier which can be regarded as his mathematical will. It contains Galois' best results written in a terse style and rather sketchily at times. Nothing less than a complete theory of solubility of algebraic equations was the outcome of his investigations. On the morning of 30th May Galois was shot through the intestines and lay where he fell until a passing peasant took him to a hospital where he died of peritonitis the following morning. He was only twenty years old at the time, the youngest mathematician ever to make such significant discoveries.

It would lead far beyond the scope of this article to describe Galois' findings. But it may be worthwhile to sketch some of his fundamental ideas which were completely new at the time. Let us consider the quadratic equation $x^2 + 1 = 0$. We know that there is no real number $x$ which satisfies this equation. Hence $x^2 + 1$ has no factorization into polynomials of smaller degree if we insist that their coefficients be reals. We say $x^2 + 1$ is *irreducible* over the reals. If, however, we allow ourselves to use the complex numbers, then $x^2 + 1 = (x + i)(x - i)$. This is actually the way one constructs the complex numbers: we postulate $i$ to be a solution of $x^2 + 1 = 0$, adjoin $i$ to the reals, i.e. create a set which consists of the reals, $i$, and the fundamental operations, observing all the time that $i^2 = -1$, and show that what we have obtained is consistent with the laws we expect to be obeyed by the fundamental operations [e.g. $(ab)c = a(bc)$, the associative law]. In particular, we find that each complex number can be written in one and only one way as $a + bi$, $a,b$ being reals. Next we observe that trading $i$ for the other solution of $x^2 + 1 = 0$, namely $-i$, has virtually no effect on the arithmetic of complex numbers: if $z = a + bi$, $\bar{z} = a - bi$ (the conjugate of $z$), then $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$. The reals are not affected at all by this "trade-in". The function which assigns $\bar{z}$ to each $z$ is called an automorphism of the complex numbers over the reals which leaves each real number unchanged. Obviously, the function which assigns to each $z$ the number $z$ itself, is another such automorphism. One can show that these two functions are the only automorphisms of the complex numbers over the reals. This idea now generalizes to any irreducible algebraic equation. Take e.g. $x^7 - 1 = 0$. This is not irreducible over the rationals, as

$$x^7 - 1 = (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1).$$

But the second factor is irreducible, i.e. it is not the product of two polynomials with rational coefficients of degree less than 6. Its roots are $w_n = \cos \frac{2\pi n}{7} + i \sin \frac{2\pi n}{7}$, $n = 1,2,3,4,5,6$ (check this!). Moreover, $w_n = w_1^n$. Again we adjoin $w_1$ to the rationals, i.e. form all rational expressions (expressions obtained by performing the fundamental operations) involving rational numbers and $w_1$. Calculation with these numbers obeys the various laws of arithmetic using the fact that $w_1^6 + w_1^5 + w_1^4 + w_1^3 + w_1^2 + w_1 + 1 = 0$. Again, if we trade $w_1$ for $w_2$ or $w_3$, etc., it turns out that the arithmetic of this domain is not affected. e.g. $(5 + w_1)(6 - w_1) = 30 + w_1 - w_1^2$ and $(5 + w_2)(6 - w_2) = 30 + w_2 - w_2^2$. Generally whenever a rational expression $R(w_1) = 0$, then also $R(w_2) = 0$ and vice versa. Hence there are six automorphisms, and one can show that these are all.

Galois assigned to each irreducible algebraic equation a set of automorphisms in this manner. Automorphisms can be "composed", i.e. if $\sigma$ and $\tau$ are automorphisms then the map which first applies $\sigma$ and then $\tau$ turns out to be again an automorphism, e.g. if $\sigma: w_1 \mapsto w_2$, $\tau: w_1 \mapsto w_3$, then $\sigma\tau: w_1 \overset{\sigma}{\mapsto} w_2 = w_1^2 \overset{\tau}{\mapsto} w_3^2 = w_1^6 = w_6$. Furthermore every "trade-in" can be reversed, e.g. if $\sigma: w_1 \mapsto w_2$, then $\mu: w_1 \mapsto w_4$ reverses the effect of $\sigma$ since $w_2 = w_1^2 \overset{\mu}{\mapsto} w_4^2 = \left(w_1^4\right)^2 = w_1^8 = w_1\, w_1^7 = w_1 . 1 = w_1$, as $w_1^7 - 1 = 0$. "Refusing a trade-in" also amounts trivially to an automorphism, *viz.* $\varepsilon: w_1 \mapsto w_1$. And finally "trading-in" is associative, i.e. if $\sigma, \tau, \mu$ denote automorphisms, then $(\sigma\tau)\mu = \sigma(\tau\mu)$. Galois called this set of automorphisms together with these laws of operation a *group*, and this is what it is still called today. He had thus invented a translation process which assigned to each irreducible algebraic equation a group (of automorphisms). He now began investigating these groups instead, and like a mirror, the information obtained from these groups reflected precisely the solution structure of the given algebraic equation, in particular whether or not the solutions could be expressed in terms of roots and the fundamental operations. Group theory is nowadays one of the best developed fields of algebra. It turned out that the applications of group theory go far beyond its original objective: quantum mechanics, crystallography, combinatorics, and indeed virtually all other fields of mathematics have employed group theory in extremely useful ways. Galois was so close to modern attitudes in algebra and yet so inarticulate that he was unable to make himself understood by the teachers of his day. His ideas were without influence until they were published in 1846, fourteen years after his death! Today we hear much about "New Maths" in schools, but it is new only in the sense that the views of Galois are finally coming into their own, more than a century after fate had dealt so cruelly with him.

## BIOGRAPHICAL SUMMARY

| | |
|---|---|
| 1811 | October 18, Evariste Galois born in Bourg-la-Reine. |
| 1823 | Galois passes his entrance examination for the Collège Louis-le-Grand and enters fourth form. |
| 1827 | Galois comes by a text by A.M. Legendre and discovers his mathematical talents. |
| 1828 | His first attempt to enter the polytechnical school fails due to insufficient training at the college. |
| 1829 | Galois publishes his first mathematical paper.<br>His father commits suicide.<br>Galois completes his college training and fails on his second attempt to enter the polytechnical school, this time because of a clash with the examiner. |
| 1830 | He enters the preparatory school of the Ecole Normale Supérieure. He publishes three papers.<br>July, revolution in Paris. Galois participates as a republican which leads to his expulsion from school in December. He joins the National Guards (artillery). |
| 1831 | Galois gives algebra courses and submits an important manuscript to the Academy.<br>May 9, Revolutionary toast to King Louis-Philippe.<br>May 10, Galois' arrest.<br>June 15, Galois' release due to skilful defence.<br>July 14, Galois arrested again by the secret police.<br>October, two mathematical papers written at the prison of Sainte-Pélagie.<br>End of October, Galois sentenced to six months in prison because of unlawful possession of arms and wearing the uniform of the Republican Guards'Artillery while prohibited. |

1832 April, Galois released from prison.
May 29, duel challenge. In the evening Galois writes his mathematical will.
May 30, the duel takes place, Galois badly injured.
May 31, Galois dies.

## FURTHER READING

Galois' writings are available at Monash University's Hargrave Library: *Ecrits et mémoires mathématiques d'Evariste Galois*. A classical text on Galois theory is: Emil Artin, *Galois Theory*.

Those who are interested in the lives of famous mathematicians will also find a chapter on Galois in E.T. Bell, *Men of Mathematics*.

