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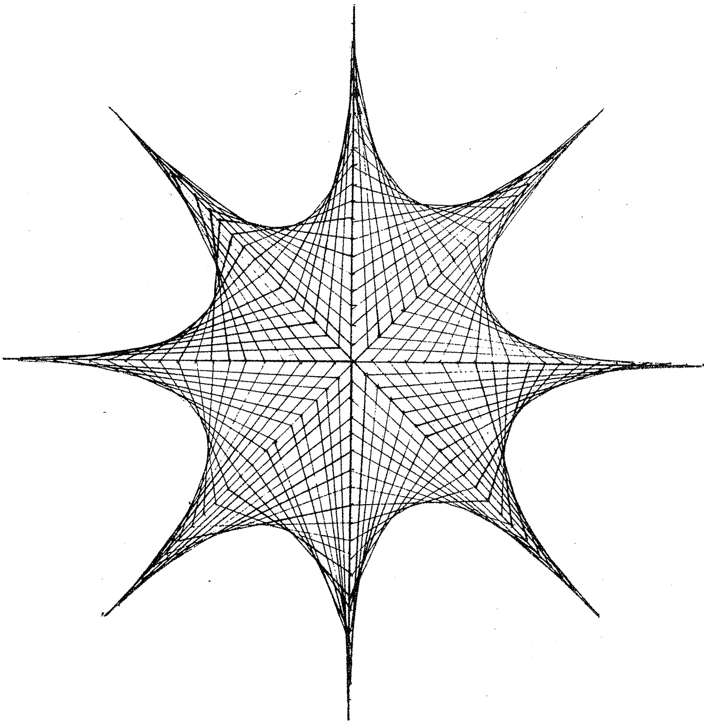

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. There is a problem section with solutions invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Most of this issue has been written by our subscribers. And this delights the editors. We have hoped from the beginning to involve our readers in the choice of material, in the writing of articles and in generally helping us to provide a magazine that you like. And the number of readers writing to us is continuing to increase. Please keep on writing to us.

Enclosed is an order form for next year's *Function*. To receive your first issue next year your order should be in before the middle of February next year. Further order forms may be obtained from the business manager.

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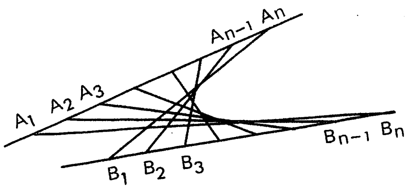
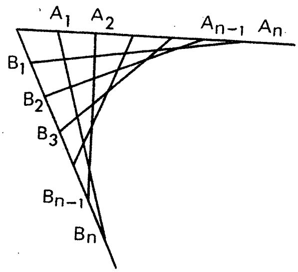
THE FRONT COVER

J. O. Murphy, Monash University

A parabola can be constructed geometrically in many different ways. The section of a right circular cone, by a plane parallel to a generating line of the curved surface of the cone, is a parabola. The geometric properties of the parabola together with those of the ellipse and hyperbola, all considered as conic sections, were first studied by the ancient Greeks. Alternatively, a parabola can be constructed as the locus of a point in a plane whose distance from a fixed point is equal to its distance from a fixed line. This is the focus (fixed point) - directrix (fixed line) property associated with the parabola.

A parabola can also be constructed from a set of enveloping tangents. Points which are equally spaced on any two straight lines are joined, as illustrated below; the intervals on line *A* do not have to be equal to the intervals on line *B*.

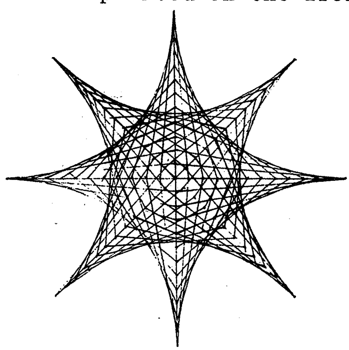




Two students from Boronia Technical School, Ken Sanders and Iain Kennedy, drew the diagrams which are printed on the front cover and on this page.

PROBLEM: Draw a parabola to touch four given straight lines.

PROJECT: Many varied and colourful parabolic designs, based on the above method, can be achieved by using nails driven into a chip board base and then joined by coloured cotton.



BOOLEAN ALGEBRA

Alasdair McAndrew

Melbourne High School

INTRODUCTION

In 1847, in England, a very small book - indeed, little more than a pamphlet - was published; its author was George Boole. Its title was *The Mathematical Analysis of Logic*, and it made new and profound contributions to the study of symbolic logic. Other people working in this field at the same time - especially Augustus De Morgan - at once hailed Boole as their master.

However, Boole's book was only the preliminary work, heralding his magnum opus: *An Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities*.† It was of this book that Bertrand Russell was to say that in it Boole had discovered pure mathematics.

What had Boole done? He had at last given logic a firm algebraic basis. Boole was not the first to give this matter some thought, but he was the first to take the decisive step into opening up what was to become one of the major fields of mathematics.

In 1904, E.V. Huntingdon proposed four postulates - now known as *Huntingdon's Postulates* - which would define a Boolean Algebra. I will now state Huntingdon's Postulates.

If we have a non-empty set B, and elements a, b, c in B, and there are two binary operations which I shall denote by $+$ and $\cdot$ (not to be confused with ordinary addition and multiplication), which are defined on B, then Huntingdon's Postulates are:

P(1) Law of Commutativity

$$a + b = b + a, \quad a \cdot b = b \cdot a$$

P(2) Law of Distributivity

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad a + (b \cdot c) = (a + b) \cdot (a + c)$$

P(3) Law of 0 and 1

There are two elements 0, and 1 in B, for which $a + 0 = a$, and $a \cdot 1 = a$, where $a \in B$.

† Now reissued in Dover paperback.

P(4) Law of Complementarity

For each element a in B there exists an element a', called the *complement* of a, such that $a \cdot a' = 0$, and $a + a' = 1$.

Any mathematical system then, that satisfies these Postulates, is called *A Boolean Algebra.*

From the Postulates we can derive many useful theorems. I will give a sample here, along with proofs for some of them.

THEOREM I (Principle of Duality). *For any theorem or postulate in Boolean Algebra, which is true, we can form another true theorem or postulate by interchanging the symbols + and ·, and also interchanging the symbols* 0 *and* 1.

Proof. This is true for all of Huntingdon's Postulates, and as any theorem can be derived from these, the Principle of Duality must hold.

THEOREM II (Law of Tautology).

$a + a = a$ *(and hence, by the Principle of Duality),* $a \cdot a = a$.

Proof.

$$\begin{aligned} a + a &= a \cdot 1 + a \cdot 1 && \text{by P(3)} \\ &= (a + a) \cdot 1 && \text{by P(2) and P(1)} \\ &= (a + a)(a + a') && \text{by P(4)} \\ &= a + (a \cdot a') && \text{by P(2)} \\ &= a + 0 && \text{by P(4)} \\ &= a && \text{by P(3).} \end{aligned}$$

Thus $a \cdot a = a$ is automatically proved to be true also.

THEOREM III. $a + 1 = 1$, *and hence,* $a \cdot 0 = 0$.

Proof.

$$\begin{aligned} a + 1 &= (a + 1) \cdot 1 && \text{by P(3)} \\ &= (a + a')(a + 1) && \text{by P(4) and P(1)} \\ &= a + (a' \cdot 1) && \text{by P(2)} \\ &= a + a' && \text{by P(3)} \\ &= 1 && \text{by P(4).} \end{aligned}$$

Hence, $a \cdot 0 = 0$ (by Principle of Duality).

The next few theorems I will give without proofs, but the reader can supply his own, using Huntingdon's Postulates, or Truth Tables (to be described shortly).

THEOREM IV (Law of Absorption).

$$a + (a \cdot b) = a, \text{ and hence, } a \cdot (a + b) = a.$$

THEOREM V. *The complement of* a *is unique and* $(a')' = a$.

THEOREM VI (Laws of Associativity). $a + (b + c) = (a + b) + c$ *and* $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

THEOREM VII (Overlap Theorem). $a + (a' \cdot b) = a + b$ and $a \cdot (a' + b) = a \cdot b$.

THEOREM VIII (De Morgan's Laws). $(a + b)' = a' \cdot b'$ so $(a \cdot b)' = a' + b'$.