## NOTES

1. Felix Klein (1849 - 1925), famous German mathematician who saw the importance of group theory for the whole of mathematics.

2. Niels Henrik Abel (1802 - 1829), Norwegian mathematician, often quoted together with Galois. Abel and Galois were the two young geniuses of the 19th century. Both made revolutionary discoveries and both died young. Galois refers to Abel's death which was mainly due to a life in poverty. Support from other mathematicians came too late.

3. Adrien Marie Legendre (1752 - 1833), French mathematician. One of the three great L's (Lagrange, Laplace, Legendre) in French mathematics during the age of the Revolution.

4. Joseph Louis Lagrange (1736 - 1813), most famous through his "Mécanique analytique".

5. Carl Friedrich Gauss (1777 - 1855), one of the greatest geniuses in mathematical history. His talents were also discovered when he was very young, but unlike Galois he came to reach old age and led a comparatively wealthy and undisturbed life. His achievements were too numerous for even the most outstanding ones to be listed here. You are invited to look at the collection of his works at Monash University. With Gauss, Göttingen became one of the world's greatest mathematical centres. (For a first-rate article on Gauss, see *Scientific American*, July 1977, p.122.)

6. Paolo Ruffini (1765 - 1822), Italian mathematician.

7. Augustin Louis Cauchy (1789 - 1857), French mathematician, one of the "fathers" of classical analysis. His works fill 27 volumes. He was the establishment figure amongst the French mathematicians. Abel, in 1826, wrote about him: "Cauchy is extremely catholic and bigoted. This is a very peculiar thing for a mathematician." Still, he was consistent: between 1830 and 1838 he lived in exile after the July Revolution, showing himself a staunch royalist.

# TOPICS IN THE HISTORY OF STATISTICAL THOUGHT AND PRACTICE

## IV. THE LANARKSHIRE MILK EXPERIMENT[†]

Peter D. Finch, Monash University

In the spring of 1930, 20 000 school children from 67 schools in Lanarkshire, Scotland, took part in a large nutritional experiment. For four months 10 000 children received 3/4 pint of milk per day and the remaining 10 000 acted as controls by not receiving milk. Of those given milk, 5000 got raw milk and 5000 pasteurised milk, in both cases Grade A, Tuberculin tested. Each of the 20 000 children was weighed and his height measured, both at the beginning and at the end of the experiment. The conclusions reported were:

(1) *The influence of the addition of milk to the diet of school children is reflected in a definite increase in the rate of growth in height and weight.*

(2) *There is no obvious contrast or constant difference in this respect between boys and girls and there is little evidence of definite relation between the age of the children and the amount of improvement. The results do not support the belief that the younger derived more benefit than the older children. As manifested merely by growth in weight and height the increase found in younger children through the addition of milk to the usual diet is certainly not greater than, and is probably not even as great as, that found in older children.*

(3) *In so far as the conditions of this investigation are concerned the effects of raw and pasteurised milk on growth in weight and height are, so far as one can judge, equal.*

The third conclusion was questioned by Fisher and Bartlett (1931) who remarked:

*'... of the 14 groups (by age and sex), pasteurised milk gave a greater increase in height in only 2 groups, the increases were equal in 1 group, while in 11 groups the raw milk gave the greater increase. If we may regard these as 14 independent experiments, the difference from expectation on the hypothesis that raw and pasteurised milk have the same effects, is such as would occur only once in about ninety trials, and*

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[†] This is the text of a talk to fifth and sixth formers given at Monash University on April 6, 1979.

*it seems evident that the conclusion should have been that the growth response in height to raw milk is significantly greater than that to pasteurised milk.'*

They then examined the magnitudes of the differences in question by calculating average excesses for weight and height by sex in respect of milk feeders as against controls. (The term 'feeders' was that used by the experimenters to describe those children given milk.) Pasteurised milk gave lower returns than raw milk in all cases though the differences were more marked for boys than for girls. Table 1 gives some relevant data for weight.

Table 1. Average Increases in Weight in Ounces

| | Boys | | | Girls | | |
|---|---|---|---|---|---|---|
| | Control | Raw Milk | Pasteur-ised | Control | Raw Milk | Pasteur-ised |
| Increase | 10·041 | 13·780 | 12·507 | 9·755 | 14·315 | 13·907 |
| Excess | | 3·739 | 2·466 | | 4·560 | 4·152 |

Though girls feeders exhibit greater excesses over controls than do boy feeders the relative effect of raw milk in comparison to pasteurised milk is greater for boys than it is for girls. Similar results hold for height increases. Some relevant data are given in table 2.

Table 2. Average Increases in Height in Hundredths of an Inch

| | Boys | | | Girls | | |
|---|---|---|---|---|---|---|
| | Control | Raw Milk | Pasteur-ised | Control | Raw Milk | Pasteur-ised |
| Increase | 73 | 81 | 77 | 73 | 81 | 79 |
| Excess | | 8 | 4 | | 8 | 6 |

Difficulties in accepting the thrust of these conclusions were pointed out by Student (1931)†. To see how they arise let us return to Fisher and Bartlett's argument purporting to show that the growth response in height to raw milk is significantly greater than that to pasteurised milk. That argument was based on the responses in fourteen groups by age and sex, it being found they were the same in one group, that pasteurised milk gave the greater response in only two groups whereas raw milk gave the greater response in eleven groups.

Suppose that the two types of milk did not affect the children's heights differently. Then for the thirteen groups in which a greater response appeared to occur for one or other milk type, it would be by chance, and hence equally likely, which milk type appeared to yield the greater response. Treat each group as a "trial" and call the trial a "success" if the group appeared to respond more to new milk than to pasteurised. The success probability would be 1/2. Moreover, if these thirteen

† "Student" was the pseudonym of William Sealy Gossett, a statistician with Guinness' Brewery in Dublin.

trials are independent the total number of successes $S$ will follow the binomial distribution. The probability of obtaining 11 or more successes in 13 such trials is, therefore,

$$\left(2^{-13}\right)\left[\binom{13}{11} + \binom{13}{12} + \binom{13}{13}\right] = \frac{92}{8192} \approx \frac{1}{89}.$$

In other words if, as the original report suggested, the effects of raw and pasteurised milk are equal, then a preponderance of raw milk effects as great or greater than that observed would occur by chance alone only about once in 90 such experiments. This is the result asserted by Fisher and Bartlett and, as they suggest, it seems to cast some doubt on the hypothesis of equal effects.

The difficulty with this argument stems from the fact that though half the milk feeders consumed raw milk and half pasteurised milk the recipients in the same school were not divided in that way. All the milk supplied to any one school was of the same type, being either raw or pasteurised but not both. As Fisher and Bartlett themselves pointed out: '*In the absence of the records from separate schools, it is impossible altogether to eliminate the doubt which this choice of method introduces ....*'. The doubt in question arises in the following way.

When we claim that raw and pasteurised milk have the same effect we mean that the increment in growth of any child will be the same, whichever type he receives. The aggregate difference between two groups of children receiving different types of milk will then be due to *other* differences between them. It might be an age difference if the two groups were boys of correspondingly different ages. It could be a sex difference if boys were in one group and girls were in the other. The grouping by age and sex referred to by Fisher and Bartlett was an attempt to remove the possibility of such extraneous causes by considering, for comparative purposes, only children who were as much alike as possible so that the type of milk diet would be the only aggregate difference between them. There would, of course, be natural variability in weights and heights but experience suggests that these should cancel out in aggregate provided they were not themselves factors determining which type of milk was received. But this is precisely the point about which we cannot be sure.