From now on, I will not be putting the $\cdot$ symbol between two elements, $a \cdot b$ will be written as ab, and I will leave out brackets if there is no ambiguity. For example, $a + (b \cdot c)$ will be written $a + bc$.

The above laws may seem difficult and abstruse at first, but after getting used to using them, you will find that they are as easy and natural to use and understand as the rules of ordinary algebra.

I will show how these laws can be used in simplifying a Boolean function.

Let $F = ac + abc + ab'c + ac' + abc' + ab'c'$

then

$$
\begin{aligned}
F &= aac + abc + ab'c + aac' + abc' + ab'c', && \text{by Theorem II} \\
&= (aa + ab + ab')c + (aa + ab + ab')c' && \text{by P(2)} \\
&= (aa + ab + ab')(c + c') && \text{by P(2)} \\
&= (aa + ab + ab') \cdot 1 && \text{by P(4)} \\
&= aa + ab + ab' && \text{by P(3)} \\
&= a(a + b + b') && \text{by P(2)} \\
&= a(a + 1) && \text{by P(4)} \\
&= a1 && \text{by Theorem III} \\
&= a && \text{by P(4).}
\end{aligned}
$$

Other methods of simplification will be introduced later on in the article.

TRUTH TABLES

In Boolean Algebra, we are concerned with things that have only two states, e.g., on or off, true or false, open or closed, belonging or not belonging to a set, etc. It may be shown that two expressions in a Boolean Algebra, formed from the variables a, b, c, etc., are equal in the algebra if and only if they are equal for every possible allocation of the value 0 or 1 to each of the variables a, b, c, etc., involved.

A truth table for an expression, or a function, simply lists the possible allocations of values 0, 1 for each of the variables involved and alongside gives the corresponding value of the function. For instance, Figure 1 is a truth table for $f(a, b) = a + b$.

a	b	$a + b$
0	0	0
0	1	1
1	0	1
1	1	1

Figure 1

A truth table can be used to prove theorems. For instance, let us prove Theorem VII that $a + a'b = a + b$. Here is the truth table for proving Theorem VII (Figure 2).

a	b	a'	$a'b$	$a + a'b$	$a + b$
0	0	1	0	0	0
0	1	1	1	1	1
1	0	0	0	1	1
1	1	0	0	1	1

Figure 2

No matter what values we assign to a and b, $a + a'b = a + b$. This proves the theorem. Other theorems can be proved similarly.

Given a truth table, we can determine the function of a and b (and all the other variables) which gives us the final values. For instance, given the truth table in Figure 3, what is the function?

a	b	F
0	0	0
0	1	1
1	0	1
1	1	0

Figure 3

To do this we use *Boole's Expansion Theorem*. In two variables, it states that $F(a, b) = F(1, 1)ab + F(0, 1)a'b + F(1, 0)ab' + F(0, 0)a'b'$. In our case, then, we have $F(a, b) = 0ab + 1ab' + 1a'b + 0a'b' = ab' + a'b$, which can also be written as $(a' + b')(a + b)$. This useful expansion theorem can be generalized to any number of variables (can you see how?) and is, of course, invaluable for determining a function.

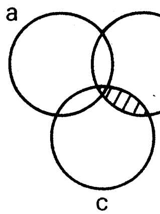
SETS

It may be thought, just by looking at Huntingdon's Postulates, and the theorems, that no system would satisfy them. Well, I shall be discussing three systems that do; the first is Sets.

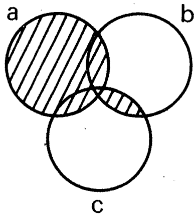
We are all acquainted with the ideas of union and intersection of sets (usually denoted by $\cup$ and $\cap$). I shall denote these operations by + and $\cdot$, so as to conform with the postulates and theorems. 0 is to be considered the set containing no elements (the null set), and 1 to be the set of all elements - although in practice we limit 1 to be something more specific. For example, if we are concerned with all Australian males, then we let 1 = {Australian males}. We also say that a' is the set of all elements not in a. Thus $a \cdot a' = 0$ is at once proved, as is $a + a' = 1$.

Given this information, I will prove a theorem, using Venn diagrams.

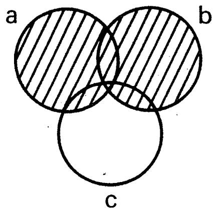
Consider P(2); $a + bc = (a + b)(a + c)$



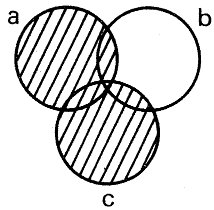
(Shaded area = bc)



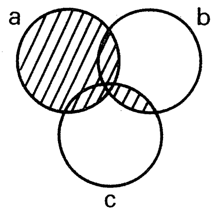
(Shaded area = $a + bc$)



(Shaded area = $a + b$)



(Shaded area = $a + c$)



(Shaded area = $(a + b)(a + c)$).

Thus it can be seen that $a + bc = (a + b)(a + c)$. All the other postulates can be proved similarly, from which it follows that the algebra of sets is a Boolean Algebra.

Sets and Venn diagrams provide us with a way of simplifying Boolean functions, but a Karnaugh Map makes things easier still.

KARNAUGH'S MAPS

Karnaugh's maps are just a modification of the Venn diagram. I will be considering the four-variable Karnaugh Map (Figure 4); the squares are numbered purely for convenience. On this map, regions are represented by groups of squares. For instance a is represented by the two columns on the left; a', then, is represented by all the other squares. Again, d is represented by the middle two rows, thus d' is all the other squares. Likewise for b and c. The intersection of two regions (or sets) is given by the squares common to both of them. Thus,

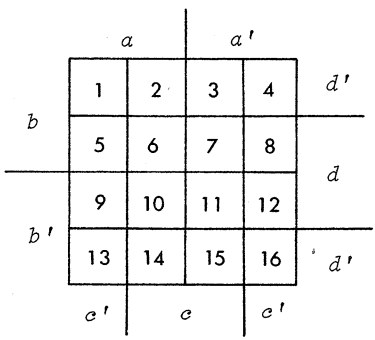


Figure 4

ad' is represented by the squares marked 1, 2, 13 and 14. The region involving intersection of three sets is represented by two squares - for example, abc is represented by the squares 2 and 6. A region involving a product of four sets is represented by one square. Thus, $abc'd'$ is represented by the square marked 1.

When marking regions on a Karnaugh Map, it is convenient to mark with a "I" all the regions we are considering, and leave the rest blank. Thus, $abc + ab'c' + abcd'$ is represented as in Figure 5. Note that $abcd'$ is contained in the region abc. We can use a Karnaugh Map to simplify a function; for instance, to simplify the function $abc'd' + abcd' + abc'd + abcd + ab'c'd + ab'cd$, we mark the regions concerned on a Karnaugh Map (Figure 6). It can immediately be seen that the region marked corresponds to $ab + ad = a(b + d)$.

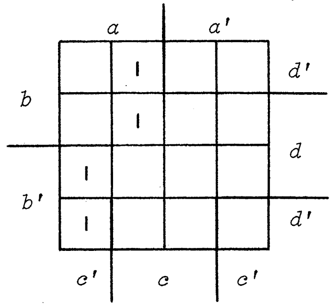


Figure 5

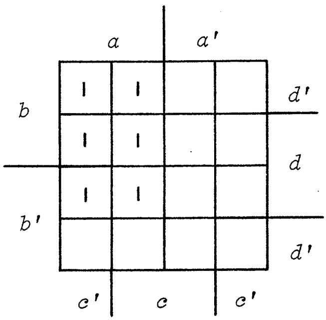


Figure 6

For 2 or 3 variables, the Karnaugh Maps are as in Figures 7 and 8. For more than four variables, no really satisfactory form of a Karnaugh map has been found. Karnaugh himself proposed maps of up to nine variables, but none of them (from 5 variables up) was really workable.

Sets really come into their own in Logic, when it is often convenient to consider a statement in logic in terms of sets. However, before I go into the subject of logic, I will discuss Switching Circuits, and the algebra of them.