Suppose, for instance, that among the schools using raw milk only those judged most in need had been allocated as milk feeders whereas in the schools with pasteurised milk the assignment to feeder and control groups had been randomised. In such a case there might well be an inherent difference between the two groups of feeders being compared, those in the raw milk group being an aggregate both smaller and lighter and, perhaps, exhibiting on the whole greater increases in weight and height. This might lead one to ascribe to raw milk the observed greater effect when, in fact, it was due to the initial disadvantage from which those feeders started. Moreover the same bias might run through several of the 14 groupings. This would cast doubt on the assumption that the individual binomial experiments were independent. These possibilities would call into question our earlier computation of the frequency with which a preponderance of raw milk effects as great or greater than that observed would occur by chance alone and thereby disqualify our previous interpretation of it.

One is faced, therefore, with the possibility that raw and pasteurised milks were tested on groups which were not strictly comparable. This was the principal point emphasised by Student. He pointed out that the selection of children was left to the head teacher of the school with the guiding principle that both controls and feeders should be representative of the average children between 5 and 12 years of age. But as noted in the original report of Leighton and McKinlay (1930): *'The teachers selected the two classes of pupils, those getting milk and those acting as controls in two different ways. In certain cases they selected by ballot and in others on an alphabetical system. In any particular school where there was any group to which these methods had given an undue proportion of well-fed or ill-nourished children, others were substituted in order to obtain a more level selection.'* It seems, however, that too large a substitution of the ill-nourished occurred among the feeders and too few among the controls for, as pointed out in the report itself, even at the beginning of the experiment, the latter were definitely superior both in weight and in height to the feeders by an amount equivalent to about 3 months' growth in weight and 4 months' growth in height. This fact led to an additional difficulty. The children were weighed in their indoor clothes but no allowance was made for the difference between their heavier winter clothing at the beginning of the experiment and their lighter summer wear at its end. This introduced the possibility of further bias.

Suppose a child had actual weight $W_1$ at the beginning of the experiment, actual weight $W_2$ at its end, let the weight of his heavy winter clothing be $H$ and that of his light summer wear be $L$. The actual increase in his weight is

$$I = W_2 - W_1$$

but the corresponding measured increase would be

$$I' = (W_2 + L) - (W_1 + H) = I - E$$

where the 'error' is the positive quantity

$$E = H - L.$$

These errors are likely to be smaller for poorer children who, in general, are not so well-clad in winter as their richer peers. Of two children with the same actual weight increase, one of whom was 'poor' and the other 'rich', the former would probably exhibit the greater *measured* weight increase by reason of his smaller 'error'. But selection into the feeder group favoured the ill-nourished and so it would, therefore, also favour greater measured weight increases for the feeders. It could be, therefore, that the observed gain in weight of feeders over control is simply a consequence of the biased selection into the two groups. In this connection it is worth noting that the gain in height was not so marked as that in weight;the former would not, of course, be affected by errors of measurement in weight due to the clothing difference.

The over-all conclusion is, therefore, that failure to control the randomised selection of controls and feeders renders the results of the experiment of doubtful value since one cannot interpret them in an unambiguous way.

Randomisation plays an important role in statistics. It is necessary to use it when we extrapolate from what we observe on a given occasion to what is typical of the general run of things. We met an instance of this in our second topic. (See Reference 1.) If the lunatics had not been allocated at random to the two groups, the possibility would exist that it was the allocation procedure rather than the diet which caused the observed difference. A similar point was implicit in our third topic. Here the two groups were supplied with water by two different companies. Snow's quoted remarks suggest that we can regard the allocation of houses to supplier as effectively random. The Lanarkshire milk experiment exemplifies lack of adequate randomisation leading to inconclusive results.

Student also pointed out that the experiment could have been conducted in a much simpler way, with fewer children and at far less cost. There is about one pair of twins in 90 births and so one might hope to get at least 160 pairs in 20 000 children. In the whole of the Lanarkshire school population there would have been some 200 to 300 pairs of twins. Of 200 pairs some 50 would be 'identicals' and, of course, of the same sex while half the remainder would be non-identical twins of the same sex. He suggested an experiment on all pairs of twins of the same sex, noting whether or not they were 'identicals'. One member of each pair was to be fed on raw milk and the other on pasteurised milk, the decision in each case to be taken at random, e.g. by tossing a coin. Weekly measurements were to be taken and weight measured without clothes. Finally Student noted that some way of distinguishing the children was necessary lest the mischievous ones play tricks. He suggested different indelible marks on a fingernail of each twin; these would grow off after the experiment was over.

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III. The communication of Cholera and an Experiment on the Grandest Scale. *Function*, Vol.3, No.1, p.22.

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3. Leighton, G. & McKinlay, P.L. Milk Consumption and the Growth of Schoolchildren. *H.M. Stationery Office*, 1930.

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# WHAT IS NON-EUCLIDEAN GEOMETRY?

## John Stillwell, Monash University

First of all, what is euclidean geometry? A quick description is as follows: let us take the *plane* to be the set of points with coordinates $(x,y)$, where $x,y$ are real numbers, and let *lines, circles, line segments, angles*, etc. be as in coordinate geometry. For example, a line is the set of points $(x,y)$ satisfying an equation

$$ax + by = c$$

for some constants $a,b,c$. A line segment is obtained by putting bounds on $x$ or $y$, and triangles and other polygons are obtained by putting line segments together. Then 2-*dimensional euclidean geometry* is simply the set of true statements about these objects. One defines 3-*dimensional* euclidean geometry similarly by starting with the space of triples $(x,y,z)$. In this case a plane is the set of points $(x,y,z)$ satisfying an equation

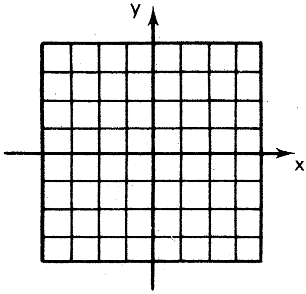
$$ax + by + cz = d$$

where $a,b,c,d$ are constants, a line is the intersection of two planes, and so on.

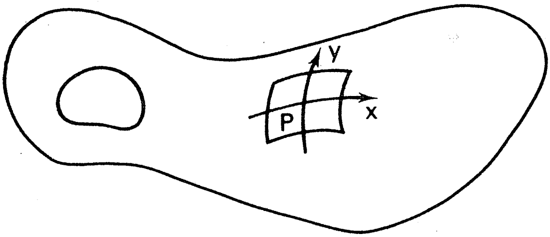
Of course, this is not the way Euclid formulated his geometry in the "Elements" of 300 B.C. The coordinate method, introduced by Descartes in the 17th century, gives a more concise description and reduces all of Euclid's geometry to algebraic calculation. (This is not to say calculation is always the easiest or best method!) Some results of this geometry that we all know are:

(1) There is exactly one line through two points;
(2) A line is infinite in length;
(3) The angle sum of a triangle is $\pi$;
(4) Similar triangles of different sizes exist;
(5) Given a line $\ell$ and a point $P$ outside it, there is exactly one line through $P$ which does not meet $\ell$ (called a *parallel* to $\ell$).

Now what is geometry if not euclidean? A very broad answer to this question, based on a general idea of coordinates, was given by Riemann in 1854. Let us begin in 2 dimensions. Any surface, not necessarily a plane, is 2-dimensional in the sense that the *neighbourhood* of any point can be given a coordinate system with 2 coordinates. One simply takes a small square patch of rubber with $x,y$ axes and a coordinate grid




and pastes it onto the surface at the given point $P$. If the patch is sufficiently small it will fit, even though the rubber may have to be stretched a bit.



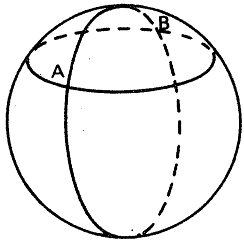

Inhabitants of a surface can map their world by applying coordinate patches at enough points to cover the whole surface and collecting small copies of the patches to form an "atlas". Like a street directory, each page of the atlas will have a note on each edge saying which page is its continuation (though in general the patches may overlap rather than fitting edge to edge). It will not necessarily be clear to the surface dwellers whether they live on a plane or not (indeed, it is still not clear to some people who live on the surface of the earth!). To find this out they have to investigate the geometry of their world and see whether it is euclidean.

At this point I must stress that the surface is assumed to be a completely self-contained world - the surface creatures and their measuring instruments are strictly confined to the surface, as are light rays and all physical events - the inhabitants study what is called the *intrinsic* geometry of the surface. Putting ourselves in their place helps us understand the limitations of our own view of 3-dimensional space - also intrinsic - and how to break away from them.

Like us, the 2-dimensional creatures define a "line segment" as the shortest route between its end points. If the two points are sufficiently close together this segment will be unique, but funny things can happen when it is prolonged. To obtain a "line" of the intrinsic geometry, prolongation must be carried out so as to obtain a curve which contains the shortest route between any two of its points which are sufficiently close together. (Light rays are assumed to travel along *geodesics* (take this as an (intuitive) definition of a geodesic), so one way to experimentally prolong a line segment from $A$ to $B$ would be to shine a beam of light from $A$ to $B$.)

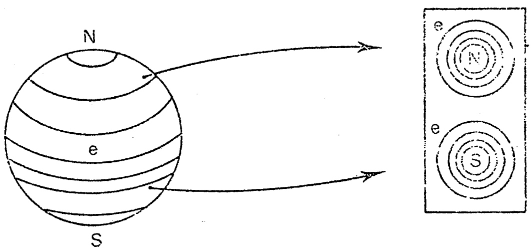
*The sphere*

After lines in the plane, the best known geodesics are great circles on the sphere. A great circle is one with the same radius as the sphere itself. Thus all meridians of longitude are great circles, but circles of latitude are not, with the exception of the equator. The fact that great circles give shortest routes is often used by airlines, who will fly over the pole from $A$ to $B$ rather than follow the latitude circle.




With the interpretation of "lines" as great circles it becomes clear that the intrinsic geometry of the sphere is highly non-euclidean: lines are finite, there are infinitely many lines through certain point pairs (which pairs?), and there are no parallels. Rather less obviously, the angle sum of a triangle exceeds $\pi$, and there are no similar triangles of different sizes.

The latter fact means that it is impossible to make maps to scale. Of course, we are familiar with this fact from maps of the earth. The larger the area to be mapped, the worse the distortion, and the sphere cannot be mapped in its entirety without certain points appearing in two different places. For example, one may map the northern and southern hemispheres separately onto discs, in which case the equator $e$ appears twice.

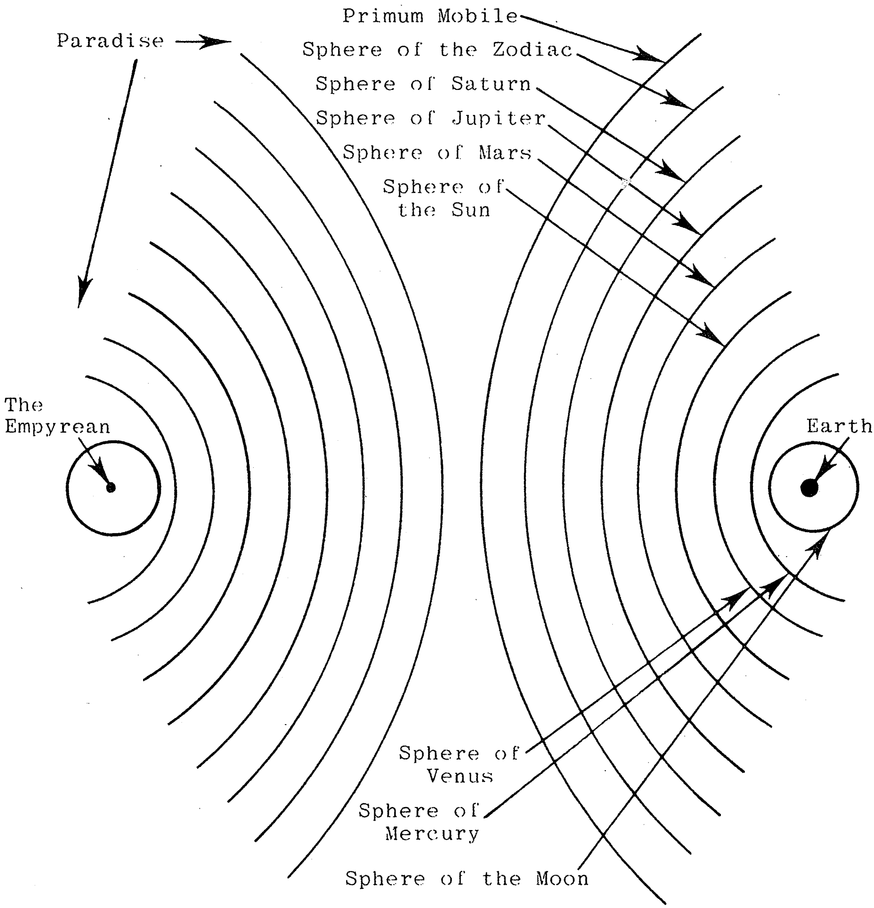



A 2-dimensional explorer carrying such a map with him would find it very confusing when he came to the boundary line, since the line is "straight" on the sphere but decidedly curved on the map (and concave or convex depending on which half of the map he looks at).

If this is what life is like in the 2-dimensional sphere, what would it be like in an analogous 3-dimensional space, a "3-sphere"? Some people believe this question was already answered by the poet Dante, when he wrote his "Paradiso" in 1321.

*Dante's vision of the 3-sphere*

Dante places the earth at the centre of a system of nine concentric spheres, carrying moon, sun, planets and stars, the outermost of which, the Primum Mobile, is the boundary of the heavens. Beyond this, Dante sees Paradise, which consists of another nine concentric spheres. The surprise is that these spheres are not concave from the viewpoint of the Primum Mobile, but convex! The Empyrean, the abode of God, in fact appears as a blinding point of light at their centre†.

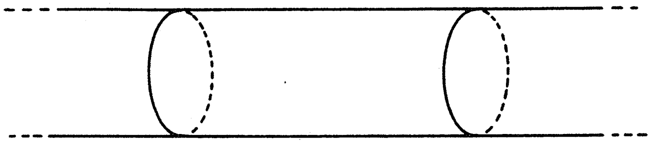



† For another version of this diagram, see *Scientific American*, August 1976, p.100.