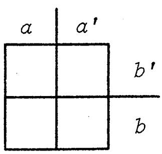


Figure 7

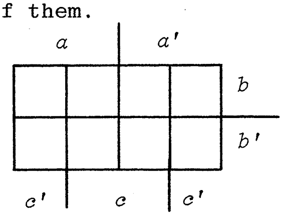


Figure 8

SWITCHING CIRCUITS

It was discovered only as recently as 1938 that the algebra of switching circuits is a Boolean Algebra. In this algebra + represents "in parallel", and · represents "in series". Thus the circuit in Figure 9 would be represented by the function $a + bc$. We call the function $a + bc$ the *transmission* of the circuit. The transmission is equal to 0 if the circuit is open, and 1 if the circuit is closed. A switch is always open or closed. A switch a' is the switch which is open when a is closed and closed when a is open. Thus if we draw the circuit in Figure 10, the circuit represented by $a \cdot a'$, it is obvious that at any time *one* of the switches must be open; thus this circuit must have a transmission of 0. If we draw the circuit with transmission $a + a'$ (Figure 11) since one of the switches is always open, then one switch must also be closed, so the circuit has a transmission of 1. Thus we have seen that $aa' = 0$, and $a + a' = 1$, showing that postulate P(4) holds. The other postulates can also be seen to be true; thus it follows that the algebra of switching circuits is a Boolean Algebra.

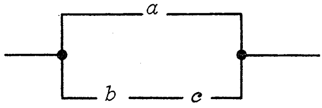


Figure 9

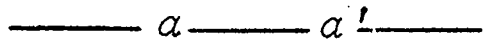


Figure 10

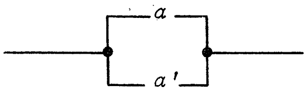


Figure 11

Boolean Algebra thus provides us with a way of simplifying a circuit. For example, consider the circuit in Figure 12. It has a transmission of $abc + a(b + c'a) + a(b + ca') + (b + a')c$. We can rewrite this as $abc + ab + ac' + ab + bc + a'c$. By applying the theorems and postulates, or using a 3-variable Karnaugh Map, we find that it all boils down to $ac' + ab + a'c$, which can be represented by the circuit in Figure 13. Notice how much simpler this is than the original circuit! Simplification of complex circuits - such as may be found in a computer - may save considerable time and money for the firm building the computer. Boolean Algebra is an immensely powerful tool for doing this sort of thing. Also, circuits can be designed from scratch using Boolean Algebra, one part of this subject is known as *control*.

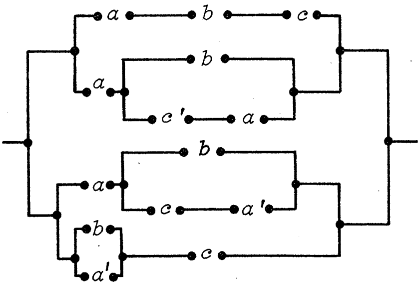


Figure 12

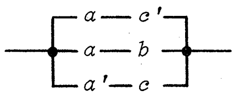


Figure 13

CONTROL

Consider the problem: "There are four switches and one light. If one switch only is on, then the light is on. If no switches are on, or if more than one switch is on, then the light stays off. Design a circuit to operate the light."

Before I demonstrate the method of solution of this problem, I must introduce the concept of a relay. Figure 14 shows a relay. If switch u is pressed on, current flows round the coil (which is an electromagnet), and pulls the switches above it down. Thus when u is on, x and y are on, but z is off. And when u is off, x and y are off, but z is on. Thus $x = y = u$, and $z = u'$. So the circuit can be represented by $u + u + u'$ $(=1)$.

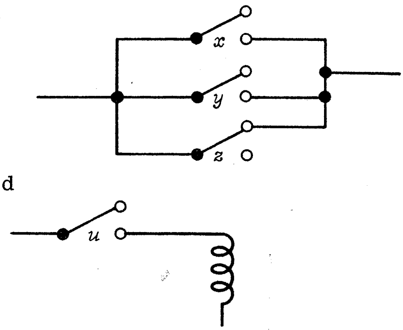


Figure 14

Now, back to the problem above. Let us call the switches a, b, c, and d. Let L stand for the light, or, more properly, let L stand for the statement "The light is on". Now, L is on (or L is true) if a is on and b, c and d are off, or if b is on, and a, c, and d are off, or if c is on, and a, b, and d are off, or if d is on, and a, b, and c are off. Using the notation above then, we can say that

$$L = ab'c'd' + a'bc'd' + a'b'cd' + a'b'c'd.$$

Thus our circuit is as in Figure 15 (opposite).

This of course, is only a simple example of the sort of problem that can be solved using Boolean Algebra.

PROPOSITIONS

A *proposition* is any statement that is either true, or false, but not both, and not neither. For example, the statement: "All pigs can fly", is a proposition, because it is false, and not true. However, the classic statement: "This statement is false" is not a proposition, because, if the statement is true, it must be false (because it says it is), and if the statement is false, then it must be true! Thus the statement can either be considered to be both true and false, or neither or them. Either way it fails to be classified as a proposition, even if, at first glance, it may appear to be one.

Let a proposition, say, "There are sentient beings on Venus", be denoted by p. We then define the symbol p' to

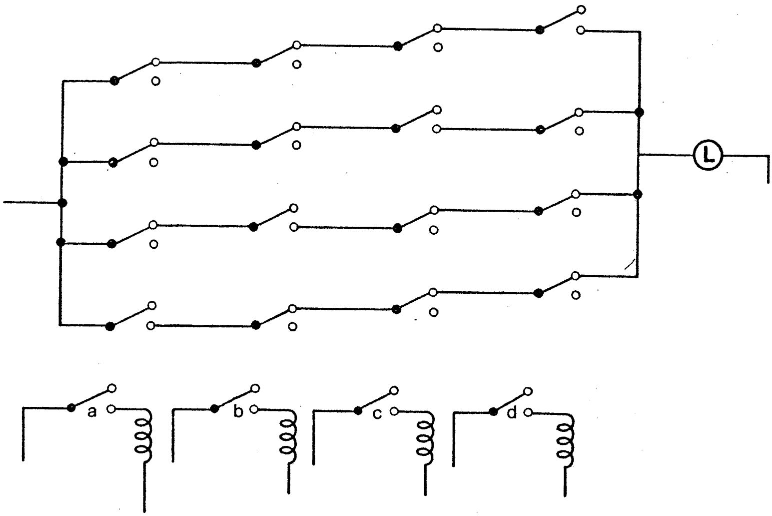


Figure 15

stand for the proposition that is true if p is false, and false if p is true. Thus, in our case, p' would be the proposition "It is false that there are sentient beings on Venus". In general, p' is the proposition "It is false that p".

Before I define the symbols + and ·, I must introduce the concept of a compound proposition. This is just a proposition made up of other, simpler propositions. For example, the proposition: "Pigs can fly and grass is blue, or else 2 = 24", is a compound proposition, because it is made up of the simpler propositions "Pigs can fly", "Grass is blue", and "2 = 24", joined by the conjunctions "and", and "or else".

The conjunctions "and", and "or", must be very strictly defined, and it is convenient to define them so: if p and q are two propositions, then the proposition "p and q" is the proposition that is true if both p and q are true, and false otherwise. The proposition "p or q", is defined to be the proposition that is true unless both of p and q are false. If we draw a truth table, letting 0 = falsity, and 1 = truthfulness (Figure 16), it can be seen then that "p and q" can be algebraically represented as $p \cdot q$, and "p or q", as $p + q$. (With this information in mind, you should be able to get a better understanding of the control problem in the section on Switching Circuits.)

p	q	p and q	p or q
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

Figure 16

We also define 0 to be the proposition that is always false, and 1 to be the proposition that is always true.

With this information in mind, it can easily be seen that all Huntingdon's Postulates are satisfied by propositions, and hence the algebra of propositions is a Boolean Algebra.

It is in the algebra of propositions, and logic, that Boolean Algebra comes into its own. Boole was not the only person to do great work in this field; Lewis Carroll (better known to most people as the author of *Alice's Adventures in Wonderland*), wrote a book called *Symbolic Logic*†. The syllogisms and soriteses invented by Lewis Carroll in this book have remained unchallenged as some of the best logic problems ever devised. Other people worthy of mention are Augustus De Morgan, and Charles Babbage. These were not the only people who worked in this field, but they are some of the greatest names. (See, for example, E.T. Bell, *Men of Mathematics*, Volume 2, Penguin.)