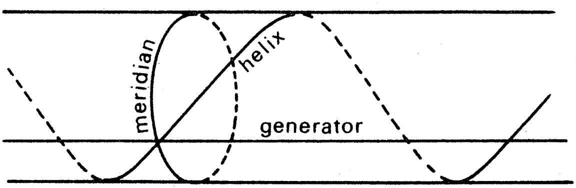
What we surely have here is a map of the 3-sphere. Earth and Empyrean are two opposite poles, and the two systems of spheres are "latitude spheres" interpolated between them, just as latitude circles are interpolated between the two poles on an ordinary sphere.

This is one way to visualize a 3-dimensional space which is finite but unbounded. The qualitative picture can easily be replaced by an exact description in terms of coordinates, but I shall not do this. If you know how to write the equation of a 2-dimensional sphere lying in $(x,y,z)$ space you might like to try writing down the equation of a 3-sphere which lies in $(x,y,z,w)$ space, and identifying its poles and latitude spheres.

*The infinite cylinder*



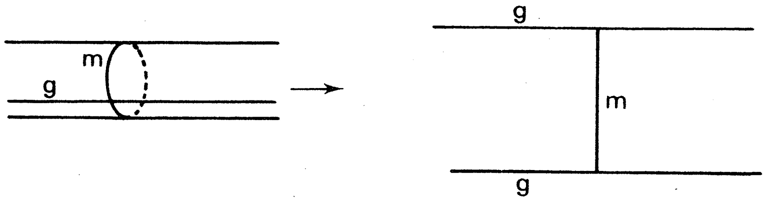
This surface is immediately seen to have non-euclidean intrinsic geometry, because it has closed geodesics and hence finite "lines". In fact there are three classes of geodesics: *generators* parallel to the axis of the cylinder, *meridian* circles perpendicular to these, and *helices*.




In this geometry it is possible for one "line" (a helix) to meet another (a generator) infinitely often. This also illustrates the fact that a geodesic need not be the shortest route between any two of its points, only those which are sufficiently close together.

Despite these bizarre "lines", the geometry is *nearly* euclidean. Pieces of the cylinder can be rolled flat on the plane without distortion - not only small pieces but also, for example, strips enclosed by two parallel helices - so such pieces could not be intrinsically distinguished from pieces of the plane. In particular, every triangle has angle sum $\pi$. Furthermore, all "lines" in the cylinder have the parallel property (5) of euclidean geometry.

Inhabitants of the cylinder could map their world onto an infinite strip, corresponding to the result of slitting the cylinder along a generator $g$ and spreading it out.

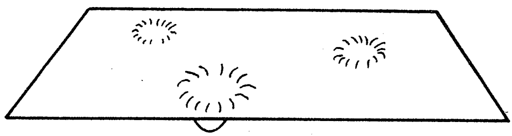



Such a map could be made true to scale, but the user would have to understand that if one passes through the top edge *g* one re-enters immediately at the corresponding point on the bottom edge.

In fact, since light rays not parallel to the axis travel round and round the cylinder indefinitely, this space would actually *look* like an infinite series of such strips placed edge to edge, to someone living inside it. This reminds us of what we see when we stand between two parallel mirrors. The only difference is that with mirrors the series of images alternates between back and front. Living in a 3-dimensional cylinder would be like looking in a mirror and seeing an endless series of images of the *back* of one's head.

*The dimpled plane*

Consider an ordinary plane with some dimples punched into it.



A geodesic which misses the dimples will be an ordinary straight line, but one which touches a dimple will be deflected like a golf ball which grazes the edge of the cup†.

This is easy to understand from our position as outside observers. To creatures within the surface, however, the dimple would not be visible, and they would be inclined to attribute the deflection to some "force", which they might call "gravity".

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† An incorrect version of this diagram appeared in *Newsweek*, 12 March 1979.

Transfer this idea to space (or more correctly to "space-time") and you have the basic idea of Einstein's theory of relativity. Einstein explains gravity by means of "space dimples" which occur in the presence of matter. As Misner, Thorne and Wheeler put it in their text book *Gravitation:* "Space tells matter how to move, matter tells space how to curve".

*Conclusion*

I hope that the above examples show that non-euclidean geometries are quite common, natural, and useful. In fact they are the rule rather than the exception. Nevertheless there are still good reasons to regard euclidean geometry as fundamental.

One reason is that euclidean spaces have the simplest possible description in terms of coordinates, so that any other space is likely to be more easily described as part of a euclidean space rather than as an independent entity. For example we can describe the 2-dimensional sphere as the set of points in $(x,y,z)$ space satisfying the equation

$$x^2 + y^2 + z^2 = 1.$$

Even when we do not write down the equation of a surface we can most easily visualize it lying in $(x,y,z)$ space.

The second reason has to do with *smoothness* and its relation to the geometry of small regions. A smooth surface is one which has a tangent plane $T_P$ at each point $P$. The smaller the neighbourhood of $P$ the more closely it approximates the plane $T_P$, and hence the more euclidean its geometry becomes. We could call this the property of being "euclidean in the limit", and it can be used as a definition of smoothness in any number of dimensions.

Smoothness is very difficult to live without, both from the standpoint of intuition and mathematical technique. For example, who can imagine a *small* triangle whose angle sum is not $\pi$. If we ever find that euclidean geometry collapses in the subatomic world then we shall need a non-euclidean geometry far more radical than those described above.

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## A PARADOX

If you check in an Encyclopaedia, you will find that two famous literary figures died on the same date. Both William Shakespeare and Miguel Cervantes died on 23 April, 1616. Yet one died on a Tuesday and the other on a Saturday! Can these two pieces of information be reconciled?

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# CALCULUS WITH A DIFFERENCE

A.D. Mattingly,

Science I, Monash University

We know that, by definition:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ where } y = f(x).$$

Define $\Delta y$ to be equal to $f(x + h) - f(x)$, where $y = f(x)$, and hence $\Delta x = x + h - x = h$, where $f(x) = x$.

Then $$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h},$$

and hence $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

In this article, I shall consider what happens if we do not take the limit, but use finite values of $\Delta x$.

The symbol $\Delta$ represents an *operator*, that is to say it performs an operation on an independent variable or a function of that variable. $\Delta f(x)$ may be expressed in words as follows: "Take the value of the function $f$ with the value of the independent variable $x$ incremented by an amount $h$ and subtract from it the original value of the function at $x$."

E.g. Suppose $f(x) = x^2$.

$$\begin{aligned} \text{Then} \quad \Delta f(2) &= f(2+h) - f(2) \\ &= (2+h)^2 - 2^2 \\ &= 4 + 2h + h^2 - 4 \\ &= h(4+h). \end{aligned}$$

(Note that $\Delta f(2)$ is a function of $h$.)

The operator $\Delta$ exhibits the following properties:

$$\Delta\{f(x) \pm g(x)\} = \Delta f(x) \pm \Delta g(x) \tag{1}$$

$$\Delta\{cf(x)\} = c\Delta f(x), \text{ where } c \text{ is a constant} \tag{2}$$

$$\Delta\{f(x)\, g(x)\} = f(x)\, \Delta g(x) + g(x+h)\, \Delta f(x) \tag{3}$$

$$= g(x)\, \Delta f(x) + f(x+h)\, \Delta g(x) \tag{4}$$

$$= g(x)\Delta f(x) + f(x)\Delta g(x) + \Delta f(x) \cdot \Delta g(x) \tag{5}$$

$$\Delta\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\, \Delta f(x) - f(x)\, \Delta g(x)}{g(x)\, g(x+h)} \tag{6}$$

The reader should verify these results.

It should be noted at this point that there is a close similarity of the rules (1) - (6) to the rules for differentiation.

Let us now consider a special function:

$$y = x^n.$$

We have the differentiation formula:

$$y' = n.x^{n-1},$$

so we would expect that

$$\frac{\Delta y}{\Delta x} = n.x^{n-1}.$$

However

$$\frac{\Delta y}{\Delta x} = \frac{(x + h)^n - x^n}{\Delta x}$$

$$= \frac{h.(n.x^{n-1} + \tfrac{1}{2}n(n - 1).x^{n-2}.h + \dots + h^{n-1})}{h}$$

$$= n.x^{n-1} + \tfrac{1}{2}n.(n - 1).x^{n-2}.h + \dots + h^{n-1}.$$

There is an inevitable discrepancy here. To get around the problem of trying to maintain the resemblance to the derivative, we can invent a new function $y = x^{(n)}$ which is such that:

$$\frac{\Delta y}{\Delta x} = n.x^{(n-1)}.$$

A function which satisfies this condition is defined as follows:

$$x^{(n)} = x.(x - h).(x - 2h)\dots(x - (n - 1)h).$$

This new function is called the "factorial function", since, letting $x = n$ and $h = 1$, we have

$$n^{(n)} = n(n - 1)(n - 2)\dots3.2.1 = n!.$$

Let us now verify that

$$\frac{\Delta x^{(n)}}{\Delta x} = nx^{(n-1)}.$$

We have

$$\frac{\Delta x^{(n)}}{\Delta x} = \frac{1}{h}\{(x + h)^{(n)} - x^{(n)}\}$$

$$= \frac{1}{h}\{(x + h)x(x - h)\dots(x - (n - 2)h)$$

$$-x(x - h)(x - 2h)\dots(x - (n - 1)h)\}$$

$$= \frac{1}{h}\{x(x-h)\dots(x-(n-2)h)\}\{(x+h)-(x-(n-1)h)\}$$

$$= \frac{1}{h}x^{(n-1)}\{x + h - x + nh - h\}$$

$$= nx^{(n-1)}, \text{ as required.}$$

It should be noted that $x^{(n)}$ is a polynomial of degree $n$. E.g. $x^{(3)} = x(x - h)(x - 2h) = x^3 - 3hx^2 + 2h^2x$.

For convenience, we shall now let $h = 1$.

Any polynomial of degree $n$ can be expressed as a "power series" of factorial functions up to and including a term in $x^{(n)}$. That is to say, if $f(x)$ is a polynomial of degree $n$, then

$$f(x) = a_0 + a_1x^{(1)} + a_2x^{(2)} + \ldots + a_nx^{(n)}. \tag{7}$$

Now

$$\frac{\Delta f(x)}{\Delta x} = a_1 + 2a_2x^{(1)} + \ldots + na_nx^{(n-1)} \tag{8}$$

and

$$\frac{\Delta^2 f(x)}{(\Delta x)^2} = \frac{\Delta}{\Delta x}\left(\frac{\Delta f(x)}{\Delta x}\right) = 2a_2 + 6a_3x^{(1)} + \ldots + n(n - 1)a_nx^{(n-2)}, \tag{9}$$

and we may continue in this way until we reach

$$\frac{\Delta^n f(x)}{(\Delta x)^n} = n!a_n. \tag{10}$$

Now, given that $\Delta x = 1$, we may find the coefficients in the power series by putting $x$ equal to zero in equations (7) - (10) to find:

$$f(0) = a_0$$
$$\Delta f(0) = a_1$$
$$\Delta^2 f(0) = 2a_2$$

and so on, until we reach

$$\Delta^n f(0) = n!a_n.$$

So now we may derive a result known as the *Gregory-Newton Formula* by substituting these expressions into Equation (7). This gives

$$f(x) = f(0) + \Delta f(0)x^{(1)} + \frac{\Delta^2 f(0)}{2!}x^{(2)} + \ldots + \frac{\Delta^n f(0)}{n!}x^{(n)}.$$

This formula has a simple and very useful application.

Suppose we have a sequence of numbers,

-2, 4, 24, 64, 130,

and we are given that these numbers correspond to $f(0)$, $f(1)$ up to $f(4)$, where $f(x)$ is a polynomial of unknown degree and coefficients.

From the definition of the $\Delta$ operator we have that,

$$\Delta f(0) = f(1) - f(0)$$

$$\Delta^2 f(0) = \Delta f(1) - \Delta f(0)$$

and in general, $\Delta^n f(0) = \Delta^{n-1} f(1) - \Delta^{n-1} f(0)$.

Now we draw up a "Difference Table" thus:

| $x$ | $f(x)$ | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| 0 | -2 | | | | |
| | | 6 | | | |
| 1 | 4 | | 14 | | |
| | | 20 | | 6 | |
| 2 | 24 | | 20 | | 0 |
| | | 40 | | 6 | |
| 3 | 64 | | 26 | | |
| | | 66 | | | |
| 4 | 130 | | | | |

Now we simply substitute the values for $f(0)$, $\Delta f(0)$ up to $\Delta^3 f(0)$ into the Gregory-Newton Formula to obtain

$$\begin{aligned} f(x) &= x^{(3)} + 7x^{(2)} + 6x^{(1)} - 2 \\ &= x(x-1)(x-2) + 7x(x-1) + 6x - 2 \\ &= x^3 - 3x^2 + 2x + 7x^2 - 7x + 6x - 2 \\ &= x^3 + 4x^2 + x - 2. \end{aligned}$$

We may check this result by setting (e.g.) $x = 2$. The formula just derived gives

$$f(2) = 8 + 16 + 2 - 2 = 24,$$

which is the correct value, as given by the table.

So we have a method for determining the original or "parent" polynomial for a sequence of numbers, thanks to a simple form of calculus - the *Calculus of Finite Differences*.

*Reference*

Murray R. Spiegel: *Calculus of Finite Differences and Difference Equations* (Schaum's Outline Series, McGraw-Hill, 1971).

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## RELATIVITY THEORY?

President Carter will land in Cairo in about half an hour, Australian time.

A.B.C. News, 8.3.79

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## C'MON AUSSIE C'MON

Darling is extremely quick along the ground. He would have several more paces per square inch than anyone else.

G. Yallop, ABV2, 24.3.79

# MATHEMATICS AND CAREERS

Lionel Parrott, Monash University

When you choose your subjects for the coming year for either further secondary study or a tertiary course, you should ask yourself several questions:

> What options am I denying myself? Am I rejecting some alternatives virtually forever?
>
> Am I choosing subjects which complement one another and allow scope for flexibility in the future?
>
> Am I over-concentrating on too few options?
>
> Am I considering every possibility?

Very few students take up mathematics again after dropping it. The same is true of physics and chemistry, but of few other subjects. An able student, uncertain of his long-term career intentions will find he retains greatest flexibility by retaining physics, chemistry, pure and applied mathematics to HSC level.† This permits the widest possible choice of tertiary courses. Although many subjects can be studied at university without prior secondary school preparation, mathematics, physics, and chemistry cannot be encountered for the first time at tertiary level. On the other hand, courses in arts, economics, and law can be completed satisfactorily by students who have prepared themselves with a science-based HSC.

I, unfortunately, was myself one of the many victims of a combination of poor teaching and inadequate career advice. As a mathematics casualty at year 11, I hesitate to comment on the relative merits of general mathematics as opposed to pure and applied mathematics. However, I recall, that even 23 years ago, general mathematics was regarded as a terminal subject, and as a result the less able students were diverted to it.