SYLLOGISMS AND SORITESES

A syllogism is a sequence of three propositions, the first two being the premises of the syllogism, while the third, the conclusion, is inferred from the first two. For example, consider the syllogism of Lewis Carroll:

No one, who exercises self-control, fails to keep his temper.
Some judges lose their tempers.
From these we can conclude that "Some judges do not exercise self control".

Now, consider this problem from an algebraic standpoint. Let C be the set of people who exercise self-control, let F be the set of people who fail to keep their temper, and let J be the set of judges.

Then the first premise of the syllogism can be rewritten (in our set notation): $C'F = F$. Likewise the second premise can be rewritten as $JF \neq 0$. Substituting from the first in the second we have $JC'F \neq 0$, whence $JC' \neq 0$, which is equivalent to our original conclusion.

† Now reissued in Dover paperback.

Another method of symbolising statements makes use of the symbol "$\rightarrow$", which means "implies"; so that "$p \rightarrow q$" means "if p, then q". Thus, rewriting the first premise above as "If a person exercises self-control, then he will not fail to keep his temper", it becomes in this notation $x \in C \rightarrow x \in F'$, or in abbreviated form, $C \rightarrow F'$. We now use this notation to solve a sorites.

A *sorites* is a more complicated form of syllogism; this time, instead of just two premises, it can have any number (one terrifyingly difficult one invented by Lewis Carroll has twenty). The method of solution however, is no different from the method of solution of a syllogism, it just takes more time.

Consider this sorites of Lewis Carroll:

(1) When I work a Logic-example without grumbling, you may be sure it is one that I understand;

(2) These Soriteses are not arranged in regular order, like the examples I am used to;

(3) No easy example ever makes my head ache.

(4) I ca'n't understand examples that are not arranged in regular order, like those I am used to,

(5) I never grumble at an example, unless it gives me a headache.

Univ. "Logic-examples worked by me"; a = arranged in regular order like the examples I am used to; b = easy; c = grumbled at by me; d = making my head ache; e = these soriteses; h = understood by me.

Here "Univ" stands for "Universe" and the set notations, supplied by Lewis Carroll, make things easier for us. Note that when it is said: "b = easy", it is meant: "b = the set of those Logic-examples that are easy", and so for all others.

Transmuting each statement into "if ... then ..." form, we get (in Carroll's notation):

(1) $c' \rightarrow h$ (or $h' \rightarrow c$)

(2) $e \rightarrow a'$ (or $a \rightarrow e'$)

(3) $b \rightarrow d'$ (or $d \rightarrow b'$)

(4) $a' \rightarrow h'$ (or $h \rightarrow a$)

(5) $c \rightarrow d$ (or $d' \rightarrow c'$)

Arranging them into one line, we get: $b \rightarrow d'$, $d' \rightarrow c'$, $c' \rightarrow h$, $h \rightarrow a$, $a \rightarrow e'$; thus our solution is $b \rightarrow e'$, or, $e \rightarrow b'$. In English: "These Soriteses are not easy".

The reader may like to solve the problem using the alternative symbolic approach that we used in considering the syllogism about judges.

Naturally, far more difficult problems can be solved than the relatively simple ones given here, but the method is the same for all of them.

* * * * * * * * * * *

That then, was an introduction to Boolean Algebra. I hope the article will be found stimulating; below is a list of references which explore the subject in greater depth.

Bell, E.T.: *Men of Mathematics*, Vol. 2† (Penguin)
Boole, George: *The Laws of Thought* (Dover)
Carroll, Lewis: *Symbolic Logic* (Dover)
Flegg, G.: *Boolean Algebra* (Transworld Student Library)
Hohn, F.E.: *Applied Boolean Algebra* (Macmillan)
Kaye, D.: *Boolean Systems* (Longman)
Pfeiffe , J.E.: *Symbolic Logic* (Sci. Amer., December 1950)
South, G.F.: *Boolean Algebra and its Uses** (Van Nostrand)
Whitesitt, J.E.: *Boolean Algebra and its Applications* (Addison-Wesley)

Watch out for different notations to the one used in this article! Watch out for $\cup$ and $\cap$, or $\vee$ and $\wedge$, instead of + and $\cdot$. Also watch out for $\bar{a}$, or $1 - a$, instead of a'.

† For the life of Boole, and for a bit of history.

* A really excellent, elementary text on the subject. One of the best.

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A FACT ABOUT THE ENGLISH LANGUAGE

The fact noted by Cynthia Kelly (*Function*, Part 3, p. 30) depends in the first instance on there being a number N such that for all $n \geqslant N$, there are fewer than n letters in the name of n. Not all languages possess this property. *Gumulgal* (an Australian language) is an example. In *Gumulgal*, if n is even, say $2m$, its name is *ukasar*^{m} (i.e. one says *ukasar* m times). If m is odd, say $2m + 1$, its name is *ukasar*^{m} - *urapon*. Such languages are (incorrectly) spoken of by anthropologists as employing base 2. Other examples are *Kiwai* (Papua), *Baikiri* (South America) and *Bushman*.

If such an N can be found, the Kelly process may terminate in a fixed number (an equilibrium) or a fixed cycle. *French* has a single 4-cycle: 3, 5, 4, 6 (*trois*, *cinq*, *quatre*, *six*). New Guinea *Tokpisin* has an equilibrium at 2 (*tu*), and a 2-cycle: 3, 4 (*tiri*, *fua*). Papuan *Polis Motu* has two equilibria: 3 (*toi*) and 4 (*hani*), and a 3-cycle: 8, 9, 11 (*taurahani*, *taurahanita*, *gwautata*).

Such attractive equilibria or cycles must exist if *N* does. This principle is related to important topological concepts applied in (for example) the theory of differential equations.

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"The generation of random numbers is too important to be left to chance."

Martin Gardner: *Mathematical Carnival*

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SLITHER

Slither is a game for two people played on a lattice of points. In the April issue of *Function* (p. 13) after introducing you to the game we explained how (p. 30) on a 5 × 6 point lattice, the player who makes the first move can always win. In this article we discuss strategies for the players on playing fields of different sizes and shapes.

One of our readers, Graham Farr, has written to us describing his attempts to analyse the game and determine winning strategies for either player. His discoveries appear in a separate article at the end of this one and can be read independently of the present article.

First we remind you of the rules of *Slither*. The rules of Slither are simple. Start with a rectangular $m \times n$ grid of points, e.g. here is a 5 × 6 point grid

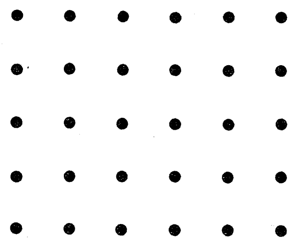


Figure 1

The players take turns marking a horizontal or vertical segment of unit length. The segments must form a continuous path but at each move a player may add to either end of the preceding path. The first player to close the path (i.e. to play so that the path has a closed loop in it) is the *loser*.

We turn now to the situation of an $m \times n$ point lattice where *both* m and n are even. We illustrate the principles on a 4 × 4 point lattice.

In a 4 × 4 point lattice there are a number of possible moves for the first player which we will refer to as his 'favoured possible moves'. One collection of such favoured moves is illustrated in Figure 2 - the favoured possible moves are indicated as dotted lines.

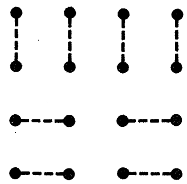


Figure 2

The strategy we now suggest for the first player is simple. At *each* move he should choose one of the dotted lines (and of course always keep to the usual rules of the game). Notice two things about the collection of favoured moves illustrated in Figure 2:-

(a) no two dotted lines have a common endpoint,
(b) each of the 16 lattice points is the beginning or end of exactly one dotted line.

Provided the first player sticks to the suggested strategy then, because of (a), the second player will never have an opportunity to use one of the dotted lines as a move. Further, because of (b), it happens that just before each move of the first player the endpoint which the other player has just added to the path will be the endpoint of a dotted line. Either that dotted line has already been used in the game, in which case the last move of the other player was a losing one, or that dotted line is still available in which case the first player can take it for his next move and it will never be a losing move for him. Eventually the second player will be unable to move without closing the path. So provided the first player abides by the strategy we have suggested he can be sure that he will always win.