As far as I am aware, little has occurred within the general mathematics syllabus to change this original intention. An unfortunate result has been that the competition for entry to courses such as medicine has caused students to substitute biology and general mathematics for pure and applied mathematics in order to improve their prospects of gaining admission.

It seems clear that any branch of mathematics originally intended as terminal constitutes an unsatisfactory preparation for courses requiring the further study of mathematics at tertiary level, despite the well-intended efforts of science and engineering faculties to provide bridging studies in first year. Students intending to pursue courses in science or engineering are obviously much better prepared for their study by taking pure and applied mathematics at HSC.

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†While this specific advice refers to Victoria, the general principle of maximum flexibility is valid for all states.

On its inside cover this magazine claims that 'there are few human endeavours ... that do not involve mathematics'. This statement can be put in another way, 'There are few tertiary courses which are not enhanced by a sound preparation in mathematics at secondary school'. For arts degrees (at least at the Universities of Melbourne and Monash), a pass in mathematics or a foreign language at Higher School Certificate is a prerequisite for entry. The ability to manipulate and interpret data is beneficial to the study of disciplines such as sociology and geography; the field of economics requires increasingly a capacity for dealing with quantitative data; psychology calls for competence with statistics; and mathematics will assume an increasingly important role in areas susceptible to rapid technological change, such as computer science and engineering. It is easy to lose sight of the fact that half the careers of today's secondary students will be located in the 21st century.

Mathematics, then, is a discipline that will enhance the study of most other disciplines, at least at tertiary level. But what can those who decide to graduate in mathematics do at the completion of their studies? The majority of mathematicians are still trained in order that they in turn may train other mathematicians.

The most recent figures for those completing science degrees in mathematics at Monash (1977) show that 44·6% of them were engaged in teaching or teacher training. This represents a slight increase over an earlier period 1969-1971 (38·9%), 17·6% were engaged in further studies (higher degree, or post-graduate diploma) 6·1% were employed by the various government bodies, and 11·5% were employed in the private sector. The remainder were mainly unavailable for employment, unemployed, returned overseas, or simply not known.

Although the number proceeding to employment other than teaching is relatively small, it is important that the nature of the work undertaken by this group should be understood. Only in isolated cases is it directly related to the actual studies taken at the university. More often the important factors are the mental qualities that a prolonged study of mathematics should have engendered.

The most important factor in the growth in employment opportunities for mathematically trained graduates has been the computer; work as a programmer or systems analyst demands precise logic. Other areas include actuarial work (still having difficulty in attracting graduates of sufficient quality), operations research (which involves the application of complicated mathematical techniques to business and other problems), market research (requires a sound feel for statistics and survey methods), quality control (which can be at times an exercise in applied probability theory) and in production planning (where mathematics can produce order for complex scheduling).

More graduates of the 'eighties will find their careers disrupted at some stage, perhaps irretrievably so, by the impact of rapidly changing and complex technology. Amongst those best equipped to survive, graduates in mathematics will certainly be numbered.

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The author is a senior careers counsellor.

# LETTERS TO THE EDITOR

## LARGEST PRIME

The largest known prime number $2^{21701} - 1$ is a 6 533 digit number discovered by two 18 year old California State University at Hayward students. Laura Nickel and Curt Noll spent three years searching for the prime, their high school having allotted 3 000 computer hours for the search. Only! 440 hours were used to find, print out, and prove that $2^{21701} - 1$ is prime. "Miss Nickel said that her 'darkest hours' were between June 1977 and August 1978 when her parents wouldn't approve of her spending hundreds of odd hours at the college when the computer was free", the Los Angeles Times reported last November 16.

The previous largest known prime is $2^{19937} - 1$ discovered in 1971 by Bryant Tuckerman; before that the record holder was the 'Illinois prime' $2^{11213} - 1$ (the Department of Mathematics, University of Illinois franked its mail with the announcement that this number is prime; now letters from Urbana-Champaign proudly announce that 'four colors suffice'). Tuckerman had in fact shown that $m = 19\,937$ is the smallest integer greater than 11 213 for which $2^m - 1$ is prime. The two Hayward students were fortunate that the next such prime occurred so close to Tuckerman's prime (in fact Tuckerman had checked all (prime) $m < 20\,000$). Primes of the form $2^m - 1$ are known as Mersenne primes.

I met Laura and Curt at the Western Number Theory Conference at Santa Barbara a few months ago; they are nice kids (and, if it matters, she is quite attractive).

Alf van der Poorten
Macquarie University.

## YET MORE ABOUT $\pi$

In *Function*, Vol.1, Part 3, Page 17, it is shown that

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi .$$

In fact, it follows from this that

$$\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630} ;$$

because $$\int_0^1 x^4(1-x)^4 dx = \frac{1}{630}$$

and $$\frac{1}{2}\int_0^1 x^4(1-x)^4\,dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx < \int_0^1 x^4(1-x)^4 dx .$$

J.M. Mack
University of Sydney

# PROBLEM SECTION

We have had some response to last issue's problems and to those still outstanding from earlier years. We would welcome more. You will notice that the problems are not of equal difficulty. This is deliberate - we hope that there is something for everyone. You may also like to submit problems to us (with or without solutions). Sometimes, problems also arise without specific numbers. The quotation from *Mrs Miniver* on page 27 of our last issue is a case in point. You are also invited to try your hand at these.

## MORE ON PROBLEM 2.2.4

This problem was, in essence, the following: if $x, y, z$ are positive and $x^2 + y^2 = z^2$, prove that $x^n + y^n < z^n$ for $n > 2$.

Professor P.D. Finch supplies the following very simple proof.

Clearly $x < z$ and $y < z$. Then $x^{n-2} < z^{n-2}$ and $y^{n-2} < z^{n-2}$. Then

$$\begin{aligned} x^n + y^n &= x^{n-2}.x^2 + y^{n-2}.y^2 \\ &< z^{n-2}.x^2 + z^{n-2}.y^2 \\ &= z^{n-2}(x^2 + y^2) \\ &= z^n. \end{aligned}$$

Last issue we published an alternative solution by Geoffrey Chappell. Note that $n$ need not be integral and that the inequality reverses if $n < 2$. Both proofs are readily modified to cover the latter case.

## SOLUTION TO PROBLEM 2.5.2

The problem asked for a formula for the sum

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}$$

and a proof of its correctness.

Otis C. Wright III, then in 8th grade at Davidson High School, New South Wales, supplied the following answer and proof, making use of the technique of *mathematical induction* or "passage to the next integer".

Otis first gives the formula for the sum as $1 - \frac{1}{(n+1)!}$ He notes that this is correct for the case $n = 1$. He now supposes it to hold for some value $N$ of $n$. That is

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{N}{(N+1)!} = 1 - \frac{1}{(N+1)!} \qquad (*)$$

He then investigates the case $n = N + 1$. We have

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{N}{(N+1)!} + \frac{N+1}{(N+2)!} = 1 - \frac{1}{(N+1)!} + \frac{N+1}{(N+2)!},$$

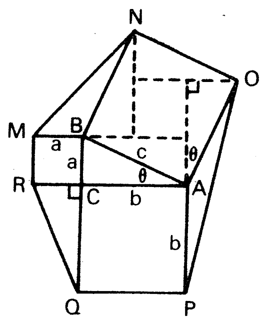
by (*). But the right hand side of this expression is equal to $1 - \frac{1}{(N+2)!}$, which is the result predicted by his formula. Hence, if his formula holds for $n = N$, it also holds for $n = N + 1$.

Because it holds for $n = 1$, it must therefore hold for $n = 2$. Because it holds for $n = 2$, it must also hold for $n = 3$. Etc. This process shows that the result is true for all positive integral values of $n$.

## SOLUTION TO PROBLEM 2.5.3

The problem read: "A right angled triangle has area $A$ and hypotenuse of length $c$. On each side of the triangle draw a square, exterior to the triangle. Now imagine a tight rubber band placed around your figure. What area would it enclose?"

Otis also solved this problem. The notation used is that of the diagram to the right. Now the square $ABNO$ is tilted from the horizontal by an angle $\theta$, and it is easily seen that four triangles, each congruent to $ABC$, may be placed along the interior sides of each of its edges (as shown by the dotted lines). We see that the triangle $MNB$ has height $b$, and the triangle $OAP$ has height $a$. It follows that both these triangles have area $\frac{1}{2}ab$, which is equal to $A$, the area of the triangle $ABC$.




Likewise, this is the area of the triangle $RCQ$. The total area is now $4A + a^2 + b^2 + c^2$. But, by Pythagoras' Theorem, $a^2 + b^2 = c^2$, and hence the area of the polygon $MNOPQR$ is equal to $4A + 2c^2$.

## SOLUTION TO PROBLEM 3.1.4

This problem read:

A man and a horse run a race, one hundred metres straight, and return. The horse leaps 3 metres at each stride and the man only 2, but then the man makes three strides to each two of the horse. Who wins the race?

Magnus Cameron, Year 7, Glen Waverley High School, writes:

For every six metres the man and horse do, the horse and man are at exactly the same place. When they come to 96 metres the horse makes 2 strides (6 metres) and brings it to 102 metres. It comes to 102 metres because at 100 metres it can't turn in mid air. As for the man, he does 3 strides (6 metres) which brings him to 98 metres (left). They keep the same pace as before. So, at 2 metres left for the man, the horse has 6 metres left. When the man finishes the horse is still making his second last stride.

## PROBLEM 3.2.1

Those who cringe at the thought of Friday 13th were undoubtedly unhappy to learn that the 13th of the month is more likely to occur on a Friday than on any other day. (See Problem 1.1.1, and its solution - Volume 1, Part 3.) You will probably be just as unhappy to know that there is at least one such Friday in *every* year. Can you explain why?

## PROBLEM 3.2.2

$P(x)$, $Q(x)$, $R(x)$ are all polynomials. They satisfy the identity

$$P(x^3) + xQ(x^3) = (1 + x + x^2)R(x).$$

Show that all three polynomials are exactly divisible by $x - 1$.

## PROBLEM 3.2.3

Prove that, of all the teenagers in the world, at least two have the same number of teenage friends.

## PROBLEM 3.2.4

A bag contains three red balls and five white ones. Balls are drawn at random from the bag without replacement, until all have been withdrawn. Show that the probability of getting a red ball on *any* particular draw (e.g. the fifth) is 3/8.

## PROBLEM 3.2.5

Let $ABCDE$ be a regular pentagon. The diagonals $AD$ and $EC$ meet at the point $Q$. Show that for the segment lengths we have

$$\frac{AD}{AQ} = \frac{AQ}{QD}$$

and hence prove that the ratio $AD/AQ$ is equal to $\frac{1}{2}(1 + \sqrt{5})$.

## PROBLEM 3.2.6

I toss three coins. I argue that the probability that they all fall heads is $(\frac{1}{2})^3 = \frac{1}{8}$. The probability that they all fall tails is also $\frac{1}{8}$, so that the probability that they all fall alike is $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$. My friend argues differently. If three coins are thrown up, at least two must come down alike; the probability that the third coin comes down the same as the other two is $\frac{1}{2}$, as it has an equal chance of being like or unlike.

Who is correct?

PROBLEM 3.2.7

$a$, $b$, $c$ are real numbers which satisfy the equation

$$3a^2 + 4b^2 + 18c^2 - 4ab - 12ac = 0.$$

Prove that $a = 2b = 3c$.

PROBLEM 3.2.8

Prove that amongst all the triangles of a given perimeter, the equilateral triangle has the largest area.

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PARABOLA

*** Parabola is a mathematics magazine whose aim is to encourage and sustain an interest in mathematics among secondary school students.

Parabola was born in 1964 and has now reached the respectable age of fifteen volumes. We have sought to provide problems and puzzles to test both ingenuity and determination, articles to divert and enrich, and opportunities for our readers to share their mathematical thoughts and activities.

*** Some highlights from the last three years:

Polyhedra: the Platonic solids and deltahedra
Areas and volumes without pain
Tessellating the plane with pentagons
How to construct magic squares
A strategy for mastermind
Indian mathematics
Testing for prime numbers with Pascal's triangle
Conway's game of life
A cosmologist's view of the universe
Palindromic numbers
Projectiles
Coincidences concerning triangles.

*** Regular features:

Problems and puzzles
Reports on mathematics competitions
Letters to the Editor.

Our plans also include an occasional series of articles on mathematics at work, giving some ideas on how mathematics is used in real life.

*** Parabola appears three times each year. The subscription rates for 1979 are \$1.50 for school students and \$2.00 for other subscribers. Back copies are also available at attractive rates.

*** For further information, write to:

The Editor, Parabola,
School of Mathematics,
University of New South Wales,
Kensington,
N.S.W. 2033.

## MATHEMATICS LECTURES

The series of lectures for 5th and 6th form students at Monash University continues. They are on Friday evenings, from 7 p.m. to 8 p.m. Your school has received a detailed programme.

Lectures are held in the Rotunda Lecture Theatre R1 (enquire at the main gate). The remaining lectures are:

| | |
|---|---|
| May 4 | Why Mathematics is Difficult. Some Interesting, Hard and Unsolvable Problems. Dr J.C. Stillwell. |
| June 8 | Mathematics of Winds and Currents. Dr C.B. Fandry. |
| June 22 | Mechanics, a Central Science. Prof. B.R. Morton. |
| July 6 | Choosing the Site of a School, to Minimize the Distance to Three Villages. Dr E Strzelecki. |
| July 20 | Prime Numbers. Dr R.T. Worley. |
| August 3 | Laputa or Tlön - How Real is the Imaginary? Dr M.A.B. Deakin. |

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## THE SCIENCE MAGAZINE

The Faculty of Science at Monash University will produce a magazine for distribution to secondary school students in 1979. It's called *The Science Magazine* and its production will be in the hands of Drs Ann Lawrie, Keith Thompson and Ian Rae.

Several issues will be prepared for 1979 and the Monash scientists hope that the magazine will become a continuing publication. Like the mathematics magazine, *Function*, the science magazine is intended to cater for keen students rather than simply to mimic the school syllabus. Students taking chemistry, physics, biology, physical science, environmental science and mathematics should find something of interest in every issue and the new editors feel that their magazine could be a valuable resource for school science students.

Some articles for the new magazine are already in hand. They deal with Australian legumes, oil from beans, mercury in the environment, and the nature of fingerprints. There will be a lot more before the year is over and the editors would welcome contributions or suggestions for topics upon which they could commission articles.

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I am glad to state that, for once, the correct solution... appears to have been given, not by an engineer on the basis of physical experience, but by a "mathematician in his armchair"... namely, myself.