Notice that any collection of dotted lines with the properties (a) and (b) above may be used by the first player as his collection of favoured possible moves. For example Figures 3 and 4 below show two different collections of favoured possible moves for the first player. If the first player plays according to the general strategy described above but bases his moves on Figure 3 or on Figure 4 (rather than on Figure 2) then he will still always win.

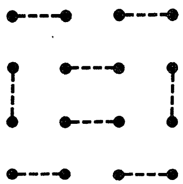


Figure 3

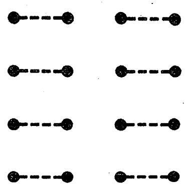


Figure 4

Any collection of dotted lines in an $m \times n$ point lattice with the properties (a) and (b) we will call a *perfect matching* for the lattice.

The argument given above for the first player (using his strategy on the 4×4 lattice based on Figure 2) works on any $m \times n$ lattice when we are given a perfect matching for that lattice. In other words, if there is a perfect matching for a particular lattice then the first player can always force a win on that lattice.

If mn is even then you will always be able to find a perfect matching for the $m \times n$ point lattice. So provided mn is even, the first player can always force a win.

Notice that in an $m \times n$ point lattice which has a perfect matching, every point in the lattice is the endpoint of exactly one (dotted) line in the perfect matching, i.e. the number of points in the lattice (mn) equals twice the number of (dotted) lines in the matching. So a lattice in which mn is odd cannot have a perfect matching. This means that the strategy we described above for the first player cannot be applied. In fact we shall see in the following paragraphs that if mn is odd then the second player can always force a win.

Suppose the game is being played on a 5×5 point lattice and the first player has just made his first move. One of the two endpoints of this first move will always be an even distance from a corner point (where 'distance' is measured in terms of the unit horizontal and vertical lines of the lattice). For example consider the situation illustrated in Figure 5.

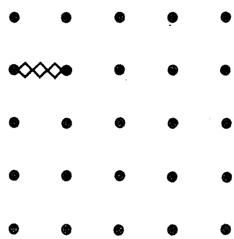


Figure 5

Here the right hand endpoint of the first move is two units from the top left corner (and the left hand endpoint is one unit from that corner). The second player should ignore the right hand endpoint (i.e. the one which is an even distance from the corner) and consider the remaining 24 lattice points. For these 24 points it is possible to find a perfect matching - one such matching is illustrated in Figure 6.

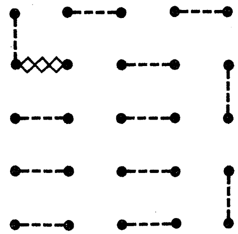
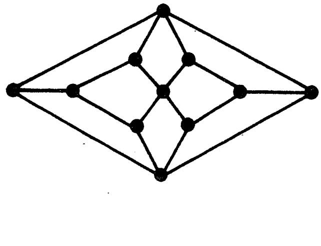


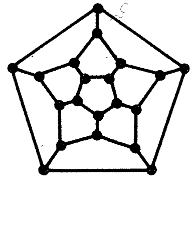
Figure 6

The second player should then treat this perfect matching as his collection of favoured possible moves and proceed just as we suggested the first player do earlier (when mn was even). Thus the second player should choose one of the favoured (dotted) lines at every move. By the way these favoured possible moves have been chosen this can always be done and moreover (just as for the first player in the situation where mn is even) by sticking to this suggested strategy the second player will always win.

The final thing to observe is that if mn is odd, whatever m and n are, and whatever the first move happens to be, by ignoring the appropriate endpoint of that move (i.e. the endpoint which is an even distance from some corner of the lattice) a perfect matching involving all the remaining $mn - 1$ lattice points can always be found. So if mn is odd the second player can always adopt the strategy we have outlined above (and described for a 5×5 point lattice) and be sure that he will win.

This completes our discussion of Slither on rectangular grids but it leaves a lot unsaid about more general versions of the game. One could apply the same rules and play Slither on a 'lattice' of many shapes. For example one could use the following 'playing fields' (the lines indicate all the possible single moves).





The strategies we have described for the players on an $m \times n$ lattice work just as well here - provided one can find suitable perfect matchings. A discussion of when suitable matchings exist and how one might go about finding them is beyond the scope of this present article so this is where we leave the story. We hope you try playing Slither for yourself - especially on unusually shaped playing fields - and perhaps you might write and tell us of any interesting discoveries you make about the game.

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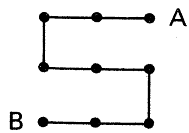
Slither - a reader's view

GRAHAM FARR (6th form, Melbourne High School) wrote to us about Slither. His letter began with a discussion of Ronald Read's symmetry strategy (see *Function*, Volume 1, Part 2, p. 14) and went on to explain a more general approach. This seemed quite a promising idea which unfortunately turned out not to solve the general Slither problem. Nevertheless we found the idea interesting and we now let Graham take up the story.

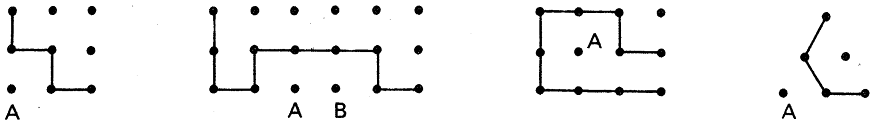
"Consider a general playing field - it may not be a field of squares; it may be a field of triangles or hexagons with the dots at the vertices. Now the first player's move does not use up or 'fill' any dots; both dots he connects can still have other lines joined to them. Thereafter each move fills one dot, as no more than two lines can ever meet at one dot. If every dot is used at least once in a game, all but 2 dots will be filled (e.g. *A* and *B* in the diagram on the right) prior to the last move.



Now if the total number of dots is odd, then a game where each dot has at least one line joined onto it (i.e. a 'completely filled' game) may be won by the second player; for after the first player makes his first move, there still remains an odd number of dots to fill and as each player now fills a dot with each move, the second player will be the last able to make a move without losing.

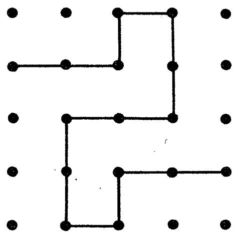
If the total number of dots is even, as in any rectangular field with at least one of the dimensions (i.e. m or n) even, then a completely filled game will be won by the first player, as after the first move there is still an even number of dots to fill, and so the first player will be the last able to move.

However, not all games are completely filled. Some dots may be 'isolated', as are the letter dots in the following diagrams.



Every dot isolated changes what I shall call the parity of the number of dots in the field. This is the evenness or oddness of the number of dots which are filled or can be filled - i.e., at any stage, all dots except those which have been isolated. On an odd field (a field with an odd number of dots) the parity is initially favourable to the second player, and so he will win a completely filled game on that field; but if an odd number of dots become isolated during the game, the parity changes and he will lose. Thus in such a game the first player will aim to isolate an odd number of dots, while the second player will try to leave no dots isolated, or to isolate an even number of dots. Similarly in an even field it is the other way round: the parity initially favours the first player and so it is the second player who tries to isolate an odd number of dots, and so on.

What, then, is the winning strategy for the second player on an odd rectangular field? I cannot find one. If the first player blunders and moves near the centre the second player plays symmetrically opposite him with respect to the central dot, as shown on the right. This, like all other symmetry-type strategies, keeps the parity odd and so favourable to the second player (in this case), but I cannot find a strategy which works no matter what first move the first player made. I suspect, however, that one of the players if he so desires can play so as to keep the parity unchanged."



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GOOD COMPANY

Alasdair McAndrew (6th form, Melbourne High School) writes to tell us (see Problem 4·3) that, in addition to our hypothetical student, Leibnitz also at one time thought that the derivative, with respect to x, of $u(x)v(x)$ was $u'(x)v'(x)$. (See E.T. Bell, *Men of Mathematics*, Pelican edition, Volume 1, p. 137.)

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A CURIOUS SET OF SERIES

D.V.A. Campbell, Monash University

Consider the series

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

The series is a geometrical progression, so that we know its sum S_n to n terms, and its sum S to infinity:

$$S_n = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}},$$

$$S = \frac{1}{1 - \frac{1}{2}} = 2.$$

Consider now the table

1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	...
$\frac{1}{2}$	$\frac{2}{4}$	$\frac{3}{8}$	$\frac{4}{16}$	$\frac{5}{32}$	$\frac{6}{64}$	...
$\frac{1}{4}$	$\frac{3}{8}$	$\frac{6}{16}$	$\frac{10}{32}$	$\frac{15}{64}$	$\frac{21}{128}$	...
$\frac{1}{8}$	$\frac{4}{16}$	$\frac{10}{32}$	$\frac{20}{64}$	$\frac{35}{128}$	$\frac{56}{256}$	...
$\frac{1}{16}$	$\frac{5}{32}$	$\frac{15}{64}$	$\frac{35}{128}$	$\frac{70}{256}$	$\frac{126}{512}$	...
$\frac{1}{32}$	$\frac{6}{64}$	$\frac{21}{128}$	$\frac{56}{256}$	$\frac{126}{512}$	$\frac{252}{1024}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The table is constructed as follows. Denote by $X_{i,j}$, i, $j = 1, 2, \ldots$, the entry in the i-th row and j-th column. Also put $X_{i,0} = 0$ and $X_{0,j} = 0$, for i, $j = 1, 2, \ldots$, and note that $X_{1,1} = 1$. Then, except for $X_{1,1}$, the entries in the table may be calculated, one by one, by the formula

$$X_{i,j} = \frac{1}{2}(X_{i,j-1} + X_{i-1,j}).$$

Despite their very different starting points, all rows have the same sums to infinity.

To demonstrate this, consider the second row: similar procedures will enable you to verify the statement for later rows. What we have done, at the first stage, is to write under each term of the sum a decomposition of that term into equal numbers to which it adds up.

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{6}{64} + \cdots$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

$$+ \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

$$+ \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

$$+ \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

$$+ \frac{1}{32} + \frac{1}{64} + \cdots$$

$$+ \frac{1}{64} +$$

$$= \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots)$$

$$+ \frac{1}{4}(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots)$$

$$+ \frac{1}{8}(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots)$$

$$+ \frac{1}{16}(1 + \frac{1}{2} + \frac{1}{4} + \cdots)$$

$$= (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots)S$$

$$= (S - 1)S = (2 - 1)2 = 2,$$

since $S = 2$.

PROBLEM: Can you prove that this must be true for all rows?

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COPS, ROBBERS and POISSON

G.A.Watterson, Monash University

In October last year, a man spent four days in Pentridge jail. He had been arrested by mistake; the police were looking for another man having the same surname.

Various probability models have been proposed to help the courts decide whether the correct man has been charged with an offence. Suppose the police know that a certain Mr Gobbledegook committed a crime. They search through the city, picking out men at random, until they find a Mr Gobbledegook. What is the probability that he is the guilty one? Or again, a maiden has lost a glass slipper at a ball; what is the probability that the first maiden that the Prince finds, whose foot fits the slipper, is the real Cinderella? Of course, if we knew how many people, say X, in the community had the correct name (or foot size), we could set the probability that the randomly chosen one was the *correct* one at $1/X$. But if we do not know X, perhaps we might be able to justify some probability distribution for X. Appropriate perhaps for the Cinderella search, but less appropriate for Mr Gobbledegook, might be the assumption that X has a Poisson† distribution.

We reason as follows. There is a large number, n, of maidens in the city but each has a very small probability, p, of having grown up to have the correct size of foot. The number, X, of maidens with correct foot size should perhaps have a binomial probability distribution, but a Poisson distribution with the mean $\mu = np$ might do just as well in these circumstances. However, we should make one modification. The Prince knows Cinderella does exist, so that $X \geqslant 1$. Conditioning the Poisson distribution by excluding the $X = 0$ possibility yields us the *truncated* Poisson distribution

$$Pr\{X = x \mid X \geqslant 1\} = \frac{Pr\{X = x\}}{1 - Pr\{X = 0\}}$$

$$= \frac{e^{-\mu}}{1 - e^{-\mu}} \frac{\mu^x}{x!} = \frac{1}{e^{\mu} - 1} \frac{\mu^x}{x!} \text{ for } x = 1, 2, 3, \ldots .$$

Let us return to our problem. The expected chance that the

†The Poisson distribution gives probability $e^{-\mu} \frac{\mu^x}{x!}$ to x, for $x = 0, 1, 2, 3, \ldots$. It has mean μ and variance μ.

first maiden whose foot fits the slipper is the real Cinderella is then

$$E(1/X) = \sum_{x=1}^{\infty} \frac{1}{x} \frac{1}{e^{\mu} - 1} \frac{\mu^x}{x!},$$

$$= \frac{1}{e^{\mu} - 1} \sum_{x=1}^{\infty} \frac{\mu^x}{x \cdot x!} .$$

A short table of values follows.

μ	·1	·2	·5	1	2	10	20
$E(1/X)$	·975	·951	·879	·767	·577	·113	·053

Table 1

Notice that as the expected number of persons, μ, having the correct attribute increases, the chance that the correct one of them is chosen at random gets nearer to $1/\mu$. But even for very small values of μ, the doubt in our minds as to whether some person other than the correct one also has the attribute means we are not absolutely sure we have discovered the correct one.

Another model may be more convincing. In the criminal situation, suppose that the criminal goes to great lengths *not* to be found by the police. So we shall assume that if there are one or more people who could be mistaken for the criminal, then the police will in fact arrest one of these innocent people. But if only the criminal has the correct attribute, then the police will arrest him. The probability of correct arrest in this model is

$$P = Pr\{X = 1 | X \geqslant 1\} = \frac{1}{e^{\mu} - 1}\mu,$$

using our truncated Poisson distribution. A short table of P follows.

μ	·1	·2	·5	1	2	10	20
P	·951	·903	·771	·582	·313	·0005	4×10^{-8}

Table 2

By taking such evasive action, the criminal has clearly reduced his chances of being the person actually arrested compared with the model in which he is as likely to be arrested as the other persons (if any) with the same attribute.

As a numerical example, suppose that it has been calculated that the probability of a person having a particular sort[†] of finger-print found at a bank robbery is $p = 0{\cdot}6 \times 10^{-9}$. Further, suppose that there are $n = 3{\cdot}33 \times 10^{9}$ people on earth. Then you would expect that there would be *two* persons ($\mu = n \times p \approx 2$) on earth having such a finger-print. If the police arrest a person with such a finger-print, then the chance that he is the robber is, from Table 1, 0·577 assuming the robber takes no evasive action, or, from Table 2, 0·313 if the robber does take evasive action.

Of course you can easily find objections to our assumptions! But it is not easy to give a really convincing discussion about the chances of wrong arrest.

For further ideas along these lines, see the articles by

Kingston, C.R.: Applications of probability theory in criminalistics, *American Statistical Association Journal*, 60(1965), 70-80, and

Kreith, K.: Mathematics, social decisions and the law, *International Journal of Mathematical Education in Science and Technology*, 7(1976), 315-330.

[†] A finger-print may be partially smudged or otherwise obscured. Finger-prints are classified according to certain characteristics and one of a particular sort is one, say, of a thumb, which displays clearly a certain number of these characteristics. Different finger-prints have certain characteristics in common. As the number of clearly displayed characteristics of a finger-print increases so the chance of the print coming from different people diminishes.

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LETTER TO THE EDITORS, FROM P.J. LARKIN, ASSUMPTION COLLEGE, KILMORE

Dear Sirs,

The program for a solution to numerical integration in the HP-25 Program Handbook is tedious and requires calculation of many $f(x)$ values "manually" before an answer can be obtained.

Here is a versatile program that requires no more than the entering of the function, the domain and the number of intervals (which must be an even integer).

There are 18 steps available that may be used to write in the function.

The program is based on Simpson's rule. Here it is: the best yet HP-25 program for numerical integration.

```
01   STO 4
02     -
03   RCL 1
04     ÷
05   STO 0
06   RCL 4

25   STO 6
26   RCL 1
27   RCL 3
28   g x=0
29   GTO 40
30   f x=y
31   GTO 44
32     2
33     ÷
34   g FRAC
35   g x≠0
36   RCL 6
37   STO+6
38   RCL 6
39   STO+6
40   RCL 0
41   RCL 6
42     Σ+
43   GTO 06
44   RCL 0
45   RCL 6
46     Σ+
47   RCL 5
48     3
49     ÷
```

The storage registers are used for:

R_0 $\delta x = \frac{b - a}{n}$

R_1 n (number of segments, which must be even)

R_2

R_3 i (particular segment number)

R_4 $a + i\ \delta x$

R_5 $3 \times \Sigma f(x)\ \delta x$

R_6 $f(x_i)$

R_7 $\Sigma f(x)$.

In terms of this program to calculate $\int_a^b f(x)dx$ for a particular $f(x)$ you then proceed as follows.

1. Key in lines 1-6 of program
2. Key in lines to calculate $f(x)$ (x initially in X register)
3. Key in branch to 25: GTO 25
4. Press SST until display shows line 24
5. Key in lines 25-49 of program
6. Switch to run mode
7. Clear registers and stack: f REG, f STK
8. Initialize (number of segments): n, STO 1
9. Enter upper terminal: b
10. Roll up: ↑
11. Enter lower terminal: a
12. Compute $\int_a^b f(x)dx$: f PRGM, R/S

Yours sincerely,

P.J. Larkin

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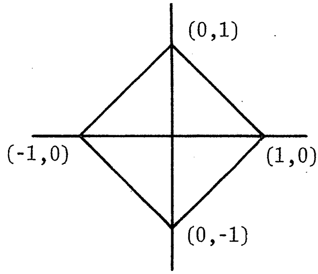
SOLUTION TO PROBLEM 2.2

In the metric space whose points are those of the cartesian plane, but for which the distance between (x, y) and (u, v) is $|x - u| + |y - v|$, the open ball centre (0, 0) and radius 1 is the interior of the square in the diagram.



SOLUTION TO PROBLEM 2.7

Calculations of $s(x) = x[1 - 0{\cdot}16605x^2 + 0{\cdot}00761x^4]$ at intervals of 0·1, from $x = 0$ to $x = \pi/2$, using a G.E. Genius 85 Scientific calculator, disagree in at most two digits in the fourth decimal place with published 4-figure tables of sin x over the range. We list the results of the calculations below, together with the values of sin x.

x	$s(x)$	sin x
0	0	0
0·1	0·0998	0·0998
0·2	0·1987	0·1987
0·3	0·2955	0·2955
0·4	0·3895	0·3894
0·5	0·4795	0·4794
0·6	0·5647	0·5646
0·7	0·6443	0·6442
0·8	0·7175	0·7174
0·9	0·7834	0·7833
1·0	0·8416	0·8415
1·1	0·8912	0·8912
1·2	0·9320	0·9320
1·3	0·9634	0·9636
1·4	0·9853	0·9854
1·5	0·9974	0·9975
$\pi/2$ = 1·5708	1·0000	1·0000

$$s(x) = x(1 + x^2(0{\cdot}00761x^2 - 0{\cdot}16605))$$

G.E. Genius 85 Scientific Calculator.

Note: $x(1 + (x^2/6)(0{\cdot}05x^2 - 1))$ (from MacLaurin) is better for lower values but much worse at upper end (% error at $\pi/2$ is 0·45).

SOLUTION TO PROBLEM 3.3

We print below a solution sent us by David Dowe (Geelong Grammar School, Corio). The problem was to show that if a and b are integers, then $ab(a^2 + b^2)(a^2 - b^2)$ is divisible by 30. The problem was also solved by Geoff Chappell (Grade 11, Kepnock High School, Bundaberg), by Christopher Stuart (Lois Street, Ringwood East) and by Rob Saunders, who set the problem.

"$30 = 2 \times 3 \times 5$

i) If a or b is even, ab is a multiple of 2.
If a and b are both odd, $a^2 + b^2$ is a multiple of 2.

ii) If a or b is a multiple of 3, ab is a multiple of 3.
If neither a nor b is a multiple of 3, then a can be written as $3x \pm 1$, and b as $3y \pm 1$.
$a^2 - b^2 = (3x \pm 1)^2 - (3y \pm 1)^2 = 9x^2 \pm 6x + 1 - 9y^2 \mp 6y - 1 = 3(3x^2 \pm 2x - 3y^2 \mp 2y)$, which is a multiple of 3.

iii) If a or b is a multiple of 5, ab is a multiple of 5.
If $a \not\equiv 0 \bmod 5$, $a^4 \equiv 1 \bmod 5$. Similarly, if $b \not\equiv 0 \bmod 5$, $b^4 \equiv 1 \bmod 5$.
Thus, $(a^2 - b^2)(a^2 + b^2) = a^4 - b^4 \equiv 0 \bmod 5$. Hence $ab(a^2 - b^2)(a^2 + b^2)$ is a multiple of $2 \times 3 \times 5 = 30$."

SOLUTION TO PROBLEM 3.5

We printed in Part 4 of *Function* an incorrect solution to this problem. The editors apologise to our readers. That the method of solution was incorrect and that it was obvious intuitively that the answer should be $\frac{1}{2}$ was pointed out by both Christopher Stuart and David Dowe. Both provided an elegant correct solution. We reproduce Christopher Stuart's.

"The problem is:-

'A die is thrown until a 6 is obtained. What is the probability that a 5 was not thrown meanwhile?'

If the statement of the problem is revised slightly, the solution is obvious:-

'What is the probability that, in successive throws of a die, the first throw that is either a 5 or a 6, is a 6?'

Clearly, the answer is a $\frac{1}{2}$.

However, I also present a more rigorous proof:-

The total probability P is equal to the probability of throwing a 6 in one throw, plus the probability of throwing a number from 1 to 4 in one throw and a 6 in the next, plus the probability of throwing numbers from 1 to 4 in the first two throws and a 6 in the next, and so on.

$$\text{Hence } P = \frac{1}{6} + \frac{4}{6}\frac{1}{6} + \frac{4}{6}\frac{4}{6}\frac{1}{6} + \frac{4}{6}\frac{4}{6}\frac{4}{6}\frac{1}{6} + \cdots$$
$$= \frac{1}{6} + \frac{4}{6}\frac{1}{6} + \left(\frac{4}{6}\right)^2\frac{1}{6} + \left(\frac{4}{6}\right)^3\frac{1}{6} \cdots.$$

This is of the form $a + ar + ar^2 + ar^3 + \cdots$ which has sum $\frac{a}{1 - r}$ if $-1 < r < 1$.

Hence $P = \frac{1/6}{1 - 4/6} = 1/2$, as before."

SOLUTION TO PROBLEM 3.6 We print the solution of David Dowe (Geelong Grammar School, Corio).

"i) If we change the speed of the record from $33\frac{1}{3}$ r.p.m. to 45 r.p.m., we will find that the frequencies of the notes multiply by $\frac{45}{33\frac{1}{3}} = \frac{135}{100} = 1\cdot35$

$1\cdot35 \simeq (\sqrt[12]{2})^5$ therefore the notes rise by 5 semitones.

C becomes F, E becomes A, G becomes C (above middle C).

ii) Again, $\frac{78}{33\frac{1}{3}} = \frac{234}{100} = 2\cdot34 \simeq (\sqrt[12]{2})^{15}$. Therefore the notes rise by an octave and 3 semitones, becoming E♭, G and B♭."

SOLUTION TO PROBLEM 3.7

This is the problem about whether further information changes the chances of a prisoner's execution. What the correct solution to this problem is has been the subject of debate. We print David Dowe's (Geelong Grammar) convincing argument.

"No, Mark is not right to feel happier. There were three groups of people who could have been executed: A) Mark and Luke, B) Mark and John, C) Luke and John. We are told that Luke is to be executed, narrowing our number of groups down to two, A and C.

If A were to be executed, the jailer would have given Luke's name.

If C were to be executed, the jailer could have said Luke or John.

Thus, Pr(jailer says Luke) = Pr(A) × Pr(jailer now saying Luke) + Pr(C) × Pr(jailer now saying Luke)

$$= \Pr(A) \times 1 + \Pr(C) \times \frac{1}{2}$$
$$= \frac{1}{3} + \frac{1}{3} \times \frac{1}{2}\ .$$

Thus, we see that when the jailer says 'Luke', there are twice as many chances of Event A as there are of Event C.

Thus, Mark's chances of being executed are still $\frac{2}{3}$, and he has no reason to be happier."

SOLUTION TO PROBLEM 4.1

The following solution is by Elijah Glenn Merlo (6th form, Taylor's College). It was also solved by David Dowe (Geelong Grammar School) and Alasdair McAndrew (Melbourne High School) who supplied the problem.

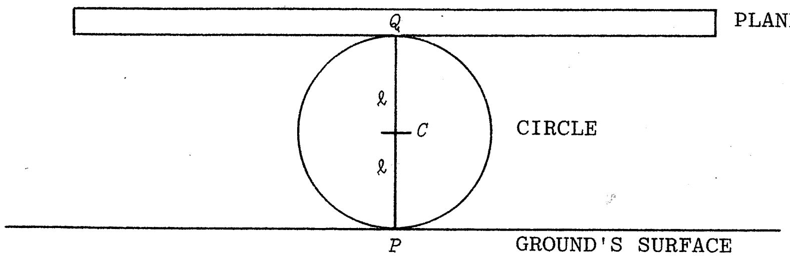
"Let the circle have radius ℓ, centre at C, and a velocity v at C. And let the velocity of the plank be V. (All velocities are taken relative to ground's surface.)

Let P be any point on circle's circumference, and Q be a point such that P, C, and Q are collinear.

Since circle does not slip, when P is in contact with ground $\overline{PQ}$ is vertical, and since plank does not slip, at that particular instant the velocity of plank, $V \equiv$ velocity of Q.

Since at this instant P is stationary (relative to ground) we can think of $\overline{PQ}$ as a radius sweeping round about point P and so using ω as the instantaneous angular velocity of $\overline{PQ}$ about P (when P on ground's surface) we have velocity of C, i.e. v, equal to $\ell\omega$ and the velocity of Q, which is the same as V, equal to $2\ell\omega$.

So $\frac{V}{v} = 2$."



SOLUTION TO PROBLEM 4.2

Solutions were received from David Dowe (Geelong Grammar School) and Geoff Chappell (Kepnock High School). We print David Dowe's.

$$"(n^2 + \tfrac{n}{2})^2 = n^4 + n^3 + \tfrac{1}{4}n^2$$

$$(n^2 + \tfrac{n}{2} + 1)^2 = n^4 + n^3 + \tfrac{9}{4}n^2 + n + 1.$$

Therefore for any integral $n (n \neq 0)$, $(n^2 + \frac{n}{2})^2 < n^4 + n^3 + n^2 + n + 1 < (n^2 + \frac{n}{2} + 1)^2$. If n is even, $n^2 + \frac{n}{2}$ and $n^2 + \frac{n}{2} + 1$ are two consecutive integers, which the (positive) square root of $n^4 + n^3 + n^2 + n + 1$ is in between. If n is odd, $n^2 + \frac{n}{2} + \frac{1}{2}$ is an integer, and the only integer between $n^2 + \frac{n}{2}$ and $n^2 + \frac{n}{2} + 1$.

If we try $(n^2 + \frac{n}{2} + \frac{1}{2})^2 = n^4 + n^3 + n^2 + n + 1$, we end up with the quadratic: $n^2 - 2n - 3 = 0$, which is satisfied by $n = -1$ or 3.

We find that as we have examined all the positive and negative integers, 0 is the only other possible solution. As is easily discovered, 0 is a solution."

SOLUTION TO PROBLEM 4.3

This problem was also solved by David Dowe (Geelong Grammar School) and Geoff Chappell (Kepnock High School). We print Geoff Chappell's solution.

"Since $\frac{d}{dx}\{u(x)v(x)\} = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x)$, if the student's belief that $\frac{d}{dx}\{u(x)v(x)\} = \frac{d}{dx}u(x) \cdot \frac{d}{dx}v(x)$ is correct, we must have the relation, $u(x) \cdot v'(x) + v(x) \cdot u'(x) = u'(x)v'(x)$, which is equivalent to $\frac{u(x)}{u'(x)} + \frac{v(x)}{v'(x)} = 1$. $\int\frac{u'(x)dx}{u(x)} = \log u(x)$, $\int\frac{v'(x)dx}{v(x)} = \log v(x)$. So if $\frac{u(x)}{u'(x)} = f(x)$ and $\frac{v(x)}{v'(x)} = g(x)$, so that $f(x) + g(x) = 1$ then $u(x) = e^{\int\frac{dx}{f(x)}}$, $v(x) = e^{\int\frac{dx}{g(x)}}$ or $u(x)v(x) = e^{\{\int\frac{dx}{f(x)}+\int\frac{dx}{g(x)}\}} = e^{\int\frac{dx}{f(x)g(x)}}$.

Examples (a) $\sin^2 x + \cos^2 x = 1$ so $f(x) = \sin^2 x$, $g(x) = \cos^2 x$; $u(x)v(x) = e^{\tan x} \cdot e^{-\cot x} = e^{\tan x - \cot x}$ can be differentiated by the student's rule.

(b) If $f(x) = x$ and $g(x) = 1 - x$ then $e^{\int\frac{dx}{x}} \cdot e^{\int\frac{dx}{1-x}} = x \cdot \frac{1}{1 - x}$ can be differentiated by the student's rule."

SOLUTION TO PROBLEM 4.4

Again David Dowe and Geoff Chappell sent in solutions to this problem. Their methods were quite different. We publish David Dowe's.

" $11 = 4 \times 2 + 3$
$111 = 4 \times (2 + 25) + 3$
$1111 = 4 \times (2 + 25 + 250) + 3$, etc.

Thus, it can be seen that all the numbers 11, 111, 1111, ... can be written in the form $4x + 3$, $x \in N$. $4x + 3$ is odd, so if it is a square, it is the square of an odd number. As all odd numbers can be written in the form, $4y \pm 1$ $(y \in N)$, we can investigate this.

$$(4y \pm 1)^2 = 16y^2 \pm 8y + 1 = 4(4y^2 \pm 2y) + 1.$$

Thus no odd square (and indeed, no even square) can be written in the form $4x + 3$, and therefore none of the numbers 11, 111, 1111, ... can be squares."

SOLUTION TO PROBLEM 4.5. David Dowe sent us the following solution.

"If we take a pedal from the box marked LEFT, and the pedal is a right-foot pedal, we cannot tell if the box is the RIGHT or the MIXED box.

Similarly, if we take a pedal from the box marked RIGHT, and the pedal is a left-foot pedal, we cannot tell which box it came from.

However, if we take a pedal from the box marked MIXED, and the pedal is

i) right-foot, this is the RIGHT box, as all the labels were changed (the box reading MIXED cannot be the MIXED box). Thus, the box marked LEFT must be the MIXED box, and the box marked RIGHT is the LEFT box.

ii) left-foot, the above argument applies with LEFT and RIGHT reversed."

PROBLEM 5.1 (Supplied by Elijah Glenn Merlo.)

If you are given a hoop, a disc and a sphere, each of uniform density and each of radius R units, and you roll them simultaneously down the slope of steepest descent of an inclined plane, which ones arrive first and last at the inclined plane's foot?

PROBLEM 5.2

Let $S = X_{11} + X_{12} + \cdots + X_{1n} + \cdots$ be a convergent series with sum S. Construct an array as in Douglas Campbell's article "A Curious Set of Series" (p. 21) where the entry X_{ij}, for $i \geqslant 2$, and $j = 1, 2, \ldots,$ in the i-th row and j-th column is given by the formula given in that article. Show that each row in this array has sum S.

PROBLEM 5.3

From a British matriculation examination in 1896:

Find the prime factors of (the single number)

5,679,431,432,056,743,205,685,679,432.

(Hint for twentieth century students: 71 is a factor.)

PROBLEM 5.4

Check that $p(x) = x^2 - x + 41$ is a prime for $x = 1, 2, \ldots$ $\ldots, 40$.

"I know what you're thinking about", said Tweedledum; "but it isn't so, nohow."

"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."

Lewis Carroll: *Through the Looking Glass*

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That was excellently observ'd, say I, when I read a Passage in an Author, where his Opinion agrees with mine. When we differ, there I pronounce him to be *mistaken*.

Jonathan Swift, 1711

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"He's remarkably average for his age."

Punch: 22 June, 1977

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"Contrary to an earlier report, one of the seven escapees was not the convicted murderer A.B."

ABC News, 1 September, 1977

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"All exact science is dominated by the idea of approximation."

Bertrand Russell

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